Particle Accelerators, 1989, Vol. 24, pp. 187–210
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Printed in the United States of America

MOMENT ANALYSIS OF SPACE CHARGE EFFECTS IN AN AVF CYCLOTRON

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(Received May 2, 1988; in final form September 30, 1988)

In comparison with linear accelerator structures the circular accelerator has the additional complication that a change in momentum spread due to longitudinal space-charge forces immediately influences the transverse particle distribution. The isochronous cyclotron is especially sensitive to this effect because of the absence of longitudinal focusing. In this paper we derive a set of differential equations for the second-order moments of the phase-space distribution function that takes into account this special feature of the circular accelerator. For the isochronous cyclotron a smoothed system of equations is also obtained that gives additional insight into the problem. The derivation is an application of the RMS (root-mean-square) approach in which only the linear part of the space-charge forces (as determined by a least-squares method) is taken into account and the charge distribution is assumed to have ellipsoidal symmetry. Since the longitudinal-transverse coupling may destroy the symmetry of the bunch with respect to the reference orbit, we allow the ellipsoid to be rotated around its vertical axis. We only consider free single bunches; i.e., we do not consider forces arising from image charges and from neighboring turns. Different integrals of motion of the moment equations are obtained including the total angular canonical momentum in the bunch, the total energy content of the bunch, and the RMS representation of the 4-dimensional horizontal phase-space volume. For bunches with a circular horizontal cross section the smoothed-moment equations reduce to RMS-envelope equations. Some numerical results obtained with the model will be presented, with the 3-MeV minicyclotron (ILEC) presently under construction at the Eindhoven University taken as an example.

1. INTRODUCTION

Analytical studies of space-charge effects in accelerators that have appeared in the literature thus far are mainly concerned with linear structures.[1–3] For circular accelerators such as the cyclotron the analysis is mostly done with numerical calculations based on many-particle codes.[4,5] In comparison with a linear structure the circular accelerator has the special feature that the transverse position of the particle depends on the longitudinal momentum due to dispersion in the bending magnets. This means that a change in longitudinal momentum spread immediately influences the transverse particle distribution. Particles in the "tail" of the bunch will lose energy due to the longitudinal space-charge forces and thus move to a smaller radius. The opposite happens for the leading particles in the bunch. The isochronous cyclotron is especially sensitive to this effect because there exists no rf focusing in the longitudinal phase space. Numerical calculations by Adam[6] show that the effect can become really important in an isochronous cyclotron.

Approximate representations of relevant properties of the bunch such as the sizes and the momentum spread are obtained from the second-order moments of the phase-space distribution function. In this paper we derive differential equations that determine the time dependence of these moments. This time dependence is directly related to the time dependence of the distribution function as determined by the Vlasow equation. This equation can be obtained from Liouville's theorem, and therefore the time dependence of the moments follows from the Hamiltonian. Now, the second-moment equations form a closed system if the equations of motion for the single particle are linear in the variables, i.e., for a Hamiltonian quadratic in the variables. Therefore we will use a linear approximation for the external forces as well as for the space-charge forces. In the derivation we will neglect the forces that arise from image charges. Moreover we will neglect the forces produced by the bunches in the neighboring turns. We expect that for the central region this will be a reasonable assumption because in this region the turns are usually well separated. In regions where the turns overlap significantly the approach will become less useful, however.

In Section 2 first of all some basic equations are presented. In Section 3 we derive a suitable Hamiltonian for the linearized particle motion in the external magnetic field without space charge. For this we make the same approximations as in Ref. 7; the most important is the assumption of an azimuthal variation of the magnetic field that is not too large. For convenience we omit the acceleration process. The derivation is such, however, that acceleration can be included in a straightforward way using methods developed by Schulte et al.[8,9] The Hamiltonian describes the particle motion in a coordinate system that moves with the bunch along a reference orbit (equilibrium orbit). Due to the azimuthal variation of the magnetic field the Hamiltonian depends explicitly on time. By a smoothing procedure this time dependence is removed, resulting in a simpler Hamiltonian for an isochronous cyclotron.

In Section 4 we then define an electric space-charge potential that is quadratic in the variables and that must be added to the unperturbed Hamiltonian. This potential also includes the magnetic self-field of the bunch. For the definition we generalize Sacherer's approach,[2] in which the linear part of the forces is determined by a least-squares method and the charge distribution is assumed to have ellipsoidal symmetry. Here we allow this ellipsoid to be rotated around the vertical axis through the bunch because the dispersion effect in the cyclotron may destroy the symmetry of the bunch with respect to the equilibrium orbit. For the calculation of the electric fields we neglect the curvature of the equilibrium orbit. This will be a good approximation as long as the transverse size of the bunch is small compared with the local radius of curvature. With these assumptions the coefficients in the potential function can be expressed in terms of the second moments of the charge distribution.

In Section 5 we derive two systems of moment equations using the results obtained in Section 3 for the particle motion in the external magnetic field and the electric space-charge potential derived in Section 4. Each system forms a set of thirteen coupled first-order differential equations. The first system corresponds to the time-dependent Hamiltonian and may be considered as being the most general of these two in the sense that the least amount of approximation has been used in the derivation. The second system corresponds to the smoothed Hamiltonian for the isochronous cyclotron. These equations contain the additional approximation that the influence of the smoothing procedure on the electric potential function is neglected. On the other hand these equations may give additional insight into the problem. Within the approximations made, the numerical integration of the moment equations will give a description of the RMS properties of the bunch under space-charge conditions.

2 BASIC EQUATIONS

We consider a canonical system with generalized coordinates $\mathbf{x} = (x, s, z)$, canonical momenta $\mathbf{p} = (p_x, p_s, p_z)$, and independent time variable t. The motion of the particle follows from the Hamiltonian $H = H(\mathbf{x}, \mathbf{p}, t)$ via the Hamiltonian equations:

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{x}}. \tag{1}$$

The phase-distribution function $f(\mathbf{x}, \mathbf{p}, t)$ is defined as the particle density in the 6-dimensional phase space:

$$dN = f(\mathbf{x}, \mathbf{p}, t)\, d\mathbf{x}\, d\mathbf{p}, \tag{2}$$

where dN is the number of particles in the phase-space volume $d\mathbf{x}\, d\mathbf{p}$. The total number of particles in the bunch then becomes

$$N = \iint_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{p}, t)\, d\mathbf{x}\, d\mathbf{p}. \tag{3}$$

According to Liouville's theorem the distribution function f remains constant for an observer who travels with an arbitrary particle in phase space. This means that f is an integral of motion of the canonical system:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} + \frac{\partial f}{\partial \mathbf{p}}\frac{d\mathbf{p}}{dt} = 0. \tag{4}$$

Substitution of Hamilton's equations [Eq. (1)] into Eq. (4) gives the Vlasow equation:

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial \mathbf{p}}\frac{\partial f}{\partial \mathbf{x}} - \frac{\partial H}{\partial \mathbf{x}}\frac{\partial f}{\partial \mathbf{p}} = 0. \tag{5}$$

Under space-charge conditions the Hamiltonian H takes the following form:

$$H(\mathbf{x}, \mathbf{p}, t) = H_0(\mathbf{x}, \mathbf{p}, t) + q\phi(\mathbf{x}, t) \tag{6}$$

where q is the charge of the particle, ϕ is the potential function due to all the other particles in the bunch, and H_0 is the Hamiltonian corresponding to the external forces.

We introduce a curved coordinate system (x, s, z), where x is the horizontal coordinate of the particle with respect to the curved orbit, s is the distance along the curved orbit, and z is the vertical coordinate. If we ignore for the moment the self-consistent magnetic field, then the potential function ϕ follows from the Poisson equation. In the curved coordinate system this equation becomes

$$\frac{1}{1+x/\rho_c}\left\{\frac{\partial}{\partial x}\left[\left(1+\frac{x}{\rho_c}\right)\frac{\partial \phi}{\partial x}\right]+\frac{\partial}{\partial s}\left(\frac{1}{1+x/\rho_c}\frac{\partial \phi}{\partial s}\right)+\left(1+\frac{x}{\rho_c}\right)\frac{\partial^2 \phi}{\partial z^2}\right\}=-\frac{\rho}{\varepsilon_0}, \tag{7}$$

where ρ is the space-charge density in the bunch and $\rho_c = \rho_c(s)$ is the local radius of curvature of the curved orbit. However, in the subsequent analysis it is assumed that the transverse size of the bunch is much smaller than the local radius of curvature $\rho_c(s)$. In this case, the coordinate system can be approximated locally as being cartesian, and the Poisson equation simplifies to

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial s^2}+\frac{\partial^2 \phi}{\partial z^2}=-\frac{\rho}{\varepsilon_0}. \tag{8}$$

The self-consistent magnetic field can be included if we neglect the velocity spread in the bunch, i.e., if we assume that all particles have the same velocity v_0. Equation (8) then must be slightly adapted to the form:

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial z^2}+\frac{1}{\gamma^2}\frac{\partial^2 \phi}{\partial s^2}=-\frac{\rho}{\varepsilon_0 \gamma^2}, \tag{9}$$

where $\gamma \equiv (1-v_0^2/c^2)^{-\frac{1}{2}}$ and c is the speed of light. Finally, the charge density ρ is simply related to the distribution function f via the formula:

$$\rho(\mathbf{x}, t)=\frac{q}{1+x/\rho_c}\int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{p}, t)\, d\mathbf{p} \approx q\int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{p}, t)\, d\mathbf{p} \tag{10}$$

The second moments of the distribution function are defined as expectation values of the products of two canonical variables. For example, for the second moment $\langle xp_x \rangle$ we obtain

$$\langle xp_x \rangle = \frac{1}{N}\iint_{-\infty}^{\infty} xp_x f(\mathbf{x}, \mathbf{p}, t)\, d\mathbf{x}\, d\mathbf{p}. \tag{11}$$

The Vlasov equation determines the time evolution of the second moments. Consider as an example a system with only one degree of freedom where $H = H(x, p_x, t)$. If we multiply the Vlasow equation by x^2, xp_x, and p_x^2 and integrate over the phase space we obtain the following system of equations:

$$\begin{aligned}\frac{d}{dt}\langle x^2\rangle &= 2\left\langle x\frac{\partial H}{\partial p_x}\right\rangle\\ \frac{d}{dt}\langle xp_x\rangle &= \left\langle p_x\frac{\partial H}{\partial p_x}\right\rangle - \left\langle x\frac{\partial H}{\partial x}\right\rangle\\ \frac{d}{dt}\langle p_x^2\rangle &= -2\left\langle p_x\frac{\partial H}{\partial x}\right\rangle,\end{aligned} \tag{12}$$

where partial integration has been used to calculate the right-hand sides. [We note that Eqs. (12) immediately follow by interchanging the averaging and the differentiation in the left-hand sides of the equations.] From Eqs. (12) it follows that for a Hamiltonian quadratic in the variables, the right-hand sides depend on second moments only. In this case the system of equations is closed, provided that the coefficients in the Hamiltonian can be expressed in terms of the second moments or that their time dependence is known.

3 THE SINGLE PARTICLE HAMILTONIAN

In this section we derive a suitable Hamiltonian H_0 for the linearized motion of nonaccelerated particles in an azimuthally-varying-field (AVF) cyclotron. It is assumed that the magnetic field has perfect symmetry with respect to the median plane and perfect S-fold rotational symmetry ($S \geqslant 3$). Furthermore, it is assumed that the amplitude of the azimuthal variation of the magnetic field is not too large. In this respect the same approximations as in Ref. 7 will be used.

In polar coordinates (r, θ) the magnetic field in the median plane can be separated into an average part and an azimuthally varying part (the flutter) that is represented by a Fourier series:

$$B(r, \theta) = \bar{B}(r)\left\{1 + \sum\left[A_n(r)\cos n\theta + B_n(r)\sin n\theta\right]\right\}, \tag{13}$$

where $\bar{B}(r)$ is the average field at radius r. Due to the assumed S-fold symmetry, only terms with $n = kS$, $k = 1, 2, 3 \ldots$, are present in the Fourier series. We consider a reference particle with kinetic momentum P_0, and associated with this kinetic momentum we define a reference radius r_0 with the relation $P_0 = qr_0\bar{B}(r_0)$. For subsequent use we introduce the following field quantities to be evaluated at the reference radius r_0:

$$\begin{aligned}\bar{\mu}' &= \left(\frac{r}{\bar{B}}\frac{d\bar{B}}{dr}\right)_{r_0}, & A_n &= A_n(r_0)\\ A_n' &= \left(r\frac{dA_n}{dr}\right)_{r_0}, & A_n'' &= \left(r^2\frac{d^2A_n}{dr^2}\right)_{r_0},\end{aligned} \tag{14}$$

and similar definitions for the sine coefficients.

For the study of space-charge effects it is convenient to use curved coordinates instead of polar coordinates. For this purpose we choose the so-called equilibrium orbit as reference orbit. The equilibrium orbit is defined as a closed orbit in the median plane with the same S-fold symmetry as the magnetic field. In polar coordinates the equilibrium orbit for the reference particle is given by the relation[7]:

$$r_e(\theta) = r_0\left\{1 - \sum\left[\frac{3n^2-2}{4(n^2-1)^2}(A_n^2+B_n^2) + \frac{1}{2(n^2-1)}(A_nA_n' + B_nB_n')\right]\right.$$
$$\left. + \sum\left(\frac{A_n}{n^2-1}\cos n\theta + \frac{B_n}{n^2-1}\sin n\theta\right)\right\}. \tag{15}$$

The effective radius R_0 is defined as the length of the equilibrium orbit divided by 2π. Using Eq. (15) we obtain for R_0:

$$R_0 = r_0\left[1 - \sum \frac{A_n^2 + B_n^2 + A_nA_n' + B_nB_n'}{2(n^2-1)}\right]. \tag{16}$$

The general expression for the Hamiltonian H_0 in the curved coordinate system is given by

$$H_0 = \left[E_0^2 + (p_x - qA_x)^2c^2 + (p_z - qA_z)^2c^2 + \left(\frac{p_s}{1+x/\rho_c} - qA_s\right)^2c^2\right]^{1/2}, \tag{17}$$

where $E_0 = m_0c^2$ is the rest energy of the particle, m_0 is the rest mass, $\rho_c(s)$ is the local radius of curvature of the curved orbit, and A_x, A_s, A_z are the components of the magnetic vector potential. We choose the coordinate system such that (x, s, z) form a left-handed system. With this definition a positively charged particle moves in the direction of increasing s when the magnetic field is pointing in the positive z direction.

If $B(x, s)$ is the median-plane field as a function of the new coordinates x and s, then a related vector potential in the left-handed system is

$$A_x(x, s, z) = \frac{-\frac{1}{2}z^2}{1+x/\rho_c}\frac{\partial B}{\partial s} + O(z^4)$$
$$A_s(x, s, z) = \tfrac{1}{2}z^2\frac{\partial B}{\partial x} - \frac{1}{1+x/\rho_c}\int_0^x\left(1+\frac{x'}{\rho_c}\right)B(x', s)\,dx' + O(z^4) \tag{18}$$
$$A_z(x, s, z) = 0.$$

Here we used the symmetry of the magnetic field with respect to the median plane $z = 0$. To calculate this vector potential we need the median-plane magnetic field B as a function of the new coordinates x and s. For this we make the transformation from the polar coordinates r and θ to the curved coordinates x and s using the expression for the equilibrium orbit [Eq. (15)] and substitute the result into the expression for the median-plane field, Eq. (13). Expanding A_x and A_s with respect to x/r_0 and z/r_0 and retaining terms up to second degree, we obtain for the vector potential:

$$A_x(x, s, z) = r_0\bar{B}(r_0)\left[\frac{1}{2}\frac{z^2}{r_0^2}\sum\left(nA_n \sin\frac{ns}{R_0} - nB_n \cos\frac{ns}{R_0}\right)\right] \tag{19}$$

$$A_s(x, s, z) = -r_0\bar{B}(r_0)\left[\frac{x}{\rho_c} - \frac{1}{2}\frac{x^2}{r_0^2}\left(Q_z + \frac{r_0^2}{\rho_c^2}\right) + \frac{1}{2}\frac{z^2}{r_0^2}Q_z\right]. \tag{20}$$

Here $Q_z(s)$ and the local radius of curvature $\rho_c(s)$ are given by

$$Q_z(s) = -\left[\bar{\mu}' + \sum\frac{-n^2(A_n^2 + B_n^2) + A_nA_n' + B_nB_n' + A_nA_n'' + B_nB_n''}{2(n^2-1)} + \sum\left(A_n' \cos\frac{ns}{R_0} + B_n' \sin\frac{ns}{R_0}\right)\right] \tag{21}$$

$$\rho_c(\sigma) = r_0\left[1 + \sum\frac{(n^2-2)(A_n^2 + B_n^2) - (A_nA_n' + B_nB_n')}{2(n^2-1)} - \sum\left(A_n \cos\frac{ns}{R_0} + B_n \sin\frac{ns}{R_0}\right)\right] \tag{22}$$

Due to the choice of the coordinate system the variables x, z, p_x, and p_z may be assumed to be small compared to r_0 and P_0. This is not so for the variables s and p_s. Therefore, we introduce a new longitudinal momentum $\tilde{p}_s$ as the deviation between the true canonical momentum p_s and the kinetic momentum P_0 of the reference particle. Furthermore, we introduce a new coordinate system that moves with the reference particle, i.e., we define a new longitudinal coordinate $\tilde{s}$ that gives the position in the bunch with respect to the "center" of the bunch defined as the position of the reference particle. The transformation becomes

$$s = v_0t + \tilde{s}; \qquad p_s = P_0 + \tilde{p}_s, \tag{23}$$

where $v_0 = P_0/\gamma m_0$ is the velocity of the reference particle. The generating function for this transformation is

$$G(p_s, \tilde{s}, t) = \tilde{s}(P_0 - p_s) - v_0p_st; \qquad \frac{\partial G}{\partial t} = -v_0p_s. \tag{24}$$

All new variables may now be considered as being small quantities, and therefore the Hamiltonian can be expanded with respect to the coordinates and the momenta. We take into account terms up to second degree in the variables. This corresponds to linear equations of motion. We find for the new Hamiltonian:

$$\begin{aligned}\tilde{H}_0 &= H_0 + \frac{\partial G}{\partial t}\\ &= v_0P_0\left[\frac{1}{2}\left(\frac{p_x}{P_0}\right)^2 + \frac{1}{2}\left(\frac{p_z}{P_0}\right)^2 + \frac{1}{2}\left(\frac{\tilde{p}_s}{\gamma P_0}\right)^2\right.\\ &\quad\left. + \frac{1}{2}Q_x(\tau)\left(\frac{x}{R_0}\right)^2 + \frac{1}{2}Q_z(\tau)\left(\frac{z}{R_0}\right)^2 - \eta(\tau)\frac{x}{R_0}\frac{\tilde{p}_s}{P_0}\right].\end{aligned} \tag{25}$$

Here we have omitted a constant term $(E_0^2 + P_0^2c^2)^{\frac{1}{2}}$ because this term does not contribute to the form of the equations of motion. The variable τ is defined below. For the quantities Q_x, Q_z, and η we find the expressions:

$$Q_x(\tau) = 1 + \bar{\mu}' + \sum \frac{-(A_n^2 + B_n^2) + A_nA_n' + B_nB_n' + A_nA_n'' + B_nB_n''}{2(n^2-1)} + \sum (2A_n + A_n')\cos n\tau + (2B_n + B_n')\sin n\tau \quad (26)$$

$$Q_z(\tau) = -\left[\bar{\mu} + \sum \frac{-n^2(A_n^2 + B_n^2) + A_nA_n' + B_nB_n' + A_nA_n'' + B_nB_n''}{2(n^2-1)} + \sum A_n'\cos n\tau + B_n'\sin n\tau\right] \quad (27)$$

$$\eta(\tau) = \frac{R_0}{\rho_c(\tau)} = 1 + \sum A_n \cos n\tau + B_n \sin n\tau. \quad (28)$$

To eliminate the constants v_0, P_0, and R_0 is the Hamiltonian [Eq. (25)] we introduce new relative variables and a new dimensionless time unit τ defined such that an increase of 2π corresponds to one revolution of the bunch in the cyclotron. The variables are normalized on quantities belonging to the reference orbit and the reference particle. To maintain Hamilton's equations, the Hamiltonian must be adjusted accordingly. It is convenient also to normalize the charge density ρ, the electric potential function ϕ, and the phase-space distribution function f. The scale transformation is defined by

$$\begin{aligned}
&\bar{x} = \frac{x}{R_0} \qquad \bar{z} = \frac{z}{R_0} \qquad \bar{s} = \frac{\gamma\tilde{s}}{R_0} \\
&\bar{p}_x = \frac{p_x}{P_0} \qquad \bar{p}_z = \frac{p_z}{P_0} \qquad \bar{p}_s = \frac{\tilde{p}_s}{\gamma P_0} \\
&\tau = \frac{v_0 t}{R_0} \qquad \bar{H} = \frac{\tilde{H}}{v_0 P_0} \\
&\bar{\rho} = \frac{R_0^3\rho}{q\gamma} \qquad \bar{\phi} = \frac{q\phi}{v_0P_0} \qquad \bar{f} = R_0^3P_0^3 f.
\end{aligned} \quad (29)$$

The Hamiltonian under space-charge conditions [Eq. (6)] and the single-particle Hamiltonian [Eq. (25)] now become

$$\bar{H}(\bar{\mathbf{x}}, \bar{\mathbf{p}}, \tau) = \bar{H}_0 + \bar{\phi}(\bar{\mathbf{x}}, \tau) \quad (30)$$

$$\bar{H}_0 = \tfrac{1}{2}\bar{p}_x^2 + \tfrac{1}{2}\bar{p}_s^2 + \tfrac{1}{2}\bar{p}_z^2 + \tfrac{1}{2}Q_x(\tau)\bar{x}^2 + \tfrac{1}{2}Q_z(\tau)\bar{z}^2 - \gamma\eta(\tau)\bar{x}\bar{p}_s. \quad (31)$$

The potential function $\bar{\phi}$ follows from Eq. (9), which now, due to the transformation [Eq. (29)], transforms into the usual Poisson equation:

$$\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\phi}}{\partial\bar{s}^2} + \frac{\partial^2\bar{\phi}}{\partial\bar{z}^2} = -\left(\frac{q^2}{\varepsilon_0\gamma v_0 R_0 P_0}\right)\bar{\rho}. \quad (32)$$

We obtain the expressions for the charge density $\bar{\rho}$ and the number of particles in the bunch N from Eqs. (3) and (10):

$$\bar{\rho} = \iiint_{-\infty}^{\infty} \bar{f}(\bar{\mathbf{x}}, \bar{\mathbf{p}}, \tau)\, d\bar{\mathbf{p}} \tag{33}$$

$$N = \iiint_{-\infty}^{\infty} \bar{\rho}(\bar{\mathbf{x}}, \tau)\, d\bar{\mathbf{x}}. \tag{34}$$

The Hamiltonian H given in Eqs. (30) and (31) describes the motion of a single particle with respect to an approximately cartesian coordinate system. This coordinate system itself moves with the bunch along the reference orbit. In Section 5 we derive from this Hamiltonian the nonsmoothed-moment equations.

The equations of motion as derived from the unperturbed Hamiltonian $\bar{H}_0$ show that $\bar{z}$ obeys an homogeneous Hill equation and $\bar{x}$ an inhomogeneous Hill equation with momentum deviation as a driving term:

$$\frac{d^2\bar{x}}{d\tau^2} + Q_x(\tau)\bar{x} = \gamma\eta(\tau)\bar{p}_s \tag{35}$$

$$\frac{d^2\bar{z}}{d\tau^2} + Q_z(\tau)\bar{z} = 0. \tag{36}$$

Equation (35) shows the influence of the longitudinal momentum deviation on the horizontal position of the particle, as mentioned in the introduction. In the absence of space charge, $\bar{p}_s$ is an integral of motion because $\bar{H}_0$ does not depend on $\bar{s}$. However, due to space charge, $\bar{p}_s$ will change, and Eq. (35) cannot be solved separately from the longitudinal motion. Also the longitudinal motion is coupled with the transverse motion as follows from Hamilton's equation for $\bar{s}$:

$$\frac{d\bar{s}}{d\tau} = \bar{p}_s - \gamma\eta(\tau)\bar{x}. \tag{37}$$

The equilibrium orbit for particles with a longitudinal momentum deviation $\bar{p}_s$ is found as the periodic solution of Eq. (35). For this we obtain

$$\bar{x}_e = \gamma\bar{p}_s\Bigg[1 - \bar{\mu}' - \sum \frac{A_n^2 + B_n^2 + 4(A_nA_n' + B_nB_n') + A_nA_n'' + B_nB_n'' + A_n'^2 + B_n'^2}{2(n^2-1)} + \sum_n \left(\frac{(A_n + A_n')}{n^2-1}\cos n\tau + \frac{(B_n + B_n')}{n^2-1}\sin n\tau\right)\Bigg]. \tag{38}$$

The cyclotron is isochronous if the average value of $\frac{d\bar{s}}{d\tau}$ over one revolution is zero for particles with deviating momentum $\bar{p}_s$ that follow the equilibrium orbit [Eq. (38)]. Substituting Eq. (38) into Eq. (37) we find the condition for isochronism:

$$\bar{\mu}' = 1 - \frac{1}{\gamma^2} - \sum \frac{3(A_nA_n' + B_nB_n') + A_nA_n'' + B_nB_n'' + A_n'^2 + B_n'^2}{2(n^2-1)}. \tag{39}$$

The quantities Q_x, Q_z, and η as given in Eqs. (26), (27), and (28) contain a time-dependent oscillating part, and therefore $\bar{H}_0$ is not an integral of motion. As has been shown in Ref. 7 the oscillating parts of the Hamiltonian can be transformed to higher order in the magnetic-field flutter (i.e. the azimuthally varying part of the magnetic field) such that, within our approximations, the new oscillating parts can be neglected. For this purpose a linear canonical transformation that changes the coordinates and momenta only slightly is applied; the difference between the old and new variables is on the order of the flutter. The new smoothed Hamiltonian $\bar{\bar{H}}_0$ has the same shape as in Eq. (31) with Q_x, Q_z, and η replaced by time-independent values ν_x^2, ν_z^2, and $\bar{\eta}$, where ν_x and ν_z are the horizontal and vertical tune, respectively. The new Hamiltonian becomes

$$\bar{\bar{H}}_0 = \tfrac{1}{2}\bar{\bar{p}}_x^2 + \tfrac{1}{2}\bar{\bar{p}}_s^2 + \tfrac{1}{2}\bar{\bar{p}}_z^2 + \tfrac{1}{2}\nu_x^2\bar{\bar{x}}^2 + \tfrac{1}{2}\nu_z^2\bar{\bar{z}}^2 - \gamma\bar{\eta}\bar{\bar{x}}\bar{\bar{p}}_s, \tag{40}$$

where $\bar{\bar{x}},\bar{\bar{z}},\bar{\bar{s}}$ are the new canonical coordinates and $\bar{\bar{p}}_x,\bar{\bar{p}}_s,\bar{\bar{p}}_z$ the new canonical momenta. For ν_x, ν_z, and $\bar{\eta}$ we find the expressions:

$$\nu_x = 1 + \tfrac{1}{2}\bar{\mu}' + \sum \frac{3n^2(A_n^2 + B_n^2) + (5n^2 - 8)(A_nA_n' + B_nB_n')}{4(n^2-1)(n^2-4)} + \sum \left(\frac{A_nA_n'' + B_nB_n''}{4(n^2-1)} + \frac{A_n'^2 + B_n'^2}{4(n^2-4)}\right) \tag{41}$$

$$\nu_z^2 = -\left[\bar{\mu}' + \sum \left(\frac{-n^2(A_n^2 + B_n^2) + A_nA_n' + B_nB_n' + A_nA_n'' + B_nB_n''}{2(n^2-1)} - \frac{A_n'^2 + B_n'^2}{2n^2}\right)\right] \tag{42}$$

$$\bar{\eta} = 1 + \sum \frac{3n^2(A_n^2 + B_n^2) + 2(n^2+2)(A_nA_n' + B_nB_n') + 3(A_n'^2 + B_n'^2)}{4(n^2-1)(n^2-4)}. \tag{43}$$

We note that the expressions for ν_x and ν_z^2 agree with the results given in Ref. (7). If Eqs. (41) and (43) are used the condition for isochronism [i.e. Eq. (39)] can be written as

$$\gamma^2 - \frac{\nu_x^2}{\bar{\eta}^2} = 0, \tag{44}$$

and with this expression the Hamiltonian $\bar{\bar{H}}_0$ for an isochronous cyclotron simplifies to

$$\bar{\bar{H}}_0 = \tfrac{1}{2}\bar{\bar{p}}_x^2 + \tfrac{1}{2}(\bar{\bar{p}}_s - \nu_x\bar{\bar{x}})^2 + \tfrac{1}{2}\bar{\bar{p}}_z^2 + \tfrac{1}{2}\nu_z^2\bar{\bar{z}}^2. \tag{45}$$

This Hamiltonian can be brought into a symmetric form with a canonical transformation to new momenta $\hat{p}_x$, $\hat{p}_s$, $\hat{p}_z$ that leaves the coordinates unchanged. The transformation is defined as

$$\begin{aligned} &G = -\hat{x}\bar{\bar{p}}_x - \hat{s}\bar{\bar{p}}_s - \hat{z}\bar{\bar{p}}_z + \tfrac{1}{2}\nu_x\hat{x}\hat{s} \\ &\bar{\bar{p}}_x = \hat{p}_x + \tfrac{1}{2}\nu_x\hat{s} \qquad \bar{\bar{p}}_s = \hat{p}_s + \tfrac{1}{2}\nu_x\hat{x} \qquad \bar{\bar{p}}_z = \hat{p}_z \\ &\bar{\bar{x}} = \hat{x} \qquad\qquad\quad\; \bar{\bar{s}} = \hat{s} \qquad\qquad\quad\; \bar{\bar{z}} = \hat{z}. \end{aligned} \tag{46}$$

With this transformation the smoothed Hamiltonian for the isochronous cyclotron takes the final form:

$$\hat{H}_0 = \tfrac{1}{2}(\hat{p}_x + \tfrac{1}{2}\nu_x\hat{s})^2 + \tfrac{1}{2}(\hat{p}_s - \tfrac{1}{2}\nu_x\hat{x})^2 + \tfrac{1}{2}\hat{p}_z^2 + \tfrac{1}{2}\nu_z^2\hat{z}^2. \tag{47}$$

In Section 5 this Hamiltonian will be used to obtain the smoothed-moment equations. We note that the part of this Hamiltonian that describes the horizontal motion has the same structure as the Hamiltonian for a particle that moves in a homogeneous magnetic field. From this it results that, for an isochronous cyclotron in the absence of space charge and within the smooth approximation made for the derivation of Eq. (40), the particle in the bunch carries out a circular motion in the coordinate system moving with the bunch. The motion will be more complicated in reality, but nevertheless it will have approximately the same characteristics. The coordinates of the center of the circle $(\hat{x}_c, \hat{s}_c)$ and also its radius $\hat{\lambda}_0$ are integrals of motion of $\hat{H}_0$. For the vertical motion the quantity $\hat{I}_z = \frac{1}{2}\hat{p}_z^2 + \frac{1}{2}\nu_z^2\hat{z}^2$ is an integral. Solving the equations of motion resulting from $\hat{H}_0$ we find for $\hat{x}_c$, $\hat{s}_c$, and $\hat{\lambda}_0$:

$$\begin{aligned}
\hat{x}_c &= \tfrac{1}{2}\hat{x} + \nu_x^{-1}\hat{p}_s \\
\hat{s}_c &= \tfrac{1}{2}\hat{s} - \nu_x^{-1}\hat{p}_x \\
\hat{\lambda}_0^2 &= (\tfrac{1}{2}\hat{x} - \nu_x^{-1}\hat{p}_s)^2 + (\tfrac{1}{2}\hat{s} + \nu_x^{-1}\hat{p}_x)^2.
\end{aligned} \tag{48}$$

In the absence of space charge, any function that depends only on the integrals $\hat{x}_c$, $\hat{s}_c$, $\hat{\lambda}_0$, and $\hat{I}_z$ would be a stationary distribution.

We also note that the coupling in the Hamiltonian $\hat{H}_0$ given in Eq. (47) can be removed with a transformation to the "Larmor frame." This frame rotates with frequency $\frac{1}{2}\nu_x$ in the horizontal plane around the vertical axis through the bunch. The transformation is defined as

$$\begin{aligned}
G &= -z_l\hat{p}_z - (\hat{p}_x x_l + \hat{p}_s s_l)\cos\tfrac{1}{2}\nu_x\tau - (\hat{p}_x s_l - \hat{p}_s x_l)\sin\tfrac{1}{2}\nu_x\tau \\
\frac{\partial G}{\partial \tau} &= \tfrac{1}{2}\nu_x(\hat{x}\hat{p}_s - \hat{s}\hat{p}_x)
\end{aligned} \tag{49}$$

$$\begin{aligned}
\hat{x} &= x_l\cos\tfrac{1}{2}\nu_x\tau + s_l\sin\tfrac{1}{2}\nu_x\tau, & \hat{p}_x &= p_{xl}\cos\tfrac{1}{2}\nu_x\tau + p_{sl}\sin\tfrac{1}{2}\nu_x\tau \\
\hat{s} &= -x_l\sin\tfrac{1}{2}\nu_x\tau + s_l\cos\tfrac{1}{2}\nu_x\tau, & \hat{p}_s &= -p_{xl}\sin\tfrac{1}{2}\nu_x\tau + p_{sl}\cos\tfrac{1}{2}\nu_x\tau \\
\hat{z} &= z_l, & \hat{p}_z &= p_{zl},
\end{aligned}$$

where x_l, s_l, z_l, p_{xl}, p_{sl}, and p_{zl} are the new variables in the Larmor frame. The Hamiltonian in the Larmor frame becomes

$$H_{0l} = \tfrac{1}{2}p_{xl}^2 + \tfrac{1}{2}p_{sl}^2 + \tfrac{1}{2}p_{zl}^2 + \tfrac{1}{8}\nu_x^2(x_l^2 + s_l^2) + \tfrac{1}{2}\nu_z^2 z_l^2. \tag{50}$$

It should be noted that the Poisson equation is invariant for the point transformation defined in Eq. (49), and furthermore that the equations of motion as derived from H_{ol} have the same shape as the equations of motion used for the study of space-charge effects in linear accelerator structures.[2–3] Consequently, the same kind of space-charge solutions as obtained in linear accelerator structures will be possible for the Hamiltonian in the Larmor frame H_{ol}. Thus, envelope equations for the RMS sizes of the bunch in the rotating Larmor frame could be derived exactly as done by Sacherer[2] for ellipsoidal bunches in linear accelerator structures. However, in the initial nonrotating frame $(\hat{x}, \hat{s}, \hat{z})$ these bunches then rotate with the frequency $\frac{1}{2}\nu_x$ around the vertical axis. Therefore, this special solution does not seem to be very useful in practice, except perhaps for very short bunches with approximately equal longitudinal and radial sizes.

4 THE ELECTRIC POTENTIAL FUNCTION

As mentioned in the introduction we want to use a linear approximation for the space-charge forces. With this condition satisfied we obtain a closed system of differential equations for the second moments, as pointed out in Section 2, provided that the coefficients in the electric potential function can be expressed in terms of the second moments. We assume that the charge distribution is symmetric with respect to the median plane and that the bunch is centred with respect to the equilibrium orbit. The most general approximation for the electric potential function, giving linear space-charge forces, then becomes

$$\bar{\phi}_0(\bar{\mathbf{x}}, \tau) = -\tfrac{1}{2}a(\tau)\bar{x}^2 - d(\tau)\bar{x}\bar{s} - \tfrac{1}{2}b(\tau)\bar{s}^2 - \tfrac{1}{2}c(\tau)\bar{z}^2. \tag{51}$$

The term $d(\tau)\bar{x}\bar{s}$ is included to take into account a possible nonsymmetric distribution of the bunch (with respect to the equilibrium orbit) that may occur as a result of the transverse-longitudinal coupling in the unperturbed Hamiltonian. The linear approximation for the electric field as derived from Eq. (51) becomes

$$\begin{aligned} \bar{E}_{xo} &= -\frac{\partial \bar{\phi}_0}{\partial \bar{x}} = a(\tau)\bar{x} + d(\tau)\bar{s} \\ \bar{E}_{so} &= -\frac{\partial \bar{\phi}_0}{\partial \bar{s}} = d(\tau)\bar{x} + b(\tau)\bar{s} \\ \bar{E}_{zo} &= -\frac{\partial \bar{\phi}_0}{\partial \bar{z}} = c(\tau)\bar{z}. \end{aligned} \tag{52}$$

We use the least-squares method, as introduced in Ref. 2, for the definition of the linear part of the electric field; i.e., we minimize the averaged difference D between the actual electric field and its linear approximation, where D is defined as

$$\begin{aligned} D &= \iint_{-\infty}^{\infty} |\bar{\mathbf{E}}_0 - \bar{\mathbf{E}}|^2 \bar{f}(\bar{x}, \bar{p}, \bar{\tau})\, d\bar{\mathbf{x}}\, d\bar{\mathbf{p}} \\ &= \iint_{-\infty}^{\infty} [(a\bar{x} + d\bar{s} - \bar{E}_x)^2 + (d\bar{x} + b\bar{s} - \bar{E}_s)^2 + (c\bar{z} - \bar{E}_z)^2]\bar{f}(\bar{\mathbf{x}}, \bar{\mathbf{p}}, \tau)\, d\bar{\mathbf{x}}\, d\bar{\mathbf{p}}, \end{aligned} \tag{53}$$

where $\bar{E}_x$, $\bar{E}_s$, and $\bar{E}_z$ are the actual components of the electric field. Differentiation of Eq. (53) with respect to a, d, b and c, respectively, gives the following system of equations for the coefficients a, b, c, and d:

$$\begin{aligned} a\langle \bar{x}^2\rangle + d\langle \bar{x}\bar{s}\rangle &= \langle \bar{x}\bar{E}_x\rangle \\ (a+b)\langle \bar{x}\bar{s}\rangle + d(\langle \bar{x}^2\rangle + \langle \bar{s}^2\rangle) &= \langle \bar{s}\bar{E}_x\rangle + \langle \bar{x}\bar{E}_s\rangle \\ d\langle \bar{x}\bar{s}\rangle + b\langle \bar{s}^2\rangle &= \langle \bar{s}\bar{E}_s\rangle \\ c\langle \bar{z}^2\rangle &= \langle \bar{z}\bar{E}_z\rangle. \end{aligned} \tag{54}$$

The solution of this system of equations gives a, b, c, and d in terms of the second moments $\langle \bar{x}^2\rangle$, $\langle \bar{x}\bar{s}\rangle$, $\langle \bar{s}^2\rangle$, $\langle \bar{z}^2\rangle$ and the yet-unknown terms $\langle \bar{x}\bar{E}_x\rangle$, $\langle \bar{s}\bar{E}_x\rangle$, $\langle \bar{x}\bar{E}_s\rangle$, $\langle \bar{s}\bar{E}_s\rangle$, and $\langle \bar{z}\bar{E}_z\rangle$. In the following we express these unknown terms as functions of the second moments $\langle \bar{x}^2\rangle$, $\langle \bar{x}\bar{s}\rangle$, $\langle \bar{s}^2\rangle$, and $\langle \bar{z}^2\rangle$. For this we assume that the charge distribution has ellipsoidal symmetry; the charge density then depends on only one parameter U as follows:

$$\bar{\rho} = \bar{\rho}(U), \qquad U = \left(\frac{\bar{x}}{\bar{A}}\right)^2 + \frac{2\bar{x}\bar{s}}{\bar{D}^2} + \left(\frac{\bar{s}}{\bar{B}}\right)^2 + \left(\frac{\bar{z}}{\bar{C}}\right)^2. \tag{55}$$

To find solutions of the electric field we rotate the coordinate frame over a yet-unknown angle φ in the horizontal plane such that in the new frame the charge density takes the more simple form:

$$\bar{\rho} = \bar{\rho}\left[\left(\frac{\tilde{x}}{A}\right)^2 + \left(\frac{\tilde{s}}{B}\right)^2 + \left(\frac{\tilde{z}}{C}\right)^2\right]. \tag{56}$$

The coordinates and the components of the electric field in both frames are then related as follows:

$$\begin{aligned} \bar{x} &= \tilde{x}\cos\varphi + \tilde{s}\sin\varphi & \bar{E}_x &= \tilde{E}_x\cos\varphi + \tilde{E}_s\sin\varphi \\ \bar{s} &= -\tilde{x}\sin\varphi + \tilde{s}\cos\varphi & \bar{E}_s &= -\tilde{E}_x\sin\varphi + \tilde{E}_s\cos\varphi \\ \bar{z} &= \tilde{z} & \bar{E}_z &= \tilde{E}_z. \end{aligned} \tag{57}$$

To find expressions for the quantities A, B, C, and φ we calculate the new second moments of the charge distribution [Eq. (56)]. These new moments are related to the old moments via transformation [Eqs. (57)]. The angle φ is determined by the requirement that in the new frame $\langle \tilde{x}\tilde{s}\rangle$ must vanish. We obtain the following expressions for φ, A, B, C:

$$\begin{aligned} \tan 2\varphi &= \frac{2\langle \bar{x}\bar{s}\rangle}{\langle \bar{s}^2\rangle - \langle \bar{x}^2\rangle} \\ (A/k)^2 &= \frac{1}{2}\left[\langle \bar{x}^2\rangle\left(1 + \frac{1}{\cos 2\varphi}\right) + \langle \bar{s}^2\rangle\left(1 - \frac{1}{\cos 2\varphi}\right)\right] \\ (B/k)^2 &= \frac{1}{2}\left[\langle \bar{x}^2\rangle\left(1 - \frac{1}{\cos 2\varphi}\right) + \langle \bar{s}^2\rangle\left(1 + \frac{1}{\cos 2\varphi}\right)\right] \\ (C/k) &= \langle \bar{z}^2\rangle. \end{aligned} \tag{58}$$

Here the parameter k still depends on the precise choice of the distribution and is defined by

$$k = \left[\frac{1}{3}\int_0^\infty r^4 h(r^2)dr\right]^{-\frac{1}{2}}, \tag{59}$$

where h specifies the distribution with the normalization

$$\int_0^\infty h(r^2)r^2 dr = 1. \tag{60}$$

Using the transformation [Eqs. (57)] the averages in the nonrotated frame $\langle \bar{x}\bar{E}_x \rangle$, $\langle \bar{x}\bar{E}_s \rangle$, $\langle \bar{s}\bar{E}_x \rangle$, and $\langle \bar{s}\bar{E}_s \rangle$ can be expressed in terms of the averages in the rotated frame as

$$\begin{aligned} \langle \bar{x}\bar{E}_x \rangle &= \quad = \langle \tilde{x}\tilde{E}_x \rangle \cos^2\varphi + \langle \tilde{s}\tilde{E}_s \rangle \sin^2\varphi \\ \langle \bar{x}\bar{E}_s \rangle = \langle \bar{s}\bar{E}_x \rangle &= -\tfrac{1}{2}(\langle \tilde{x}\tilde{E}_x \rangle - \langle \tilde{s}\tilde{E}_s \rangle) \sin 2\varphi \\ \langle \bar{s}\bar{E}_s \rangle &= \quad = \langle \tilde{x}\tilde{E}_x \rangle \sin^2\varphi + \langle \tilde{s}\tilde{E}_s \rangle \cos^2\varphi. \end{aligned} \tag{61}$$

The averages in the rotated frame $\langle \tilde{x}\tilde{E}_x \rangle$ and $\langle \tilde{s}\tilde{E}_s \rangle$ now result from the rotated charge distribution [Eq. (56)] and have been given by Sacherer.[2] Using those results we find for the unknown terms:

$$\begin{aligned} \langle \bar{x}\bar{E}_x \rangle &= \quad = \frac{I}{I_0}\left[\frac{k}{A} g\left(\frac{B}{A}, \frac{C}{A}\right) \cos^2\varphi + \frac{k}{B} g\left(\frac{C}{B}, \frac{A}{B}\right) \sin^2\varphi\right] \\ \langle \bar{s}\bar{E}_x \rangle = \langle \bar{x}\bar{E}_s \rangle &= -\frac{I}{I_0}\left[\frac{k}{A} g\left(\frac{B}{A}, \frac{C}{A}\right) - \frac{k}{B} g\left(\frac{C}{B}, \frac{A}{B}\right)\right] \sin\varphi \cos\varphi \\ \langle \bar{s}\bar{E}_s \rangle &= \quad = \frac{I}{I_0}\left[\frac{k}{A} g\left(\frac{B}{A}, \frac{C}{A}\right) \sin^2\varphi + \frac{k}{B} g\left(\frac{C}{B}, \frac{A}{B}\right) \cos^2\varphi\right] \\ \langle \bar{z}\bar{E}_z \rangle &= \quad = \frac{I}{I_0}\frac{k}{C} g\left(\frac{A}{C}, \frac{B}{C}\right), \end{aligned} \tag{62}$$

where I is the beam current averaged over one turn. The function $g(p, q)$ is defined by the integral expression:

$$g(p, q) = \frac{3}{2}\int_0^\infty \frac{d\mu}{(1+\mu)^{3/2}(p^2+\mu)^{1/2}(q^2+\mu)^{1/2}}. \tag{63}$$

The characteristic current I_0 contains all the constants that appear as a result of the scaling transformation [Eqs. (29)] and as a result of the introduction of the average current I in Eqs. (62). It is defined as

$$I_0 = \frac{2nm_0\varepsilon_0 c^3}{\lambda_3 q}\frac{(\gamma^2-1)^{3/2}}{\gamma}, \tag{64}$$

where n denotes the number of bunches per turn. The parameter λ_3 in Eq. (64) still depends on the type of distribution chosen in Eq. (56). However, as shown in Ref. 2, this dependence is very weak for practical distributions, and we can take $\lambda_3 = 1/(5\sqrt{5})$, which corresponds to a uniform distribution. With this approximation the coefficients of the potential function a, b, c, and d are completely specified in terms of the moments $\langle \bar{x}^2 \rangle$, $\langle \bar{x}\bar{s} \rangle$, $\langle \bar{s}^2 \rangle$, and $\langle \bar{z}^2 \rangle$. If Eqs. (62) is used the solution of Eqs. (54) becomes

$$
\begin{aligned}
a &= \frac{\langle \bar{s}^2 \rangle \langle \bar{x}\bar{E}_x \rangle - \langle \bar{x}\bar{s} \rangle \langle \bar{s}\bar{E}_x \rangle}{\langle \bar{x}^2 \rangle \langle \bar{s}^2 \rangle - \langle \bar{x}\bar{s} \rangle^2} \\
b &= \frac{\langle \bar{x}^2 \rangle \langle \bar{s}\bar{E}_s \rangle - \langle \bar{x}\bar{s} \rangle \langle \bar{s}\bar{E}_x \rangle}{\langle \bar{x}^2 \rangle \langle \bar{s}^2 \rangle - \langle \bar{x}\bar{s} \rangle^2} \\
c &= \langle \bar{z}\bar{E}_z \rangle / \langle \bar{z}^2 \rangle \\
d &= \frac{\langle \bar{x}\bar{s} \rangle (a - b)}{\langle \bar{x}^2 \rangle - \langle \bar{s}^2 \rangle}.
\end{aligned}
\tag{65}
$$

For subsequent use in Section 5 we calculate the field energy of the bunch. The total electric field energy of a free charge distribution is given by the integral:

$$W = \frac{\varepsilon_0}{2} \iiint_{-\infty}^{\infty} E^2 \, d\mathbf{x}, \tag{66}$$

where E is the electric field strength. Consider the following vector relation between $\mathbf{E}$ and the position vector $\mathbf{r}$, valid for any vector $\mathbf{E}$ for which $\nabla \times \mathbf{E} = 0$:

$$\tfrac{1}{2}E^2 = (\mathbf{r} \cdot \mathbf{E})(\nabla \cdot \mathbf{E}) - \nabla \cdot [\tfrac{1}{2}E^2\mathbf{r} + (\mathbf{r} \times \mathbf{E}) \times \mathbf{E}]. \tag{67}$$

If we substitute this expression into Eq. (66) we can convert the second term on the right-hand side of Eq. (67) into a surface integral that goes to zero. With $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$ we obtain the following general expression for the total field energy of a free-charge distribution:

$$W = \iiint_{-\infty}^{\infty} (\mathbf{r} \cdot \mathbf{E})\rho \, d\mathbf{x} = qN(\langle xE_x \rangle + \langle sE_s \rangle + \langle zE_z \rangle), \tag{68}$$

with q the charge of a particle and N the total number of particles. For the ellipsoidal charge distribution given in Eq. (55) this expression for W can be reduced to the following form:

$$\bar{W} = \frac{W}{v_0 P_0} = \frac{3N}{2} \frac{I}{I_0} \frac{k}{A} \int_0^{\infty} \frac{d\mu}{(1+\mu)^{1/2} \left(\frac{B^2}{A^2} + \mu\right)^{1/2} \left(\frac{C^2}{A^2} + \mu\right)^{1/2}}, \tag{69}$$

with A, B, and C given in Eqs. (58) and I_0 given in Eq. (64). We note that the magnetic field energy is also included in Eq. (69).
Furthermore we note that this expression for W is invariant for interchanges of A, B, and C, as it must be. Finally, we note that for $A = B$ the integral in Eq. (69) can be calculated analytically, and the result obtained agrees with the expression for the field energy of a uniformly charged ellipsoid with rotational symmetry, as calculated by Hofmann and Struckmeier.[3] An alternative expression for the field energy of the charge distribution is obtained by adding the first, third, and fourth

of Eqs. (54). We then obtain with Eq. (68):

$$\bar{W} = N(a\langle\bar{x}^2\rangle + 2d\langle\bar{x}\bar{s}\rangle + b\langle\bar{s}^2\rangle + c\langle\bar{z}^2\rangle). \tag{70}$$

This expression may be useful for calculating $\bar{W}$ in the numerical program when a, b, c, and d are already known.

The time derivative of $\bar{W}$ is obtained by differentiation of Eq. (69) with respect to A, B, and C, followed by a differentiation of the quantities A, B and C with respect to time using Eqs. (58). We find the following expression for the time derivative of $\bar{W}$:

$$\frac{d\bar{W}}{d\tau} = -\frac{N}{2}\left(a\frac{d}{d\tau}\langle\bar{x}^2\rangle + 2d\frac{d}{d\tau}\langle\bar{x}\bar{s}\rangle + b\frac{d}{d\tau}\langle\bar{s}^2\rangle + c\frac{d}{d\tau}\langle\bar{z}^2\rangle\right). \tag{71}$$

A similar result was obtained by Hofmann and Struckmeier[3] for an ellipsoidal bunch with uniform density.

5 MOMENT EQUATIONS

In Section 3 we derived the time-dependent Hamiltonian $\bar{H}_0$ [Eq. (31)] for a particle moving in the azimuthally varying magnetic field in the absence of space charge. In Section 4 we defined the potential function $\bar{\phi}_0$ [Eq. (51)] for a bunch with ellipsoidal symmetry and with a linear approximation of the space-charge forces. The total Hamiltonian under space-charge conditions $\bar{H}$ is found by adding $\bar{\phi}_0$ to the unperturbed Hamiltonian:

$$\bar{H} = \tfrac{1}{2}\bar{p}_x^2 + \tfrac{1}{2}[Q_x(\tau) - a(\tau)]\bar{x}^2 + \tfrac{1}{2}\bar{p}_s^2 - \gamma\eta(\tau)\bar{x}\bar{p}_s - \tfrac{1}{2}b(\tau)\bar{s}^2 - d(\tau)\bar{x}\bar{s} + \tfrac{1}{2}\bar{p}_z^2 + \tfrac{1}{2}(Q_z(\tau) - c(\tau))\bar{z}^2. \tag{72}$$

Substituting this Hamiltonian into Eq. (5) we obtain for the Vlasow equation:

$$\frac{\partial\bar{f}}{\partial\tau} + \bar{p}_x\frac{\partial\bar{f}}{\partial\bar{x}} + (\bar{p}_s - \gamma\eta\bar{x})\frac{\partial\bar{f}}{\partial\bar{s}} - [(Q_x - a)\bar{x} - \gamma\eta\bar{p}_s - d\bar{s}]\frac{d\bar{f}}{\partial\bar{p}_x} + (b\bar{s} + d\bar{x})\frac{\partial\bar{f}}{\partial\bar{p}_s} + \bar{p}_z\frac{\partial\bar{f}}{\partial\bar{z}} - (Q_z - c)\bar{z}\frac{\partial\bar{f}}{\partial\bar{p}_z} = 0. \tag{73}$$

From this equation we obtain the second-moment equations, as shown in Section 2. In a system with three degrees of freedom, 21 independent second moments can be formed. However, since in our linear approximation the vertical motion of a single particle is not coupled with the horizontal motion, we need not consider cross terms between horizontal and vertical variables. We then have ten independent second moments for the horizontal variables and three for the vertical variables. From Eq. (73) we derive the following system of differential equations for the second moments:

$$\frac{d}{d\tau}\langle\bar{x}^2\rangle = 2\langle\bar{x}\bar{p}_x\rangle \tag{74}$$

$$\frac{d}{d\tau}\langle\bar{x}\bar{p}_x\rangle = \langle\bar{p}_x^2\rangle - (Q_x - a)\langle\bar{x}^2\rangle + \gamma\eta\langle\bar{x}\bar{p}_s\rangle + d\langle\bar{x}\bar{s}\rangle \tag{75}$$

$$\frac{d}{d\tau}\langle\bar{p}_x^2\rangle = -2(Q_x - a)\langle\bar{x}\bar{p}_x\rangle + 2\gamma\eta\langle\bar{p}_x\bar{p}_s\rangle + 2d\langle\bar{s}\bar{p}_x\rangle \tag{76}$$

$$\frac{d}{d\tau}\langle\bar{s}^2\rangle = 2\langle\bar{s}\bar{p}_s\rangle - 2\gamma\eta\langle\bar{x}\bar{s}\rangle \tag{77}$$

$$\frac{d}{d\tau}\langle\bar{s}\bar{p}_s\rangle = \langle\bar{p}_s^2\rangle - \gamma\eta\langle\bar{x}\bar{p}_s\rangle + b\langle\bar{s}^2\rangle + d\langle\bar{x}\bar{s}\rangle \tag{78}$$

$$\frac{d}{d\tau}\langle\bar{p}_s^2\rangle = 2b\langle\bar{s}\bar{p}_s\rangle + 2d\langle\bar{x}\bar{p}_s\rangle \tag{79}$$

$$\frac{d}{d\tau}\langle\bar{x}\bar{s}\rangle = \langle\bar{s}\bar{p}_x\rangle + \langle\bar{x}\bar{p}_s\rangle - \gamma\eta\langle\bar{x}^2\rangle \tag{80}$$

$$\frac{d}{d\tau}\langle\bar{p}_x\bar{p}_s\rangle = -(Q_x - a)\langle\bar{x}\bar{p}_s\rangle + \gamma\eta\langle\bar{p}_s^2\rangle + d\langle\bar{s}\bar{p}_s\rangle + b\langle\bar{s}\bar{p}_x\rangle + d\langle\bar{x}\bar{p}_x\rangle \tag{81}$$

$$\frac{d}{d\tau}\langle\bar{x}\bar{p}_s\rangle = \langle\bar{p}_x\bar{p}_s\rangle + b\langle\bar{x}\bar{s}\rangle + d\langle\bar{x}^2\rangle \tag{82}$$

$$\frac{d}{d\tau}\langle\bar{s}\bar{p}_x\rangle = \langle\bar{p}_x\bar{p}_s\rangle - \gamma\eta\langle\bar{x}\bar{p}_x\rangle - (Q_x - a)\langle\bar{x}\bar{s}\rangle + \gamma\eta\langle\bar{s}\bar{p}_s\rangle + d\langle\bar{s}^2\rangle \tag{83}$$

$$\frac{d}{d\tau}\langle\bar{z}^2\rangle = 2\langle\bar{z}\bar{p}_z\rangle \tag{84}$$

$$\frac{d}{d\tau}\langle\bar{z}\bar{p}_z\rangle = \langle\bar{p}_z^2\rangle - (Q_z - c)\langle\bar{z}^2\rangle \tag{85}$$

$$\frac{d}{d\tau}\langle\bar{p}_z^2\rangle = -2(Q_z - c)\langle\bar{z}\bar{p}_z\rangle \tag{86}$$

[*Remark*: We note that the second-moment equations can also be written in matrix form by making use of the σ-matrix notation

$$(\langle x^2\rangle = \sigma_{11},\ \langle xp_x\rangle = \sigma_{12},\ \langle xs\rangle = \sigma_{13},\ \text{etc.})$$

as introduced by K. Brown[11] for the TRANSPORT computer code. The first-order differential equation for this σ-matrix has been given by Sacherer.[2] The matrix method is also used in TRANSOPTR.[12]]

The time-dependent quantities Q_x, Q_z, and η are specified in Eqs. (26), (27), and (28), and the coefficients a, b, c, and d are given in terms of the second moments $\langle\bar{x}^2\rangle$, $\langle\bar{x}\bar{s}\rangle$, $\langle\bar{s}^2\rangle$, $\langle\bar{z}^2\rangle$ via Eqs. (58), and (62)–(65). The system therefore forms a closed set of differential equations for the second moments and can be integrated numerically for a given set of initial conditions. We note that the equations for the horizontal and vertical moments are mutually coupled due to the space-charge effect, i.e., via the coefficients a, b, c, and d.

Furthermore we note that the three first-order differential equations for the vertical moments can be reduced to one second-order differential equation for the vertical RMS envelope $\bar{z}_m = \langle \bar{z}^2 \rangle^{1/2}$:

$$\frac{d^2\bar{z}_m}{d\tau^2} + Q_z(\tau)\bar{z}_m - \frac{\bar{\varepsilon}_z^2}{16\bar{z}_m^3} - \frac{\langle \bar{z}\bar{E}_z \rangle}{\bar{z}_m} = 0, \tag{87}$$

where $\bar{\varepsilon}_z$ is the vertical RMS emittance defined as

$$\bar{\varepsilon}_z = 4(\langle \bar{z}^2 \rangle \langle \bar{p}_z^2 - \langle \bar{z}\bar{p}_z \rangle^2)^{1/2}. \tag{88}$$

Due to our linear approximation of the space-charge forces this RMS emittance is constant, as can be verified with Eqs. (84)–(86). Equation (87) is of the same form as the RMS-envelope equations derived by Sacherer[2] for linear-accelerator structures. In fact, this equation is valid for the general case, where the space-charge forces may have a nonlinear part. However, the problem then lies in the fact that the RMS emittance is no longer a constant. Recently, Hofmann and Struckmeier[3] derived differential equations that relate the change of the RMS emittances to the change of the nonlinear field energy in the bunch.

In Section 3 we also derived the smoothed Hamiltonian $\hat{H}_0$ for an isochronous cyclotron, as given in Eq. (47). This Hamiltonian was obtained from the Hamiltonian $\bar{H}_0$ [Eq. (31)] via a smoothing procedure and via the canonical transformation defined in Eqs. (46). In principle, these transformations also must be applied on the electric potential function defined in Eq. (51). The transformation given in Eqs. (46) is a point transformation that leaves the electrical potential function unchanged. Note however that the second moments that contain $\hat{p}_x$ and $\hat{p}_s$ will have a different meaning due to this transformation. As for the smoothing procedure, we already noted that this transformation gives only a small difference, on the order of the flutter between the old and new variables. We neglect the change of the potential function due to this transformation. Consequently the moment equations resulting from the smoothed Hamiltonian $\hat{H}_0$ are less accurate than the system given in Eqs. (74)–(86). On the other hand, these equations may give additional insight into the problem. The equations for the vertical moments remain the same as in Eqs. (74)–(86) but with $Q_z(\tau)$ replaced by the average value ν_z^2. For the horizontal moments we find the following system of equations:

$$\frac{d}{d\tau}\langle \hat{x}^2 \rangle = 2\langle \hat{x}\hat{p}_x \rangle + \nu_x \langle \hat{x}\hat{s} \rangle \tag{89}$$

$$\frac{d}{d\tau}\langle \hat{x}\hat{p}_x \rangle = \langle \hat{p}_x^2 \rangle - \tfrac{1}{4}(\nu_x^2 - 4a)\langle \hat{x}^2 \rangle + \tfrac{1}{2}\nu_x(\langle \hat{x}\hat{p}_s \rangle + \langle \hat{s}\hat{p}_x \rangle) + d\langle \hat{x}\hat{s} \rangle \tag{90}$$

$$\frac{d}{d\tau}\langle \hat{p}_x^2 \rangle = \nu_x \langle \hat{p}_x\hat{p}_s) - \tfrac{1}{2}(\nu_x^2 - 4a)\langle \hat{x}\hat{p}_x \rangle + 2d\langle \hat{s}\hat{p}_x \rangle \tag{91}$$

$$\frac{d}{d\tau}\langle\hat{s}^2\rangle = 2\langle\hat{s}\hat{p}_s\rangle - \nu_x\langle\hat{x}\hat{s}\rangle \tag{92}$$

$$\frac{d}{d\tau}\langle\hat{s}\hat{p}_s\rangle = \langle\hat{p}_s^2\rangle - \tfrac{1}{4}(\nu_x^2 - 4b)\langle\hat{s}^2\rangle - \tfrac{1}{2}\nu_x(\langle\hat{x}\hat{p}_s\rangle + \langle\hat{s}\hat{p}_x\rangle) + d\langle\hat{x}\hat{s}\rangle \tag{93}$$

$$\frac{d}{d\tau}\langle\hat{p}_s^2\rangle = -\nu_x\langle\hat{p}_x\hat{p}_s\rangle - \tfrac{1}{2}(\nu_x^2 - 4b)\langle\hat{s}\hat{p}_s\rangle + 2d\langle\hat{x}\hat{p}_s\rangle \tag{94}$$

$$\frac{d}{d\tau}\langle\hat{x}\hat{s}\rangle = \tfrac{1}{2}\nu_x(\langle\hat{s}^2\rangle - \langle\hat{x}^2\rangle) + \langle\hat{x}\hat{p}_s\rangle + \langle\hat{s}\hat{p}_x\rangle \tag{95}$$

$$\begin{aligned}\frac{d}{d\tau}\langle\hat{p}_x\hat{p}_s\rangle = {} & \tfrac{1}{2}\nu_x(\langle\hat{p}_s^2\rangle - \langle\hat{p}_x^2\rangle) - \tfrac{1}{4}(\nu_x^2 - 4a)\langle\hat{x}\hat{p}_s\rangle + \\ & - \tfrac{1}{4}(\nu_x^2 - 4b)\langle\hat{s}\hat{p}_x\rangle + d(\langle\hat{x}\hat{p}_x\rangle + \langle\hat{s}\hat{p}_s\rangle)\end{aligned} \tag{96}$$

$$\frac{d}{d\tau}\langle\hat{x}\hat{p}_s\rangle = \langle\hat{p}_x\hat{p}_s\rangle + \tfrac{1}{2}\nu_x(\langle\hat{s}\hat{p}_s\rangle - \langle\hat{x}\hat{p}_x\rangle) - \tfrac{1}{4}(\nu_x^2 - 4b)\langle\hat{x}\hat{s}\rangle + d\langle\hat{x}^2\rangle \tag{97}$$

$$\frac{d}{d\tau}\langle\hat{s}\hat{p}_x\rangle = \langle\hat{p}_x\hat{p}_s\rangle + \tfrac{1}{2}\nu_x(\langle\hat{s}\hat{p}_s\rangle - \langle\hat{x}\hat{p}_x\rangle) - \tfrac{1}{4}(\nu_x^2 - 4a)\langle\hat{x}\hat{s}\rangle + d\langle\hat{s}^2\rangle \tag{98}$$

If we subtract Eq. (97) from Eq. (98) and use the relation between a, b and, d as given in Eqs. (65) we find that the quantity

$$\bar{L} = N(\langle\hat{s}\hat{p}_x\rangle - \langle\hat{x}\hat{p}_s\rangle) \tag{99}$$

is an integral of motion. In fact, this means that in the final coordinate system for which the unperturbed Hamiltonian $\hat{H}_0$ is time independent, the total angular canonical momentum of the bunch is conserved. We note that this result holds not only for an ellipsoidal charge distribution but also for any charge distribution with nonlinear space-charge forces. [In this more general case we have $d\bar{L}/d\tau = -N(\langle\hat{s}\partial\bar{\phi}/\partial\hat{x}\rangle) - \langle\hat{x}\partial\phi/\partial\hat{s}\rangle) = 0.$]

The kinetic energy of a single particle in the bunch is equal to the Hamiltonian $\hat{H}_0$ as given in Eq. (47). Thus, for the total kinetic energy $\bar{T}$ of the bunch, we have

$$\bar{T} = \frac{N}{2}[\langle\hat{p}_x^2\rangle + \langle\hat{p}_s^2\rangle + \langle\hat{p}_z^2\rangle + \tfrac{1}{4}\nu_x^2(\langle\hat{x}^2\rangle + \langle\hat{s}^2\rangle) + \nu_z^2\langle\hat{z}^2\rangle + \nu_x(\langle\hat{s}\hat{p}_x\rangle - \langle\hat{x}\hat{p}_s\rangle)], \tag{100}$$

and for the time derivative of $\bar{T}$ we obtain with Eqs. (89)–(98)

$$\begin{aligned}\frac{d\bar{T}}{d\tau} &= \frac{N}{2}[2a\langle\hat{x}\hat{p}_x\rangle + 2b\langle\hat{s}\hat{p}_s\rangle + 2c\langle\hat{z}\hat{p}_z\rangle + 2d(\langle\hat{s}\hat{p}_x\rangle + \langle\hat{x}\hat{p}_s\rangle)] \\ &= \frac{N}{2}\left[a\frac{d}{d\tau}\langle\hat{x}^2\rangle + 2d\frac{d}{d\tau}\langle\hat{x}\hat{s}\rangle + b\frac{d}{d\tau}\langle\hat{s}^2\rangle + c\frac{d}{d\tau}\langle\hat{z}^2\rangle\right].\end{aligned} \tag{101}$$

Comparing this expression with the time derivative of the field energy as given in Eq. (71) we find that, for the final time-independent Hamiltonian $\hat{H}_0$, the total energy $\bar{U}$ of the bunch

$$\bar{U} = \bar{T} + \bar{W} \tag{102}$$

is conserved.

Apart from the total angular momentum of the bunch $\bar{L}$ and the total energy of the bunch $\bar{U}$ there is a third integral of Eqs. (89)–(98), namely the quantity

$$\hat{\varepsilon} = 4[\langle \hat{x}^2 \rangle \langle \hat{p}_x^2 \rangle - \langle \hat{x}\hat{p}_x \rangle^2 + \langle \hat{s}^2 \rangle \langle \hat{p}_s^2 \rangle - \langle \hat{s}\hat{p}_s \rangle^2 + 2(\langle \hat{x}\hat{s} \rangle \langle \hat{p}_x\hat{p}_s \rangle - \langle \hat{x}\hat{p}_s \rangle \langle \hat{s}\hat{p}_x \rangle)]^{1/2}. \tag{103}$$

For an uncoupled canonical system each of the three terms $\langle \hat{x}^2 \rangle \langle \hat{p}_x^2 \rangle - \langle \hat{x}\hat{p}_x \rangle^2$, $\langle \hat{s}^2 \rangle \langle \hat{p}_s^2 \rangle - \langle \hat{s}\hat{p}_s \rangle^2$, and $\langle \hat{x}\hat{s} \rangle \langle \hat{p}_x\hat{p}_s \rangle - \langle \hat{x}\hat{p}_s \rangle \langle \hat{s}\hat{p}_x \rangle$ is conserved, where the first term corresponds to the transverse emittance and the second term with the longitudinal emittance. For a coupled system only the sum of these three terms is conserved. We note that the quantity ε is a constant not only for the system Eqs. (89)–(98) but also for all linear canonical systems with two degrees of freedom and arbitrary coupling. It can easily be verified that it is an integral of motion of Eqs. (74)–(86) also.

According to Liouville's theorem the total volume occupied by the particles in phase space is conserved. For a system with one degree of freedom the RMS representation of the phase-space area takes the form given in Eq. (88). For a system with two degrees of freedom we find the following expression for the RMS representation of the four-dimensional phase space volume:

$$\begin{aligned} \hat{\tau} = 16\{&[\langle \hat{x}^2 \rangle \langle \hat{p}_x^2 \rangle - \langle \hat{x}\hat{p}_x \rangle^2][\langle \hat{s}^2 \rangle \langle \hat{p}_s^2 \rangle - \langle \hat{s}\hat{p}_s \rangle^2] \\ &+ [\langle \hat{x}\hat{s} \rangle \langle \hat{p}_x\hat{p}_s \rangle - \langle \hat{x}\hat{p}_s \rangle \langle \hat{s}\hat{p}_x \rangle]^2 - [\langle \hat{x}^2 \rangle \langle \hat{s}^2 \rangle \langle \hat{p}_x\hat{p}_s \rangle^2 \\ &+ \langle \hat{p}_x^2 \rangle \langle \hat{p}_s^2 \rangle \langle \hat{x}\hat{s} \rangle^2 + \langle \hat{x}^2 \rangle \langle \hat{p}_s^2 \rangle \langle \hat{s}\hat{p}_x \rangle^2 + \langle \hat{s}^2 \rangle \langle \hat{p}_x^2 \rangle \langle \hat{x}\hat{p}_s \rangle^2] \\ &+ 2[\langle \hat{x}^2 \rangle \langle \hat{s}\hat{p}_s \rangle \langle \hat{p}_x\hat{p}_s \rangle \langle \hat{s}\hat{p}_x \rangle + \langle \hat{s}^2 \rangle \langle \hat{x}\hat{p}_x \rangle \langle \hat{p}_x\hat{p}_s \rangle \langle \hat{x}\hat{p}_s \rangle \\ &+ \langle \hat{p}_x^2 \rangle \langle \hat{s}\hat{p}_s \rangle \langle \hat{x}\hat{s} \rangle \langle \hat{x}\hat{p}_s \rangle + \langle \hat{p}_s^2 \rangle \langle \hat{x}\hat{p}_x \rangle \langle \hat{x}\hat{s} \rangle \langle \hat{s}\hat{p}_s \rangle] \\ &- 2[\langle \hat{x}\hat{p}_x \rangle \langle \hat{s}\hat{p}_s \rangle \langle \hat{x}\hat{s} \rangle \langle \hat{p}_x\hat{p}_s \rangle + \langle \hat{x}\hat{p}_x \rangle \langle \hat{s}\hat{p}_s \rangle \langle \hat{x}\hat{p}_s \rangle \langle \hat{s}\hat{p}_x \rangle]\}^{1/2}. \end{aligned} \tag{104}$$

The quantity $\hat{\tau}$ is conserved for all linear canonical systems with two degrees of freedom and arbitrary coupling.

A special solution of the system of Eqs. (89)–(98) is obtained if we consider bunches that have rotational symmetry with respect to the vertical axis through the bunch, i.e., bunches with a circular horizontal cross section. To obtain this solution the moments have to be chosen as follows:

$$\begin{aligned} &\langle \hat{s}^2 \rangle = \langle \hat{x}^2 \rangle, \qquad \langle \hat{s}\hat{p}_s \rangle = \langle \hat{x}\hat{p}_x \rangle, \qquad \langle \hat{p}_s^2 \rangle = \langle \hat{p}_x^2 \rangle \\ &\langle \hat{x}\hat{s} \rangle = \langle \hat{p}_x\hat{p}_s \rangle = 0, \qquad \langle \hat{s}\hat{p}_x \rangle = -\langle \hat{x}\hat{p}_s \rangle = \bar{L}/2N. \end{aligned} \tag{105}$$

The moment equations for the circular bunch can be reduced to two second-order differential equations for the horizontal and vertical RMS envelopes $\hat{x}_m = \langle \hat{x}^2 \rangle^{1/2}$ and $\hat{z}_m = \langle \hat{z}^2 \rangle^{1/2}$:

$$\begin{aligned} &\frac{d^2\hat{x}_m}{d\tau^2} + \frac{\nu_x^2}{4}\hat{x}_m - \frac{(\hat{\varepsilon}^2 - 8\bar{L}^2/N^2)}{32\hat{x}_m^3} - \frac{I}{I_0}\frac{1}{\hat{x}_m^2} g\left(1, \frac{\hat{z}_m}{\hat{x}_m}\right) = 0 \\ &\frac{d^2\hat{z}_m}{d\tau^2} + \nu_z^2\hat{z}_m - \frac{\hat{\varepsilon}_z^2}{16\hat{z}_m^3} - \frac{I}{I_0}\frac{1}{\hat{z}_m^2} g\left(\frac{\hat{x}_m}{\hat{z}_m}, \frac{\hat{x}_m}{\hat{z}_m}\right) = 0, \end{aligned} \tag{106}$$

with I the average beam current and the function g defined in Eq. (63).

Another special solution of the system of Eqs. (89)–(98) is the stationary solution, which is obtained by putting the right-hand sides of the equations equal to zero. In doing so one finds that the moments $\langle \hat{x}^2 \rangle$, $\langle \hat{x}\hat{s} \rangle$, $\langle \hat{s}^2 \rangle$, and $\langle \hat{z}^2 \rangle$ can be chosen freely and that all other moments can be expressed in terms of $\langle \hat{x}^2 \rangle$, $\langle \hat{x}\hat{s} \rangle$, $\langle \hat{s}^2 \rangle$, and $\langle \hat{z}^2 \rangle$. We note, however, that, under space-charge conditions, only the stationary solution for the circular bunch is physically realistic. For noncircular stationary bunches one finds that physical quantities that must be positive (such as, for example, the second moment $\langle \bar{p}_x^2 \rangle \cong \langle \hat{p}_x^2 \rangle + \nu_x \langle \hat{s}\hat{p}_x \rangle + \frac{1}{4}\nu_x^2 \langle \hat{s}^2 \rangle$) become negative. In view of the rotational symmetry of the unperturbed Hamiltonian $\hat{H}_0$ it is not so surprising that nonrotational symmetric solutions cannot be stationary.

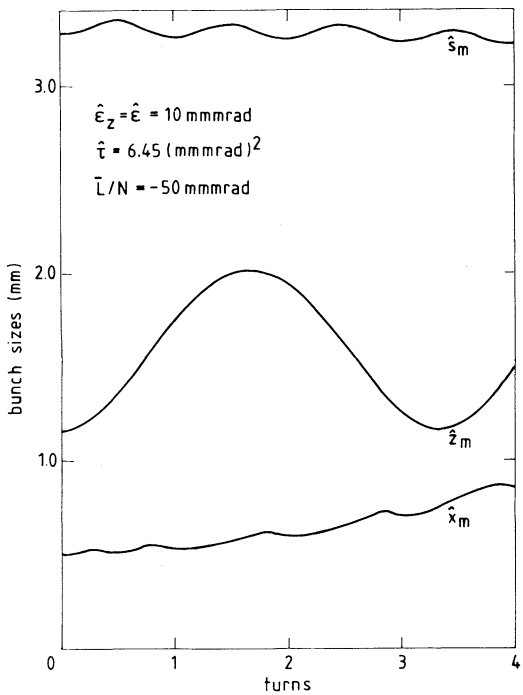

FIGURE 1 The RMS sizes of a 1-mA/0.97-MeV coasting beam making four turns in the minicyclotron ILEC, calculated numerically with the smoothed-moment equations, Eqs. (89)–(98).

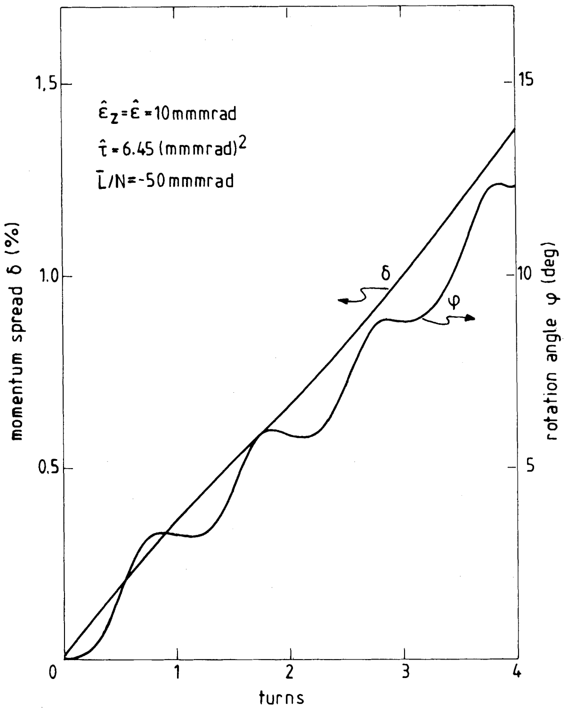

FIGURE 2 The RMS rotation angle φ [Eqs. (58)] and the longitudinal momentum spread δ of a 1-mA/0.97-MeV coasting beam in the minicyclotron ILEC as calculated with Eqs. (89)–(98).

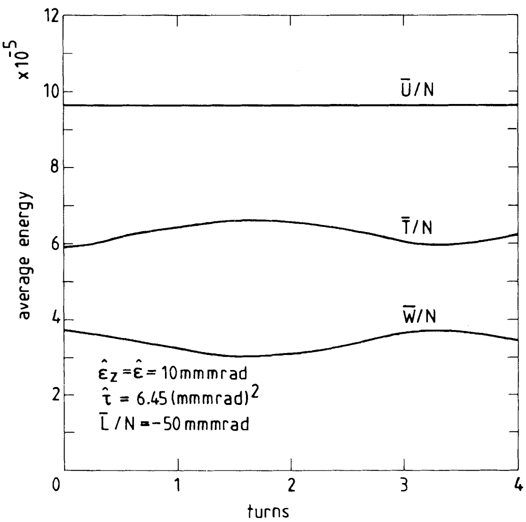

FIGURE 3 The average kinetic energy $\bar{T}/N$ [Eq. (100)], space-charge energy $\bar{W}/N$ [Eq. (70)], and total energy $\bar{U}/N$ [Eq. (102)] of a 1-mA/0.97-MeV coasting beam in the minicyclotron ILEC as calculated with Eqs. (89)–(98). The energies are normalized with respect to the nominal energy of the particles, $E = 0.97$ MeV.

Figures 1, 2, and 3 show some results obtained with the smoothed-moment equations [Eqs. (89)–(98)] for the minicyclotron ILEC[10] under construction at the Eindhoven University. The calculations were done for a coasting beam making four turns at a nominal radius r_0 of 10 cm [$\bar{B}(r_0) = 1.42\ T$, $\nu_r = 1.0004$, $\nu_z = 0.1861$, $E = 0.97$ MeV] with an average beam current I of 1 mA and two bunches per turn. The initial values of the moments were taken such that the beam would be stationary if there were no space-charge effect. The integrals of motion were taken as follows: $\hat{\varepsilon}_z = \hat{\varepsilon} = 10$ mm-mrad, $\bar{L}/N = -50$ mm-mrad, $\hat{\tau} = 6.45$ (mm-mrad)^2.

Figure 1 shows the time evolution of the RMS sizes of the bunch $\hat{x}_m$, $\hat{s}_m$, and $\hat{z}_m$ during the four turns. The relatively strong increase of the horizontal beam size $\hat{x}_m$ is due to the rotation of the bunch around its vertical axis.

Figure 2 depicts the RMS rotation angle φ of the bunch as defined in Eqs. (58) and the longitudinal momentum spread $\delta = 2\langle \bar{p}_s^2 \rangle^{1/2} \cong 2(\langle \hat{p}_s^2 \rangle + \nu_x \langle \hat{x}\hat{p}_s \rangle + \frac{1}{4}\nu_x^2 \langle \hat{x}^2 \rangle)^{1/2}$. Under the assumed conditions the momentum spread increases $\approx 0.3\%$ per turn and the rotation angle φ increases ≈ 3 degrees per turn.

In Fig. 3 we give the average kinetic energy in the bunch $\bar{T}/N$ as defined in Eq. (100) and the average space-charge energy in the bunch $\bar{W}/N$ as defined in Eq. (70).

Both quantities are normalized with respect to the nominal energy of the particles $E = 0.97$ MeV. It can be seen from Fig. 3 that there is an exchange between the two forms of energy but that the total energy $\bar{U}/N$ is conserved.

Finally, we note that we made the same calculations as described above with the nonsmoothed system given in Eqs. (74)–(86). The initial values of the old moments, needed for the integration of Eqs. (74)–(86), were calculated from the initial values of the new moments by applying the transformation given in Eqs. (46), which relates both systems if one neglects the smoothing transformation. Under the assumed conditions the deviation between the results obtained with both systems of moment equations was, in general, less than one per cent.

6 CONCLUSION

We have derived moment equations for the particle distribution of a bunched beam in an AVF cyclotron. Within the approximations made, the numerical integration of these equations will give the time development of the RMS properties of the bunch under space-charge conditions. The most important approximations in our model are the assumption of an ellipsoidal charge distribution and the assumption of linear space-charge forces. For linear accelerator structures the assumption of an ellipsoidal charge distribution seems to be a good approximation in practice.[2,3] For the AVF cyclotron this assumption is not trivial because of the coupling between the transverse and longitudinal variables in the unperturbed Hamiltonian. In the multiparticle code used at GANIL the bunch is simulated by a Gaussian ellipsoidal distribution, and the results obtained with this code seem to be satisfactory.[4] On the other hand, Adam[6] found numerically that under certain conditions the bunch shape starts to deviate considerably from the ellipsoidal shape. The choice of the initial distribution, the energy of the particles, and the average beam current seem to be important factors in this aspect.

As for our second assumption, we note that in linear-accelerator structures the nonlinear parts of the space-charge forces are responsible for RMS emittance growth. For not-too-high beam currents this effect could be neglected in a first approximation. However, in an AVF cyclotron the nonlinear part of the electric field will be determined first of all by the deviation between the actual geometrical bunch shape and the ellipsoidal shape. Therefore, we expect that the validity of the second assumption mainly depends on the accuracy of the first assumption. We conclude that the possibilities and the restrictions of the model presented in this paper should be further evaluated by comparing the results with numerical many-particle calculations.

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