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EVOLUTION OF RMS BEAM ENVELOPES IN TRANSPORT SYSTEMS WITH LINEAR $X - Y$ COUPLING†

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Beam transport systems in which the two transverse degrees of freedom of particle motion (x and y) are coupled are often used to transport high-current, low-energy beams. Consideration of the matching problem for such systems requires a formalism for the description of the evolution of the beam envelope, including the effect of space charge. By defining a 4×4 beam covariance matrix, it is possible to give a simple prescription for following the rms beam ellipse through an arbitrary linear transport system, allowing the length of the axes of the ellipse and its angle of tilt to vary; the prescription trivially reproduces the usual K–V envelope equations in the decoupled case. The matching condition for the general coupled system is given, and the stability problem for the matched solution in the presence of space charge is formulated in general and discussed in a particular example.

1. INTRODUCTION

The familiar K–V envelope equations[1] are applicable to beams confined by focusing systems that do not couple the two transverse degrees of freedom of particle motion, x and y. The beam ellipse remains upright in these systems, and the axes merely vary in length along the direction of beam propagation. Certain focusing systems used for the transport of high-current, low-energy beams, however, do couple the x and y motion. In some simple cases with azimuthal symmetry, e.g. a solenoid, a transformation to a rotating frame decouples the two transverse degrees of freedom, but in the general case the problem is most directly treated in the coupled, "laboratory-frame" variables. Systems to which the following analysis may be applied include the modified betatron[2] (for all stable values of field index), the bumpy torus,[3] the reversing solenoid lens,[4] and the stellarator-focused betatron, or stellatron.[5] For this last case, matched beam solutions have been constructed,[6,7] but the stability of these solutions in the presence of space charge remains to be studied. The formalism outlined here allows such an analysis to be carried through; this is done, as an example, below.

The stability of K–V distributions in quadrupole and solenoid systems has been analyzed by Hofmann, Laslett, Smith, and Haber[8] and by Struckmeier and Reiser.[9] Hofmann, *et al.* carried out a thorough analysis of the electrostatic

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modes of the K–V distribution, using the linearized Vlasov equation, whereas Struckmeier and Reiser studied the linear stability of the matched solutions of the K–V envelope equations themselves; the envelope oscillation modes they found corresponded to the "second-order even" modes of Ref. 8. The stability analysis reported here is the analog of that of Ref. 9, for coupled systems. Since we make no assumption regarding the symmetry of the focusing system, this analysis includes the "second-order–odd" modes of Ref. 8.

Three main sections follow: In the first, Section 2, the 4×4 beam covariance matrix, Σ, is defined, and its evolution equation is given; the size and orientation of the rms beam ellipse at any location along the beam axis are obtained from the spatial components of Σ. For the special case of the $x-y$ coupled analog of the K–V distribution, the space-charge term in the evolution equation is obtained.

Section 3 includes a discussion of the matching problem for a certain class of distribution functions that includes the K–V distribution as a special case. We go on to describe the stability of the matched solution for the K–V distribution in the presence of space charge. As an example of envelope evolution in a coupled system we discuss in some detail in Section 4 the propagation of an electron beam in an $l=2$ stellarator field, a continuously rotated quadrupole superimposed on a constant longitudinal magnetic field. The nature of the matched solution is described, including the dependence of the beam radii on beam current, beam energy, longitudinal field strength, and quadrupole field strength. The stability of the matched solution is then analyzed, and plots are presented of the growth rates of the unstable modes as functions of these same parameters.

2. ENVELOPE EQUATION, WITH SPACE-CHARGE TERM

The single-particle equations in the paraxial approximation may be written in matrix form as

$$v'(s) = M(s)v(s), \tag{1}$$

where $v(s)$ is a 4×1 column vector with the elements $x(s)$, $x'(s)$, $y(s)$, and $y'(s)$; $M(s)$ is a 4×4 matrix, independent of v; s is the independent variable measuring distance from some reference point on the optic axis; and a prime mark denotes d/ds. We consider a monoenergetic distribution of particles, forming a beam moving in the s direction, and construct the averages

$$\Sigma_{ij}(s) = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle, \tag{2}$$

where $i, j = 1, 2, 3, 4$, corresponding to x, x', y, and y', respectively, and the averages are over all particles in a slice of the beam at s. [One may consider formulating Eq. (2) in terms of canonical momenta instead of the variables x' and y'. However, the gauge dependence of the canonical momenta and the resulting unphysical nature of the Σ matrix and its determinant (related to beam emittance) prompted the author's decision to formulate the present theory in terms of the physical quantities x' and y'. It should be noted, however, that Gluckstern[6] has employed emittances defined using canonical momenta in an interesting example, discussed here in Section 4.] No assumption is made here, of course, about the exact form of the distribution of particle initial conditions. Using Eq. (1) it follows that the matrix Σ obeys the equation

$$\Sigma'(s) = M\Sigma + (M\Sigma)^T, \tag{3}$$

where ^{T} denotes transpose.

The matrix M generally consists of two parts, one due to applied, extenal fields and another due to self, space-charge fields. For magnetic focusing with $\mathbf{B} = (B_x, B_y, B_z)$,

$$M^{\text{ext}}(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \bar{B}_{yx} & 0 & \bar{B}_{yy} & -\bar{B}_z \\ 0 & 0 & 0 & 1 \\ -\bar{B}_{xx} & \bar{B}_z & -\bar{B}_{xy} & 0 \end{bmatrix}, \tag{4}$$

where $\bar{B}_{ab} \equiv -q(\partial B_a/\partial b)/pc$, $\bar{B}_z \equiv -qB_z/pc$, q is the particle charge, p is its momentum, c is the speed of light, and cgs units are used.

The space-charge contribution ot $M(s)$ will depend on the spatial elements of Σ, viz., $\Sigma_{11} \equiv \sigma_{xx}$, $\Sigma_{33} \equiv \sigma_{yy}$, and $\Sigma_{13} \equiv \Sigma_{31} \equiv \sigma_{xy}$, that is, on the size and orientation of the beam ellipse. The space-charge forces are linear, of course, only when the ellipse is uniformly populated, as for the K–V distribution, which here takes the form

$$f(v^TWv) = \frac{|W|^{1/2}}{\pi^2}\,\delta[v^T(s_0)W(s_0)v(s_0) - 1], \tag{5}$$

where W is a positive definite, real symmetric matrix. If we define $V(s \mid s_0)$ as that 4×4 matrix solution to the single-particle equations of motion satisfying $V(s_0 \mid s_0) = 1$, the identity matrix, then

$$W(s) = V^{-1T}(s \mid s_0)W(s_0)V^{-1}(s \mid s_0) \tag{6}$$

and[10]

$$\Sigma(s) = \frac{1}{4}W^{-1}(s) = V(s \mid s_0)\Sigma(s_0)V^T(s \mid s_0). \tag{7}$$

The space-charge part of M is evaluated in terms of σ_{xx}, σ_{yy}, and σ_{xy} for the K–V distribution in the Appendix. The result may be expressed as

$$M^{\text{sc}}(s) = \frac{\bar{I}}{(\beta\gamma)^3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ q_{xx} & 0 & q_{xy} & 0 \\ 0 & 0 & 0 & 0 \\ q_{xy} & 0 & q_{yy} & 0 \end{bmatrix}, \tag{8}$$

where $\bar{I} =$ beam current in $\hat{s}$ direction$/(mc^3/q)$, m is the particle mass, β and γ are the usual relativistic factors, $q_{xx} = S_y/D$, $q_{yy} = S_x/D$, $q_{xy} = -\sigma_{xy}/D$, $D = S_0(S_x + S_y)$, $S_x = S_0 + \sigma_{xx}$, $S_y = S_0 + \sigma_{yy}$, and $S_0 = (\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)^{1/2}$.

With $M(s)$ given by the sum of Eqs. (4) and (8), Eq. (3) describes the evolution of the x–y coupled K–V beam envelope in the presence of space charge. In the decoupled system, with M^{ext} in 2×2 block diagonal form ($\bar{B}_{xx} = \bar{B}_{yy} = 0 = \bar{B}_z$) and $\sigma_{xy} = 0$, some algebra shows that Eq. (3) reproduces the usual K–V envelope equations, for the radii $r_x = 2\sigma_{xx}^{1/2}$ and $r_y = 2\sigma_{yy}^{1/2}$. For $\sigma_{xy} \neq 0$, the K–V beam radii ($r_\pm$) and the angle (α) that the major beam axis makes with the x-axis are given by

$$\tfrac{1}{2}r_\pm^2 = \sigma_{xx} + \sigma_{yy} \pm \sigma_0 \tag{9a}$$

$$\sin 2\alpha = 2\sigma_{xy}/\sigma_0 \tag{9b}$$

$$\cos 2\alpha = (\sigma_{xx} - \sigma_{yy})/\sigma_0, \tag{9c}$$

where $\sigma_0 = [(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2]^{1/2}$.

In the decoupled case, the quantities $\sigma_{xx}\sigma_{x'x'} - \sigma_{xx'}^2 \equiv \varepsilon_{x\,\text{rms}}^2$ and $\sigma_{yy}\sigma_{y'y'} - \sigma_{yy'}^2 \equiv \varepsilon_{y\,\text{rms}}^2$ are conserved, in the absence of acceleration. When coupling is present it follows from Eq. (7) or, more generally, from Eq. (3) that the determinant of Σ is conserved. In the decoupled case, $|\Sigma| = \varepsilon_{x\,\text{rms}}^2 \varepsilon_{y\,\text{rms}}^2$, which suggests that the quantity

$$\varepsilon_{xy\,\text{rms}} \equiv |\Sigma|^{1/4} \tag{10}$$

may be a useful definition of beam quality in the coupled case; such a quantity is easily calculated by particle-tracking codes.

3. MATCHED SOLUTIONS AND THEIR STABILITY

We consider the simplest definition of a matched solution to Eq. (3) in a periodic focusing system with period L; for all s

$$\Sigma(s + L) = \Sigma(s). \tag{11}$$

A slightly more general definition, which specifies that the beam ellipse rotates by some angle θ every period, is possible but greatly complicates the analysis in the general case. This more general definition is important to consider for focusing by a longitudinal field; in that case one can find matched solutions that rotate at the Larmor frequency. We will assume here, however, that the beam ellipse returns to its initial shape and orientation every period, as specified by Eq. (11). If Eq. (7) is used, the matching condition for any distribution of the form $f(Q)$, where $Q = v^T W v$, is

$$\Sigma_0 = V_0 \Sigma_0 V_0^T, \tag{12}$$

where $\Sigma_0 = \Sigma(s_0)$ and $V_0 = V(s_0 + L \mid s_0)$. The solution of Eq. (12) for Σ_0 gives the matched launching condition for the beam matrix.

The eigenvalues of V_0 all have complex modulus 1 (since, by assumption, the single-particle motion is stable) and occur in conjugate pairs. When the eigenvalues are all distinct the eigenvectors of V_0 are complete, which we assume to be the case. If we define a matrix U that has, as columns, the normalized eigenvectors of V_0, then any 4×4 matrix, in particular Σ_0, may be written

$$\Sigma_0 = UDU^T \tag{13}$$

for some matrix of coefficients D. Substituting Eq. (13) into Eq. (12) and ordering the eigenvalues of V_0 such that $\lambda_1\lambda_2 = \lambda_3\lambda_4 = 1$, one obtains the result that only the elements D_{12}, D_{21}, D_{34}, and D_{43} of D are non-vanishing. Requiring that Σ_0 be symmetric and real means that D is symmetric and real; that is, the matched solution Σ_0 depends on just two arbitrary real parameters, D_{12} and D_{34}. In the decoupled case, of course, these parameters are related to the x and y emittances.

For $s > s_0$ the matched solution, from Eq. (7), is

$$\Sigma_0(s) = F(s)DF^T(s), \tag{14}$$

where $F(s) \equiv V(s \mid s_0)U$ is the matrix containing the four Floquet solutions of Eq. (1); F satisfies

$$F(s_0 + L) = F(s_0)\Lambda, \tag{15}$$

where Λ is diagonal and contains the eigenvalues of V_0.

In the presence of the K–V space-charge term, Eq. (8), one cannot calculate V_0 from Eq. (1) before knowing the matched solution for Σ, and an iterative solution is necessary; the most direct technique is to neglect space charge at first, calculate V_0 and the matched solution for Σ, then recalculate V_0, and so on. This method is used successfully in the example treated in Section 4.

Once a matched solution is obtained one may ask whether it is stable; that is, if a beam is launched with a distribution that is close to being matched, does it remain close to the matched solution for $s > s_0$? This is an important question for highly artificial distributions such as the K–V distribution that are never exactly (but that may be approximately) realized in practice.

In the absence of space charge the matched solution is clearly stable with respect to slight variations in Σ_0; replacing Σ_0 by $\Sigma_0 + \delta\Sigma_0$ in Eq. (7) clearly does not affect the stability of $\Sigma(s)$ since the single-particle solutions $V(s \mid s_0)$ are assumed to be stable.

In the presence of space charge one examines stability by linearizing Eq. (3). Writing $\Sigma(s) = \Sigma_0(s) + \Sigma_1(s)$ and $M(s) = M_0(s) + M_1(s)$, where a subscript 1 denotes the perturbation, one obtains the linearized equation for Σ_1:

$$\Sigma_1'(s) = M_0\Sigma_1 + (M_0\Sigma_1)^T + M_1\Sigma_0 + (M_1\Sigma_0)^T. \tag{16}$$

Only the space-charge part of M, Eq. (8), contributes to M_1 in Eq. (16). The actual linearization of M^{sc} is completely straightforward, if a bit tedious. The nine quantities needed are

$$\frac{1}{q_{xx}}\frac{\partial q_{xx}}{\partial \sigma_{xx}} = \frac{\sigma_{yy}}{2S_0S_y} - d_{xx} \tag{17a}$$

$$\frac{1}{q_{xx}}\frac{\partial q_{xx}}{\partial \sigma_{yy}} = \frac{1 + \sigma_{xx}/2S_0}{S_y} - d_{yy} \tag{17b}$$

$$\frac{1}{q_{xx}}\frac{\partial q_{xx}}{\partial \sigma_{xy}} = \frac{-\sigma_{xy}}{S_0S_y} - d_{xy} \tag{17c}$$

$$\frac{1}{q_{yy}}\frac{\partial q_{yy}}{\partial \sigma_{xx}}=\frac{1+\sigma_{yy}/2S_0}{S_x}-d_{xx} \tag{17d}$$

$$\frac{1}{q_{yy}}\frac{\partial q_{yy}}{\partial \sigma_{yy}}=\frac{\sigma_{xx}}{2S_0S_x}-d_{yy} \tag{17e}$$

$$\frac{1}{q_{yy}}\frac{\partial q_{yy}}{\partial \sigma_{xy}}=\frac{-\sigma_{xy}}{S_0S_x}-d_{xy} \tag{17f}$$

$$\frac{1}{q_{xy}}\frac{\partial q_{xy}}{\partial \sigma_{xx}}=-d_{xx} \tag{17g}$$

$$\frac{1}{q_{xy}}\frac{\partial q_{xy}}{\partial \sigma_{yy}}=-d_{yy} \tag{17h}$$

$$\frac{1}{q_{xy}}\frac{\partial q_{xy}}{\partial \sigma_{xy}}=\frac{1}{\sigma_{xy}}-d_{xy}, \tag{17i}$$

where

$$d_{xx}=\frac{\sigma_{yy}}{2S_0^2}+q_{xx} \tag{18a}$$

$$d_{yy}=\frac{\sigma_{xx}}{2S_0^2}+q_{yy} \tag{18b}$$

$$d_{xy}=\frac{-\sigma_{xy}}{S_0^2}+2q_{xy}. \tag{18c}$$

Equation (16) represents ten coupled linear equations with periodic coefficients; the ten quantities are the distinct elements of the 4×4 symmetric matrix Σ_1. By reassembling the ten quantities into a column vector w, Eq. (16) may be formally written

$$w'(s)=N(s)w(s), \tag{19}$$

where $N(s)$ is a 10×10 matrix with period L. In the usual way, one now defines another 10×10 matrix $T(s\mid s_0)$, each column of which satisfies Eq. (19); T is defined to satisfy the initial condition $T(s_0\mid s_0)=1$, the identity matrix. The eigenvalues of $T(s_0+L\mid s_0)$ then determine the stability of the system; an envelope mode is (stable, unstable) if the modulus of the corresponding eigenvalue is (less than or equal to, seems greater than) 1. There are ten eigenvalues of $T(s_0+L\mid s_0)$, the product of which may be shown to be 1. If λ is an eigenvalue then so is λ^*; $1/\lambda$ is *not* also necessarily an eigenvalue, as it is in the decoupled case.[9] The ten eigenvalues are accounted for as follows: In the decoupled system, Eqs. (3) and (16) both reduce to two 2×2 matrix equations, one each for x and y. Each 2×2 submatrix of Σ is symmetric so there are $2\times 3=6$ independent variables, giving 6 eigenvalues in the decoupled case. (Struckmeier and Reiser restrict perturbations to those that do not change the emittances ε_x and ε_y, and so find $6-2=4$ eigenvalues.) The remaining four eigenvalues are attributable to the $x-y$ coupling.

4. AN EXAMPLE: BEAM TRANSPORT IN AN $l=2$ STELLARATOR FIELD

It has been suggested that an $l=2$ stellarator field may have some advantages for the transport of high-current, relativistic electron beams.[5] Such a field consists of a continuously twisted magnetic quadrupole superimposed on a constant longitudinal field. The analysis of the form of the matched K–V beam in the continuous quadrupole case has been treated by Gluckstern;[6] inclusion of a longitudinal field requires only a redefinition of certain quantities in his analysis. Here we sketch out the solution in our own notation[7] and go on to study the envelope stability problem.

The quadrupole field takes the form

$$B_x(x, y, s) \approx kB_0(-x \sin 2ks + y \cos 2ks) \tag{20}$$

$$B_y(x, y, s) \approx kB_0(x \cos 2ks + y \sin 2\text{ks}), \tag{21}$$

where k and B_0 are constants. In the presence of the space-charge fields Eqs. (A-1, -2) the paraxial equations for a particle near the axis may be written compactly as

$$\xi'' + ib_1\xi' - n_{s1}\xi + \left(\mu_1 e^{2iks} + n_{s1}\frac{a-b}{a+b}e^{2i\alpha}\right)\xi^* = 0, \tag{22}$$

where $\xi = x + iy$, $b_1 = -eB_s/pc$, $\mu_1 = -kB_0e/pc$, $n_{s1} = (\omega_b/\beta\gamma c)^2/2$, $\omega_b =$ beam-plasma frequency, $-e$ and p are the electron charge and momentum, and a, b, and α are the radii and orientation angle of the ellipse (see Appendix).

The matched beam consists of an ellipse of fixed radii that rotates with the quadrupole field; that is, a and b are constants, and $\alpha = ks$. The substitution $\xi = \psi e^{iks}$ then allows the following solution to Eq. (22):

$$\psi = A_+e^{i\kappa_+ s} + \sigma_+A_+^*e^{-i\kappa_+ s} + A_-e^{i\kappa_- s} + \sigma_-A_-^*e^{-i\kappa_- s}, \tag{23}$$

where $A_\pm$ are arbitrary complex numbers,

$$\kappa_\pm^2 = -k_2^2 + \frac{1}{2}b_2^2 \pm \left[b_2^2\left(\frac{1}{4}b_2^2 - k_2^2\right) + \mu_2^2\right]^{1/2}, \tag{24}$$

$$\sigma_\pm = \frac{\kappa_\pm^2 + b_2\kappa_\pm + k_2^2}{\mu_2} = \frac{\mu_2}{\kappa_\pm^2 - b_2\kappa_\pm + k_2^2}, \tag{25}$$

where $k_2^2 = k^2 + kb_1 + n_{s1}$, $b_2 = b_1 + 2k$, and $\mu_2 = \mu_1 + n_{s1}(a-b)/(a+b)$. In the interesting special case where $b_2 = 0$, corresponding to the situation in which the period of the quadrupole field equals the electron cyclotron wavelength,[7] one finds $\sigma_\pm = \pm\text{sgn}(\mu_2)$.

The betatron oscillations are stable ($\kappa_\pm$ are real) when

$$b_2^2\left(\frac{1}{4}b_2^2 - k_2^2\right) + \mu_2^2 > 0 \tag{26a}$$

$$\frac{1}{2}b_2^2 - k_2^2 > 0 \tag{26b}$$

$$|k_2^2| - |\mu_2| > 0. \tag{26c}$$

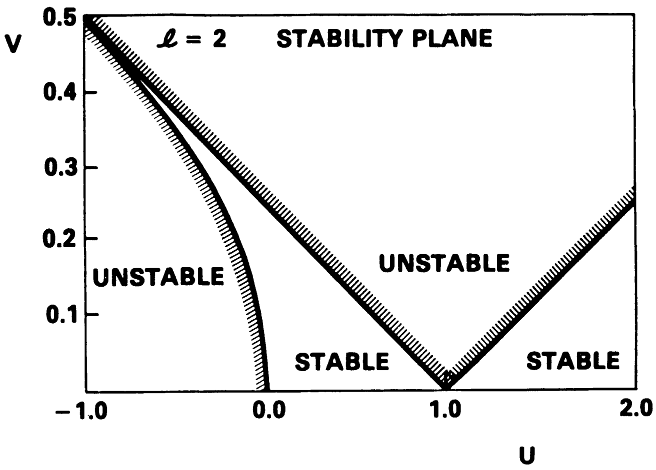

FIGURE 1 Stellarator stability plane. The quantities u and v are defined in the text following Eq. (26c).

These conditions may be illustrated using the auxiliary quantities

$$u \equiv 1 - 4k_2^2/b_2^2, \qquad v \equiv |\mu_2|/b_2^2,$$

as shown in Fig. 1. In the absence of space charge, only Eq. (26c) represents a nontrivial constraint.

We construct a K–V distribution that is independent of the phases of $A_\pm$:

$$f = f_0\delta(f_+\,|A_+|^2 + f_-\,|A_-|^2 - 1], \tag{27}$$

where f_0, $f_\pm$ are constants. The amplitudes may be expressed as functions of the real and imaginary parts of ψ:

$$|A_+|^2 = \frac{1}{D_1^2}[\kappa_-(1-\sigma_-)\psi_r - (1+\sigma_-)\psi_i']^2 + \frac{1}{D_2^2}[\kappa_-(1+\sigma_-)\psi_i + (1-\sigma_-)\psi_r']^2 \tag{28a}$$

$$|A_-|^2 = \frac{1}{D_1^2}[\kappa_+(1-\sigma_+)\psi_r - (1+\sigma_+)\psi_i']^2 + \frac{1}{D_2^2}[\kappa_+(1+\sigma_+)\psi_i + (1-\sigma_+)\psi_r']^2, \tag{28b}$$

where $D_1 = -\kappa_+(1-\sigma_+)(1+\sigma_-) + \kappa_-(1+\sigma_+)(1-\sigma_-)$, and $D_2 = \kappa_+(1+\sigma_+)\times(1-\sigma_-) - \kappa_-(1-\sigma_+)(1+\sigma_-)$. One may alternately[6] express the amplitudes in terms of ψ_r, ψ_i, and the canonical momenta $P_r \equiv \psi_r' - \frac{1}{2}b_2\psi_i$ and $P_i \equiv \psi_i' + \frac{1}{2}b_2\psi_r$.

Using Eqs. (28) in Eq. (27) and integrating over ψ_r' and ψ_i' one finds that the beam radii in the real and imaginary directions are

$$a^2 = \frac{(1+\sigma_+)^2}{f_+} + \frac{(1+\sigma_-)^2}{f_-} \tag{29a}$$

$$b^2 = \frac{(1-\sigma_+)^2}{f_+} + \frac{(1-\sigma_-)^2}{f_-}. \tag{29b}$$

Similarly, one may calculate the emittances in the real and imaginary directions by integrating Eq. (27) over (ψ_r, ψ_r') and (ψ_i, ψ_i'):

$$\varepsilon_r^2 = a^2\left[\frac{\kappa_+^2(1+\sigma_+)^2}{f_+} + \frac{\kappa_-^2(1+\sigma_-)^2}{f_-}\right] \tag{30a}$$

$$\varepsilon_i^2 = b^2\left[\frac{\kappa_+^2(1-\sigma_+)^2}{f_+} + \frac{\kappa_-^2(1-\sigma_-)^2}{f_-}\right]. \tag{30b}$$

In the quadrupole-only case, Gluckstern has defined the emittances as areas in the $P_r - \psi_r$ and $P_i - \psi_i$ planes instead. For the present problem these are given by

$$\varepsilon_{Gr}^2 = a^2\left(\frac{c_+^2}{f_+} + \frac{c_-^2}{f_-}\right) \tag{31a}$$

$$\varepsilon_{Gi}^2 = b^2\left(\frac{d_+^2}{f_+} + \frac{d_-^2}{f_-}\right), \tag{31a}$$

where $c_+ = \kappa_+(1+\sigma_+) + \frac{1}{2}b_2(1-\sigma_+)$, $c_- = \kappa_-(1+\sigma_-) + \frac{1}{2}b_2(1-\sigma_-)$, $d_+ = \kappa_+(1-\sigma_+) + \frac{1}{2}b_2(1+\sigma_+)$, and $d_- = \kappa_-(1-\sigma_-) + \frac{1}{2}b_2(1+\sigma_-)$.

Using the definitions of Σ and ψ and Eqs. (27) and (28), the value of the matched Σ matrix may now be calculated. The nonzero elements at $s = 0$ are

$$\begin{aligned}
\Sigma_{11} &= \frac{1}{4}a^2 \\
\Sigma_{14} = \Sigma_{41} &= \frac{1+\sigma_+}{f_+}[k(1+\sigma_+) + \kappa_+(1-\sigma_+)] + (+\rightarrow -) \\
\Sigma_{22} &= \frac{1}{f_+}[k(1-\sigma_+) + \kappa_+(1+\sigma_+)]^2 + (+\rightarrow -) \\
\Sigma_{23} = \Sigma_{32} &= -\frac{1-\sigma_+}{f_+}[k(1-\sigma_+) + \kappa_+(1+\sigma_+)] + (+\rightarrow -) \\
\Sigma_{33} &= \frac{1}{4}b^2 \\
\Sigma_{44} &= \frac{1}{f_+}[k(1+\sigma_+) + \kappa_+(1-\sigma_+)]^2 + (+\rightarrow -),
\end{aligned} \tag{32}$$

where the notation $+ (+\rightarrow -)$ means add the previous term with all $+$ subscripts replaced by $-$. The matched Σ matrix is clearly divided into a sum of two contributions, one from the fast $(+)$ betatron oscillation mode and one from the slow $(-)$ mode, corresponding to the decomposition in Eq. (13). The determinant of Σ, a constant of motion, is given by

$$|\Sigma| = \left(\frac{D_1 D_2}{16 f_+ f_-}\right)^2. \tag{33}$$

The specification of Σ depends on the two unspecified constants $f_\pm$, which may be determined in a variety of ways. One reasonable choice would seem to be to specify a value for $|\Sigma|$ and to require that $\varepsilon_r = \varepsilon_i$ in Eq. (30). This choice, however, gives the unsatisfactory result that the matched beam radii are *decreasing* functions of beam current unless the operating point is in the "second stable region" of Fig. 1, where $u > 1$. Curiously, instead of specifying $\varepsilon_r = \varepsilon_i$, the choice $\varepsilon_{Gr} = \varepsilon_{Gi}$, from Eq. (31), avoids this problem, leading to beam radii that are increasing functions of current; we make this choice in the numerical example, below. The corresponding condition on $f_\pm$ is, using Eq. (31),

$$|1 - \sigma_-^2| f_+ = |1 - \sigma_+^2| f_-, \tag{34}$$

except in the special case[7] $b_2 = 0$, for which $\sigma_\pm = \pm\, \mathrm{sgn}\,(\mu_2)$; in that case Eq. (34) is replaced by the condition $\kappa_- f_+ = \kappa_+ f_-$.

Figures 2a, 3a, 4a, and 5a illustrate the dependence of beam radii on beam current, beam energy, longitudinal field strength, and quadrupole field strength, respectively. All cases have $k = -2\pi/18\ \mathrm{cm}^{-1}$ and $\beta\gamma\, |\Sigma|^{1/4} = 2.78 \times 10^{-3}$ rad-cm, which correspond to typical values for a beam-transport experiment presently being carried out.[11] Each point on the plots, except those for zero current, requires an iterative procedure to obtain because the betatron wavenumbers $\kappa_\pm$ depend on current density, that is on the beam radii, which makes Eqs. (29) implicit relations for the beam radii, for nonzero current. Once the beam radii are found, the elements of the matched Σ matrix at $s = 0$ follow from Eq. (32).

One may study the stability of the matched solution, using Eq. (16). One might expect that since the matched solution is a constant in the frame rotating with the stellarator field, one could carry through the stability analysis to calculate oscillation frequencies and growth rates analytically for this example. Actually, even in the rotating frame [the frame of the ψ-variable of Eq. (23)] the two degrees of freedom, ψ_r and ψ_i, are coupled, requiring the calculation of the eigenvectors of a full 4×4 matrix to find the matched solution. Note however that in the special case where $b_2 = 0$, described following Eq. (25), the coupling in the rotating frame vanishes, and the formalism described here may be shown to reproduce the K–V envelope equations, with constant focusing terms, in the rotating frame; in this case the envelope oscillation frequencies are easily calculated[9] analytically, and agree with those obtained numerically from Eq. (16). To study the linear stability problem in the coupled case, Eq. (16) must be integrated simultaneously with Eq. (3) for the matched solution, using Eq. (32) as the initial condition. One then forms the matrix $T(L \mid 0)$, defined following Eq. (19) and finds its eigenvalues λ_j, $j = 1, 2, \ldots, 10$. We define the growth rate per period as

$$\Gamma_j \equiv \ln |\lambda_j|. \tag{35}$$

Figures 2b–5b show plots of the growth rates for the unstable ($\Gamma_j > 0$) envelope modes as functions of system parameters. One sees that, depending on parameter values, the dominant mode has a fairly large peak growth rate, which suggests the probability of large emittance growth for a real system at this point. The growth-rate curves of Figs. 2b–5b are typical of an instability due to a system resonance, which is detuned when system parameters are varied slightly; in fact, this is an example of a "confluent resonance" for the stellarator system.[9] We note that calculation of linear growth rates for the K–V beam, although not

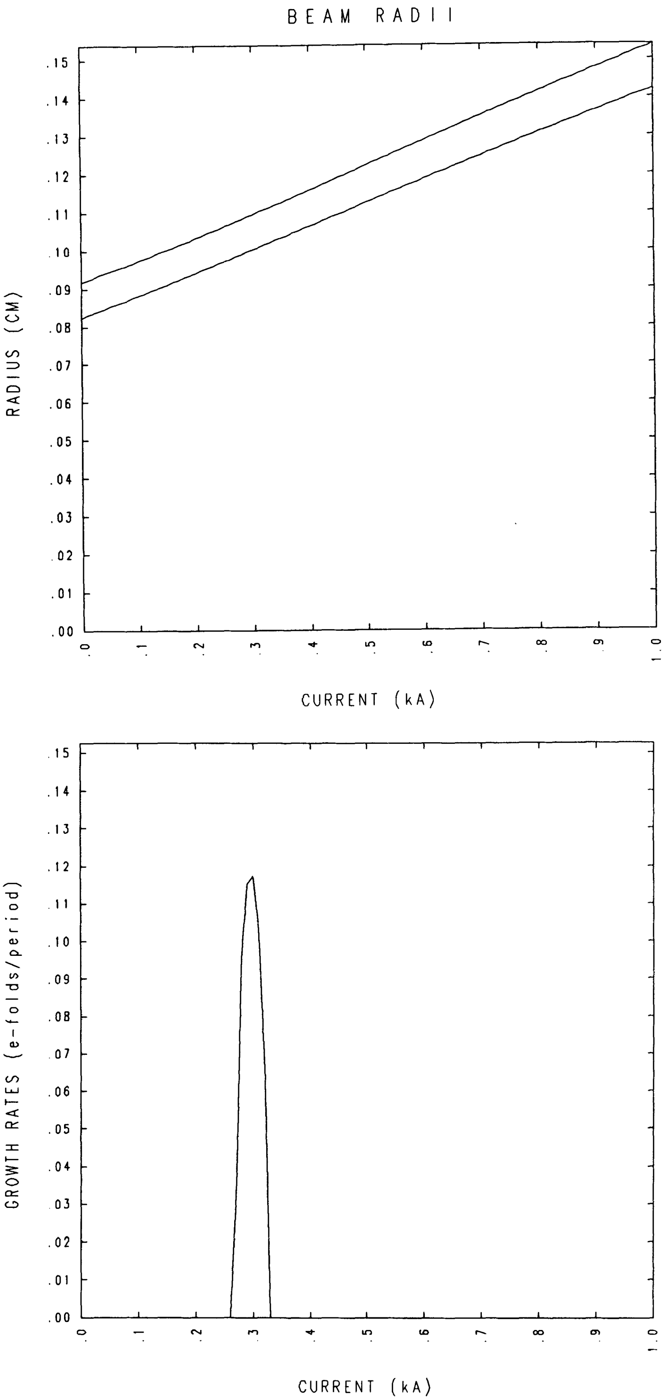

FIGURE 2 Equilibrium beam radii and growth rates for unstable envelope modes versus current for $E_{\mathrm{beam}} = 1\,\mathrm{MeV}$, $B_s = 5\,\mathrm{kG}$, and $kB_0 = 500\,\mathrm{G/cm}$.

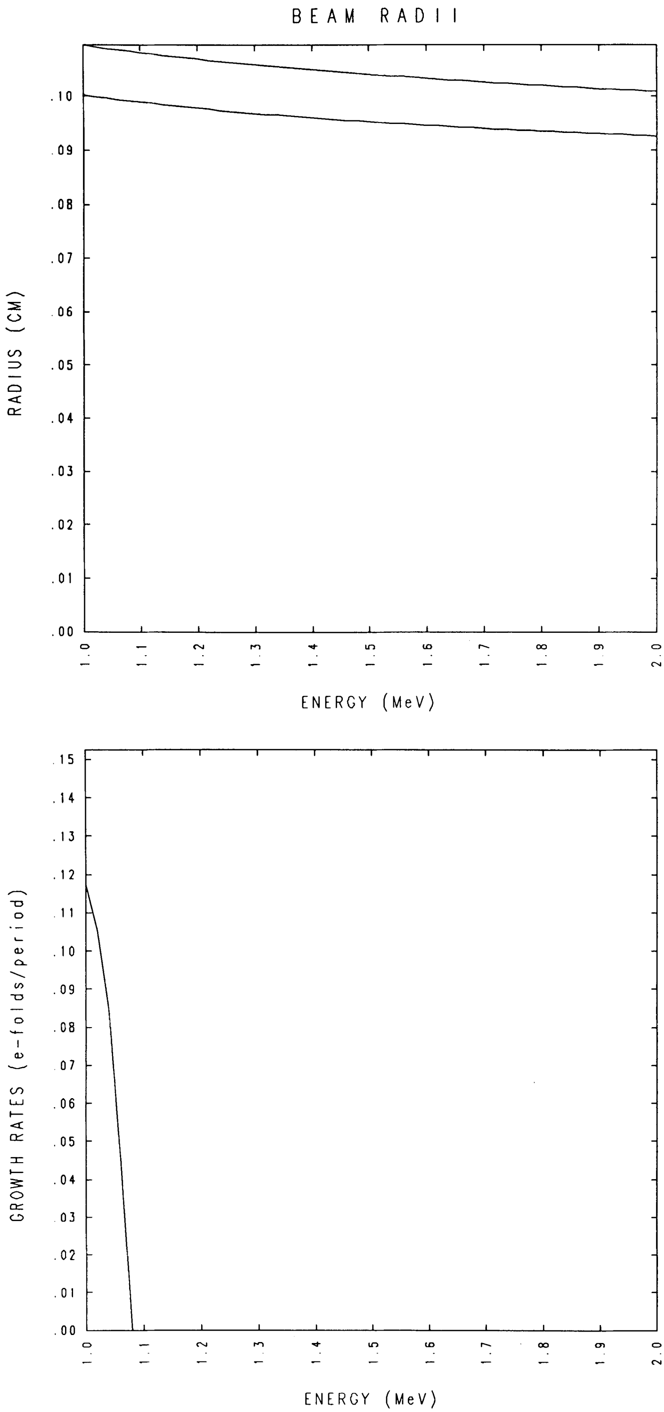

FIGURE 3 Equilibrium beam radii and growth rates for unstable envelope modes versus beam energy for $I_{\text{beam}} = 300$ A, $B_s = 5$ kG, and $kB_0 = 500$ G/cm.

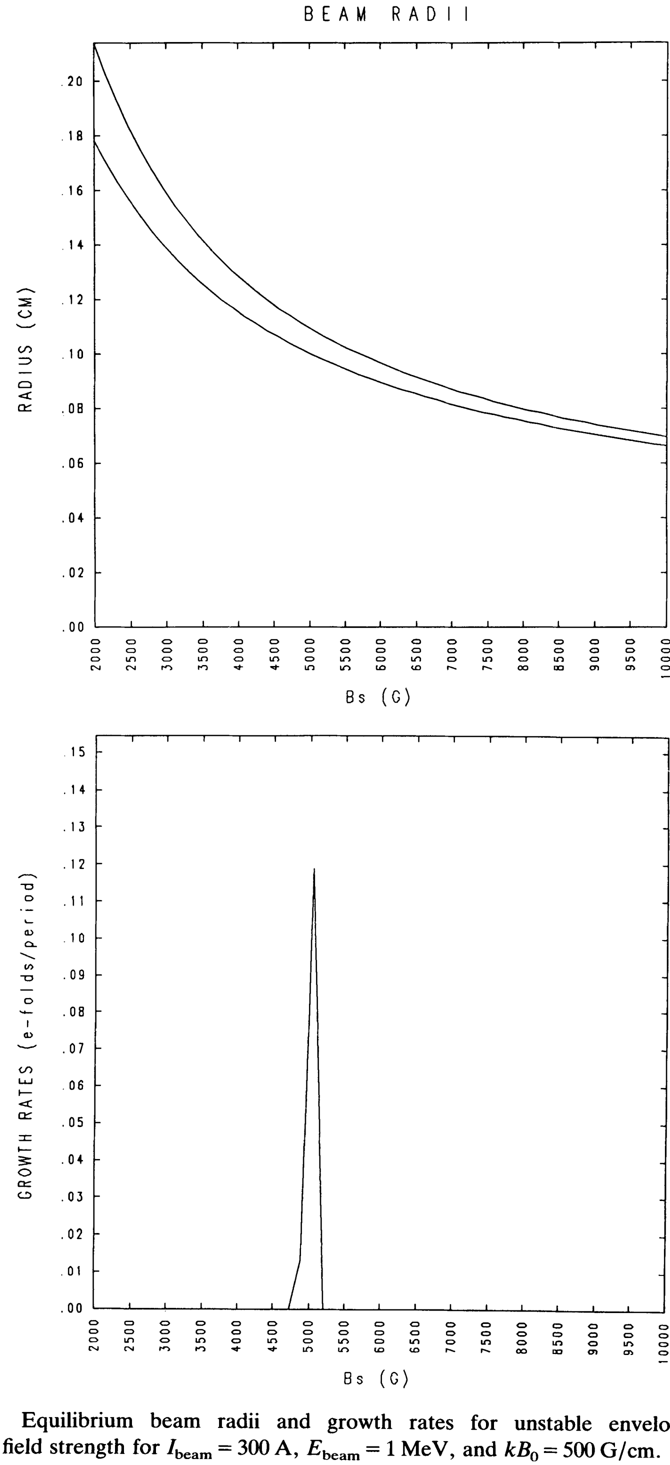

FIGURE 4 Equilibrium beam radii and growth rates for unstable envelope modes versus longitudinal field strength for $I_{\text{beam}} = 300$ A, $E_{\text{beam}} = 1$ MeV, and $kB_0 = 500$ G/cm.

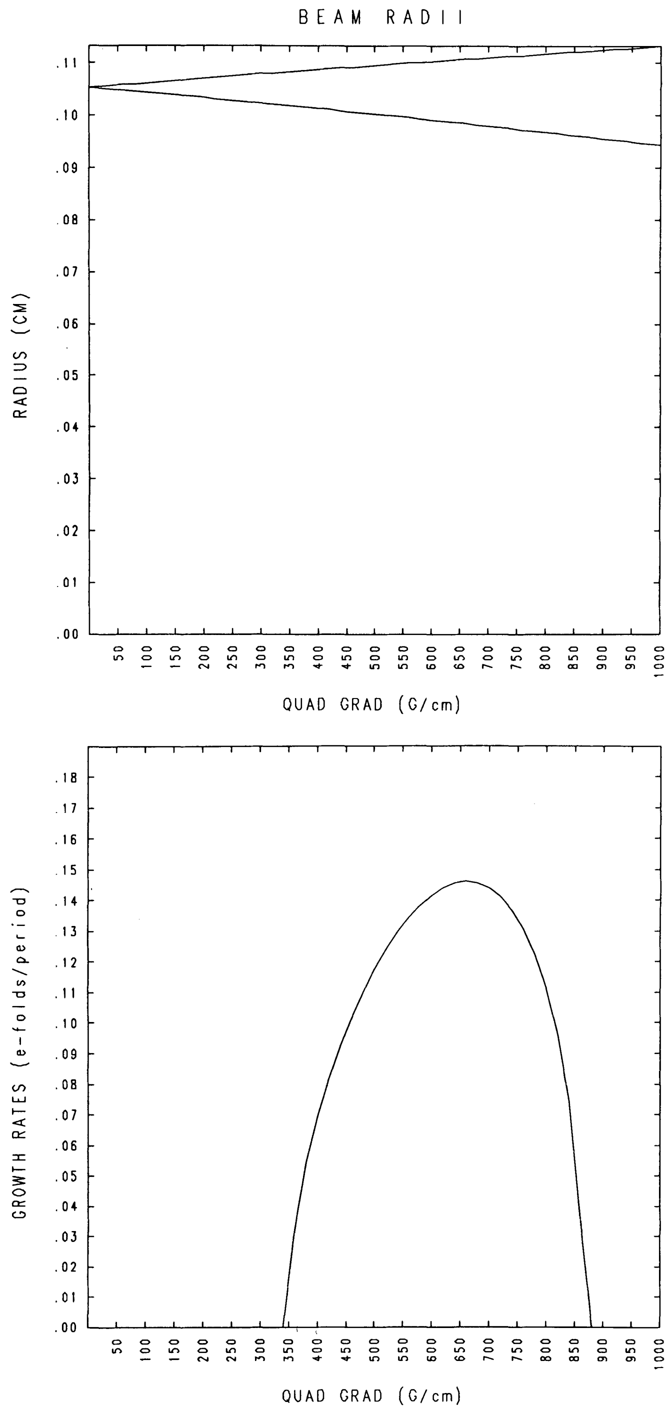

FIGURE 5 Equilibrium beam radii and growth rates for unstable envelope modes versus quadrupole field strength for $I_{\text{beam}} = 300$ A, $E_{\text{beam}} = 1$ MeV, and $B_s = 5$ kG.

quantitatively important for real beams, may suggest values of parameters (external field strengths, beam currents and energies, etc.) required to avoid significant emittance growth in real beams. Numerical simulations need to be done to test this conjecture for the stellarator system, though simulations in the decoupled case show that emittance growth is observed in simulations of beams with realistic profiles when the growth rates for corresponding K–V beams are large.[8,9]

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APPENDIX

Space-Charge Contribution to Matrix *M* for K–V Distribution

We consider the self-fields of an elliptical beam of axes a and b; axis a is tilted at an angle α to the x-axis. The electro- and magneto-static potentials are simply obtained by rotation of coordinates:

$$\phi(x, y) = -\frac{\lambda}{2}(q_{xx}x^2 + 2q_{xy}xy + q_{yy}y^2) \tag{A-1}$$

$$\mathbf{A}(x, y) = \beta\phi\hat{s}, \tag{A-2}$$

where λ is the line-charge density and

$$q_{xx} = \frac{4}{a+b}\left(\frac{\cos^2\alpha}{a} + \frac{\sin^2\alpha}{b}\right) \tag{A-3a}$$

$$q_{yy} = \frac{4}{a+b}\left(\frac{\cos^2\alpha}{b} + \frac{\sin^2\alpha}{a}\right) \tag{A-3b}$$

$$q_{xy} = \frac{4}{a+b}\sin\alpha\cos\alpha\left(\frac{1}{a} - \frac{1}{b}\right). \tag{A-3c}$$

The values of the spatial σ's for a uniformly populated ellipse are

$$\sigma_{xx} = \frac{1}{4}(a^2\cos^2\alpha + b^2\sin^2\alpha) \tag{A-4a}$$

$$\sigma_{yy} = \frac{1}{4}(b^2\cos^2\alpha + a^2\sin^2\alpha) \tag{A-4b}$$

$$\sigma_{xy} = \frac{1}{4}(a^2 - b^2)\sin\alpha\cos\alpha. \tag{A-4c}$$

It is now simply a matter of algebra to eliminate a, b, and α from Eqs. (A-3) in favor of σ_{xx}, σ_{yy}, and σ_{xy}, using Eqs. (A-4). The intermediate results

$$ab = 4(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)^{1/2}, \tag{A-5}$$

$$a^2 + b^2 = 4(\sigma_{xx} + \sigma_{yy}) \tag{A-6}$$

are useful for this exercise. The results for q_{xx}, q_{yy}, and q_{xy}, cited in the text, follow.

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10. For other distributions $f(Q)$, functions of the quadratic form $Q = v^T W v$, a straightforward calculation shows that Eqs. (6) and (7) continue to hold, with the replacement of the factor 1/4 in Eq. (7) by $\alpha/4$, where $\alpha = \int_0^\infty dQ Q^2 f(Q) / \int_0^\infty dQ Q f(Q)$.
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