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STRONG SPIN MATCHING WITH AND WITHOUT SNAKES: A SCHEME FOR PRESERVING POLARIZATION IN LARGE RING ACCELERATORS

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The so-called strong spin matching is proposed as a localized compensation scheme for spin-orbit coupling, producing spin transparency, not only globally for the ring as a whole but locally for each of its many subsections. Distortions of the spin motion will then accumulate to a lesser extent, and polarization will be more stable. The method amounts to looking for a resonant compensation of errors instead of merely avoiding resonant accumulation. It is shown how Siberian snakes can be used to obtain strong spin matching, independent of energy, as a key to polarized proton acceleration to very high energies. As an example, the method is applied to the SSC ring, and a preferred snake design is proposed.

It has now been more than 10 years since "Siberian snakes," originally called "spin flips," were proposed by Derbenev and Kondratenko (D + K)[1–3,4a,5] for eliminating the intrinsic depolarizing resonances which are crossed by the beam in a proton ring during acceleration. They suggested two types of snakes, subsequently called the first and the second kind, with intrinsic spin precessions of 180° and 0°, respectively (see Section 2.2), and showed that such a pair of snakes inserted between half rings will make the spin tune equal to one-half, independent of energy, and will thus remove the intrinsic spin resonances. D + K[4a,5] also pointed out that polarization could even be made much more stable by inserting $2M$ snakes, an odd number of each kind, but it did, to my knowledge, not become generally understood what the optimum lattice configuration and parameters would be, and which gain factor could be obtained.

Meanwhile, people began inventing snakes[4b,6–16,18–20] and thereby discovered snakes of a more general type,[17] with continuously variable intrinsic spin precession (see Section 2.2), that could be employed instead of the particular kinds proposed by D + K initially. Numerical tracking studies indicated that in a particular ring configuration employing these new snakes polarization was much more stable,[21,22] but a quantitative evaluation of the general ring layout with snakes has, to my knowledge, not been obtained so far.

It is the aim of this paper to help to fill this need by presenting the concept of "strong spin matching" and derive from it the quantitative design criteria for an optimum ring, first without snakes, and then with snakes. Here it will turn out that, in designing the ring for stable spin motion at all energies, the important parameter is the intrinsic snake precession angle which, for a given configuration, must be adjusted to particular values.

In Section 3 of this paper, the rules obtained are applied to an example, where I show how the next large accelerator for protons, the SSC ring in the U.S., would want to be equipped with snakes in case polarized beams were considered a worthwhile goal in this machine.

1. ENERGY-DEPENDENT STRONG SPIN MATCHING WITHOUT SNAKES

1.1. *Strong Spin Matching for Vertical Betatron Oscillations in the Standard Periodic Cell Lattice*

Even in an ideal planar ring without errors, the spin motion is generally coupled to the vertical orbital motion. A particle performing vertical betatron oscillations sees horizontal field components which rotate the spin away from its vertical equilibrium position. If the betatron frequency is in resonance with the spin precession frequency, these rotations will accumulate, and the spin may approach and precess near the horizontal plane and, thereby, due to energy variation, lose its phase synchronism. An initially polarized beam will thus get depolarized when crossing one of the "intrinsic" linear resonances:

$$\nu \pm Q_z = p, \quad p \text{ integer}. \tag{1}$$

But even away from strong resonances, at a stationary working energy, migrations of the spin away from the vertical will be detrimental to polarization, and the spin should be prevented from straying too far in one period. Initially for electron rings, Chao and Yokoya[23,24a] have proposed, with good success, to match the optics such that the spin will return after one revolution. For proton rings, tracking results[21,22] have also indicated that there are large differences in spin rigidity between comparable lattice configurations, probably again due to differences in optical matching. Therefore, we want to find those particular optical settings which tie the spin most closely and allow for minimum migrations only.

Quantitatively, in a linear approximation in an flat, perfect ring, the change of vertical spin component, i.e., of the polar angle between spin direction and vertical precession axis, is given by the integral over the quadrupole strength $k(s)$ times the betatron oscillation amplitude, modulated by the spin precession:[23,24a]

$$I_\pm(\sigma) = \int_0^\sigma e^{i\psi(s)} k(s)\sqrt{\beta_z(s)}\, e^{\pm i\psi_z(s)}\, ds = \int_0^\sigma kb e^{i(\psi \pm \psi_z)}\, ds. \tag{2}$$

Following Yokoya's notation,[24b] we have written this "spin-orbit coupling integral" in complex form, with both signs of ψ_z, in order to include in its 4 components (real and imaginary) the two orthogonal phases for each of the two oscillations. If this integral vanishes over a certain period, then the change of vertical spin component over this period is zero for all phases, and the period is said to be "spin transparent for *vertical betatron oscillations*," which means that it causes zero spin-orbit coupling.

In a periodic lattice, the integral [Eq. (2)] over a string of n identical cells may be written

$$I_{\pm} = \sum_{k=0}^{n-1} e^{k \cdot 2\pi i(\nu_0 \pm Q_{z0})} \int_{1\text{st cell}} kbe^{i(\psi \pm \psi_z)}\, ds, \tag{3}$$

using the quasi-periodicity of the integrand, with $2\pi\nu_0$ and $\pm 2\pi Q_{z0}$ being the phase advances of spin precession and betatron oscillation per cell. Thus, summing up,

$$I_{\pm} = \frac{1 - e^{n \cdot 2\pi i(\nu_0 \pm Q_{z0})}}{1 - e^{2\pi i(\nu_0 \pm Q_{z0})}} \cdot I_{\text{cell}\,\pm}. \tag{4}$$

This formula tells how, for a given betatron tune per cell, one can choose certain spin tunes, i.e., working energies, which make the string of n cells spin-transparent. This procedure is called "spin matching over n cells." Technically, it means achieving a resonant cancellation of spin-orbit coupling, instead of merely avoiding a resonant enhancement by considering the resonant deonominator of Eq. (4). The modulus of the fraction in Eq. (4) is

$$\left|\frac{1 - e^{in\delta}}{1 - e^{i\delta}}\right| = \left|\frac{\sin \frac{1}{2}n\delta}{\sin \frac{1}{2}\delta}\right|, \quad \text{with} \quad \delta = 2\pi(\nu_0 \pm Q_{z0}), \tag{5}$$

and is shown for the example of $n = 8$ in Fig. 1. It has, on the resonance at $\delta = 0$ or 2π, the value n and is zero at $\delta = k \cdot 2\pi/n$, $k = 1, 2, \ldots, n-1$. We are, of course, already familiar with related behavior: In a perfect machine with superperiodicity N, we only expect linear resonances of the form $\nu = k \pm Q_z$ to appear with $k = pN$; p, k integer. If k is not a multiple of N, i.e., for $k = 1, 2, \ldots, N-1$, resonances are suppressed by the superperiodicity.[25]

To maintain maximum polarization in a collider ring, the working energy

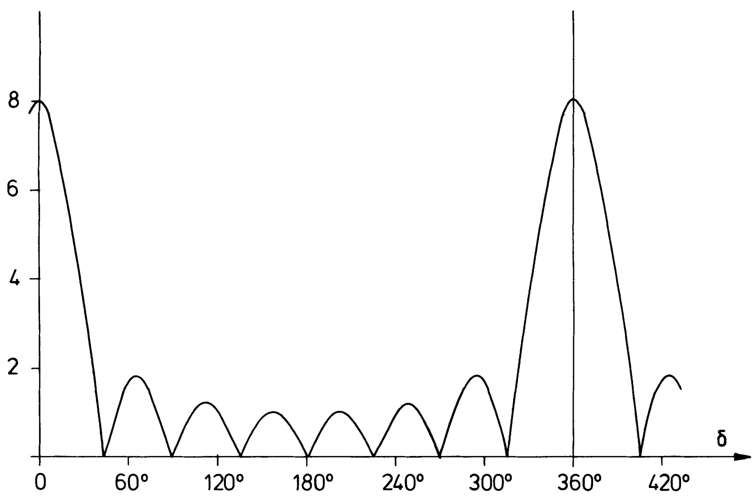


FIGURE 1 The function $\left|\frac{\sin 1/2n\delta}{\sin 1/2\delta}\right|$ for $n = 8$.

should be chosen such that the spin-orbit coupling integral $I_\pm$ vanishes over each string composed of the smallest possible number, or at least a small number *n*, of cells. Then, the vertical component of the spin will return as quickly as possible to its initial value, independent of betatron phase and amplitude, and the intervening deviations which give rise to depolarizaton will be kept minimal. Expressed differently, it means that the beat of the spin-orbit coupling integral is being kept small. The spin matching over a small number *n* of cells, as compared to the standard spin matching over, say, one revolution, may be seen to be analogous to strong focusing in comparison to weak orbit focusing. Matching over a small number of cells means that we have strong "focusing" of the spin, and we thus propose to call this technique *strong spin matching.* In large rings, it means a localized compensation of spin-orbit coupling instead of a global one, and it thus goes beyond the original spin-matching scheme that was proposed by Chao and Yokoya[23,24a] for the ring as a whole and that has meanwhile become standard. This new localized compensation technique may possibly lead into a qualitatively new regime where the vertical match will, at the same time, also serve to partly cure the depolarizing effect of horizontal oscillations in the case of magnet errors, as will be explained below.

If one wants to compare the stability of various proton ring designs against depolarization, the concept of strong spin matching permits a quantitative evaluation, suggesting as an inverse quality factor q the mean square value of the spin-orbit coupling integral of Eq. (2), taken over one revolution:

$$q = \langle I_+(s) \cdot I_+^*(s) + I_-(s) \cdot I_-^*(s) \rangle_{\text{ring}}. \tag{6}$$

If q varies significantly with ring position, its average value must be used.

In practice, typical values of the betatron phase advance per cell are $\varphi = 90°$, $72°$, and $60°$; i.e., $\varphi = 360°/c$, with $c = 4, 5, 6$, thus, $Q_{z0} = 1/4, 1/5, 1/6$, respectively. Considering a string of n cells, with $n = 1, 2, \ldots, 20$, all possible values of ν_0 for obtaining spin transparency over this string are given in Table I,

TABLE I

Local spin tunes ν_0 per cell for strong spin matching over a string of n cells. ($p = 0, 1, 2, \ldots$)

betatron phase per cell	number of cells												
	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 8$	$n = 9$	$n = 10$	$n = 12$	$n = 15$	$n = 16$	$n = 18$	$n = 20$
$\varphi = 90°$ ($Q_{z0} = \frac{1}{4}$)	—	—	$\nu_0 = \frac{p}{2}$	—	$\nu_0 = \frac{p}{2} \pm \frac{1}{12}$	$\nu_0 = \frac{p}{2} \pm \frac{1}{8}$ and $\nu_0 = \frac{p}{2}$	—	$\nu_0 = \frac{p}{2} \pm \frac{1}{20}$ and $\nu_0 = \frac{p}{2} \pm \frac{3}{20}$	$\nu_0 = \frac{p}{12}$, but $\nu_0 \neq p \pm \frac{1}{4}$	—	$\nu_0 = \frac{p}{16}$, but $\nu_0 \neq p \pm \frac{1}{4}$	$\nu_0 = \frac{2p+1}{36}$, but $\nu_0 \neq p \pm \frac{1}{4}$	$\nu_0 = \frac{p}{20}$, but $\nu_0 \neq p \pm \frac{1}{4}$
$\varphi = 72°$ ($Q_{z0} = \frac{1}{5}$)	—	—	—	$\nu_0 = \frac{p}{5}$, but $\nu_0 \neq p \pm \frac{1}{5}$	—	—	—	$\nu_0 = \frac{p}{10}$, but $\nu_0 \neq p \pm \frac{1}{5}$	—	$\nu_0 = \frac{p}{15}$, but $\nu_0 \neq p \pm \frac{1}{5}$	—	—	$\nu_0 = \frac{p}{20}$, but $\nu_0 \neq p \pm \frac{1}{5}$
$\varphi = 60°$ ($Q_{z0} = \frac{1}{6}$)	—	$\nu_0 = p \pm \frac{1}{2}$	—	—	$\nu_0 = p + \frac{1}{2}$ and $\nu_0 = \frac{p}{3}$	—	$\nu_0 = p + \frac{1}{2}$ and $\nu_0 = \frac{p}{3} \pm \frac{1}{18}$	—	$\nu_0 = \frac{p}{12}$, but $\nu_0 \neq p \pm \frac{1}{6}$	$\nu_0 = \frac{2p+1}{30}$, but $\nu_0 \neq p \pm \frac{1}{6}$	—	$\nu_0 = \frac{p}{18}$, but $\nu_0 \neq p \pm \frac{1}{6}$	—

where ν_0 is the local spin tune per cell. The table shows that the minimum numbers of cells in the string are $n=3$ or 6 for $\varphi=60°$, $n=5$ for $\varphi=72°$, and $n=4$ or 6 for $\varphi=90°$, but corresponding ν_0 values may not lie within the energy range of the machine, and one will then go to somewhat higher n values. In any case, the recipe for strong spin matching is to satisfy one of the matching conditions by choosing the corresponding number n of cells in the string, the local betatron tune Q_{z0}, and that particular working energy which, in the periodic cell lattice, will give the corresponding local spin tune ν_0.

This holds for the group of strings that can be placed in each section of the periodic arc lattice in the ring. Between these, the insertions must also be made spin-transparent for vertical betatron oscillations at the chosen particular energy, using standard spin-matching procedures. Thereby, the overall spin tune of the ring can be adjusted to be away from resonances, no matter what the contribution of the spin-transparent groups of regular cells to the spin tune will add up to by themselves.

In the ideal flat ring considered so far, the spin motion is not coupled to horizontal orbit oscillations, since the periodic spin direction $\mathbf{n}$ is vertical everywhere. Magnet errors, however, will distort the closed orbit, and the periodic spin direction $\mathbf{n}(s)$ will then vary along this orbit and will locally precess about the vertical with the spin precession frequency, and so will its horizontal component. If this precession stays sufficiently uniform over one period of strong spin matching (n cells), the spin-orbit coupling integral for horizontal betatron oscillations will assume the same form as in Eq. (2), since the change in polar angle between spin direction and $\mathbf{n}(s)$ is again given by the quadrupole strength $k(s)$ times the betatron oscillation amplitude, modulated by the precession of the projection of $\mathbf{n}(s)$ onto the horizontal plane:

$$I_{x\pm} \cong \int e^{i\psi(s)}k(s)\sqrt{\beta_x(s)}e^{\pm i\psi_x(s)}\,ds = \int kb_x e^{i(\psi\pm\psi_x)}\,ds = \frac{1-e^{n\cdot 2\pi i(\nu_0\pm Q_{x0})}}{1-e^{2\pi i(\nu_0\pm Q_{x0})}}\cdot I_{x\,\text{cell}\,\pm}. \tag{7}$$

Considering here only the simplest case with $Q_{x0}=Q_{z0}$, which is often chosen in practice, we see that strong spin matching for vertical betatron oscillations may imply that at the same time one also obtains an approximate strong spin matching for horizontal betatron oscillations in a machine with magnet errors.

Carrying this further, one observes that a similar argument also holds for the horizontal orbit deviations due to an offset in energy. In this case, the betatron amplitude in the integral [Eq. (7)] must be replaced by the horizontal dispersion trajectory which, although not looking quite sinusoidal, can in a FODO structure effectively be approximated by its main Fourier component that has one oscillation per cell. Spin-orbit coupling is then approximately given by

$$I_{s\pm} \cong \frac{1-e^{n\cdot 2\pi i(\nu_0\pm 1)}}{1-e^{2\pi i(\nu_0\pm 1)}}\cdot I_{s\,\text{cell}\,\pm} \tag{8}$$

and therefore becomes small for noninteger $\nu_0=p/n$; $p=1, 2, \ldots.$ Comparing this local spin tune per cell with the ν_0 values for strong spin matching composed in Table I, it appears that in almost all listed cases with noninteger ν_0 the integral given by Eq. (8) will vanish together with that given by Eq. (4), and we may have an approximate strong spin matching also for the effect of the horizontal dispersion in a machine with errors.

In the vertical direction, the closed orbit deviations generated by errors will, similarly to betatron oscillations, also cause a spin-orbit coupling. To the extent that, over one strong spin-matching period, the vertical closed orbit will maintain its similarity to a vertical betatron oscillation trajectory, the coupling of this orbit to the spin motion will also cancel over this period.

Finally, it should be mentioned that even the spin motion due to nonlinear orbit effects may be subjected to strong spin matching. As an example, let us assume that the field of the cell quadrupoles has a 12-pole component giving a spin rotation proportional to the fifth power of the vertical betatron amplitude. Then, in a lowest-order approximation, where one ignores the effect of the nonlinearity on the orbit, the coupling of this nonlinearity to the spin over a string of n cells is given by

$$I_{\pm}^{(5)} = \int e^{i\psi} k b^5 e^{\pm 5i\psi_z}\, ds = \int k b^5 e^{i(\psi \pm 5\psi_z)}\, ds = \frac{1 - e^{n\cdot 2\pi i(\nu_0 \pm 5Q_{z0})}}{1 - e^{2\pi i(\nu_0 \pm 5Q_{z0})}} \cdot I_{\text{cell}\,\pm}^{(5)} \qquad (9)$$

and therefore is made to vanish at all noninteger values of

$$\nu_0 \pm 5Q_{z0} = \frac{p}{n}; \qquad p = 1, 2, \ldots$$

1.2. *Particular Energies for Strong Spin Matching in LEP and HERA*

As an example of strong spin matching without snakes, we shall select from Table I the particular local spin tunes that lie within the energy ranges of the LEP ring at Geneva and the HERA electron ring at Hamburg. Generally, the spin tune ν_0 and the orbit deflection angle Δ per cell are related by

$$\nu_0 = \frac{E[\text{GeV}]}{0.44065} \cdot \frac{\Delta}{2\pi} \quad \text{for electrons,} \qquad (10a)$$

and

$$\nu_0 = \frac{E[\text{GeV}]}{0.52335} \cdot \frac{\Delta}{2\pi} \quad \text{for protons.} \qquad (10b)$$

With the deflection angles per cell, Δ, equal to 0.03020762 rad in the HERA electron ring and 0.02261280 rad in the LEP regular cell lattice,[26] and the energy ranges of, say, 27.5 GeV $< E <$ 35 GeV for HERA and 45 GeV $< E <$ 60 GeV for LEP, Phase 1, we find from Table I the particular energies at which, in these machines, strong spin matching is obtained over a string of n cells ($n \leq 20$). They are given in Table II. At these particular energies, polarization is expected to be particularly stable, provided that the insertions between the regular arc sections are also matched to be spin-transparent at these energies.

TABLE II

Particular energies for strong spin matching over n cells in the HERA and LEP electron rings

spin tune per cell ν_0	HERA energy E [GeV]	betatron phase per cell φ	n = 3	4	5	6	8	9	10	12	15	16	18	20	LEP energy E [GeV]
3/10	27.497	90°												X	
		72°							X					X	
		60°									X				
11/36	28.006	90°											X		
5/16	28.642	90°										X			
1/3	30.552 GeV	90°								(X)					
		72°									X				
		60°				(X)				X	X		X		
7/20	32.079 GeV	90°							(X)					X	
		72°												X	
13/36	33.098 GeV	90°											(X)		44.214
11/30	33.607	60°									X				44.894
3/8	34.371 GeV	90°					(X)					X			45.915
7/18	35.644	60°						X					X		47.615
2/5	36.662	90°												X	48.976
		72°			X				X		X			X	
		60°									X				
5/12		90°				X				X			X		51.016
		60°								X					
13/30		60°									X				53.057
7/16		90°										X			53.567
4/9		60°											X		54.417
9/20		90°							X					X	55.097
		72°												X	
7/15		72°									X				57.138
17/36		90°											X		57.818
1/2		90°		X			X		X	X		X		X	61.219
		72°							X					X	
		60°	X			X		X		X			X		

HERA, as well as LEP, are intended to start operation at a betatron phase advance of $\varphi = 60°$ per cell, with the option of changing to $\varphi = 90°$ later, when going to higher energies. In HERA, the spin rotators for longitudinal polarization at the interaction points must, for each period of operation at a fixed working energy, be geometrically set for this energy. It appears from Table II that, to start polarized beam work, 30.35 GeV should be a good choice, since there, with $\varphi = 60°$, every string of $n = 6$ cells will be spin-transparent. With 52 cells in each of the four arcs, eight such strings can be placed in each arc, provided that there will remain enough optical flexibility for spin-matching in the insertions outside. The question of which tolerances on energy and phase advance must be met[27] for an effective strong spin matching, will not be answered here and needs a separate investigation.

2. ENERGY-INDEPENDENT STRONG SPIN MATCHING WITH SNAKES

2.1. *How to Obtain Strong Spin Matching Independent of Energy?*

The spin-transparency conditions obtained in the previous section have the principal drawback of depending on spin tune and thus on energy. They are very useful in a collider ring with a polarized beam at a fixed working energy, but during acceleration, the ring does not remain strongly spin-matched. How then can this be achieved?

As a first step in this direction, resonance crossing can be avoided by applying the ingenious topological trick invented by Derbenev and Kondratenko (D + K).[4a,5] It consists of introducing into a ring composed of two identical half rings a so-called "Siberian snake" at the end of each half ring. Each snake inverts the spin from upwards to downwards, or vice versa, such that the energy-dependent spin precession accumulated in the first half ring is spun backwards to zero again in the second half ring. If, then, an energy-independent spin precession of 180° is designed into one of the snakes, the spin tune of the ideal ring without errors will be $\nu = 1/2$ at all energies. The intrinsic resonances will thus be avoided if the betatron tune is not half-integer, and the spin tune spread due to energy spread in the beam will vanish over one revolution. It was pointed out by J. Buon that spin flips by snakes are used in a similar way as the flips of magnetization employed in the well-known NMR spin-echo technique; the underlying principle is the same in both cases.[28] But staying away from resonances does not mean that one has a spin-transparent machine; by itself, it does not even ensure that the spin-orbit coupling over one revolution will be small.

The ingenious second step toward an energy-independent strong spin matching is the notion of D + K[4a,5] that by increasing the number of superperiods from one to M, with $2M$ snakes in the ring, the sensitivity of polarization against magnet errors can be reduced by a factor $1/M$. They wrote in 1978:[5] "Using multiple flips of the vertical polarization for increasing spin stability can be compared to using strong focusing instead of the weak one for betatron oscillations of the particles. For a storage ring with $2M$ spin flip sections, . . . the requirements to the magnetic system are M times weaker."

This points in the direction of strong spin matching with snakes and thus is at the root of it, but again a localized half-integer spin tune for each superperiod does not guarantee that a short string of superperiods will be spin-transparent; it only ensures that resonance accumulation of errors in spin motion is avoided locally, although with a spin-orbit coupling that may not be small.

To make the integral vanish in the general periodic lattice with snakes, the snake precession angle (see Section 2.2) must be adjusted to the betatron tune and number of cells. In doing this third and last step, we will, as in Section 1.1 for the case without snakes, write down the spin-orbit coupling integral for a string of m superperiods that each include a pair of snakes, and will thereby obtain a quantitative formulation of energy-independent strong spin matching as a recipe for an optimum ring design with snakes.

2.2. *General Properties of Snakes and Snake Configurations*

By "snake" we mean any composite system of magnets that, in the orbit-following spin coordinate system, rotates the spin by 180° about an arbitrary axis that lies in the horizontal plane. Then, the overall rotation of the spin can be thought to be obtained by two consecutive rotations:

1st rotation: by 180° about the transverse horizontal axis,

2nd rotation: by an angle α about the vertical axis.

If both rotations are combined into one, the effective rotation axis is at an angle $\alpha/2$ with respect to the transverse horizontal axis. α is called the precession angle of the snake.[17] Snakes for any positive or negative value of α can be designed,[17] and examples are given in Section 3.

In the most general ring with $2M$ snakes, as shown in Fig. 2 for $2M = 4$, the equilibrium spin direction between snakes will alternately point upwards and downwards, and the spin tune is

$$\nu = \frac{1}{360°} \sum_{i=1}^{n} (-1)^i (\alpha_i + \psi_i), \tag{11}$$

where the spin precession angles between snakes, ψ_i, depend on energy, while the snake precession angles α_i do not. The alternating signs in Eq. (11) account for the fact that the spin coordinate system is flipped in every snake, and the direction of the spin precession is reversed. To make ν independent of energy, the places for the snakes must be chosen such that

$$\sum_i (-1)^i \psi_i = 0,$$

and then the spin tune is the alternating sum of the snake precession angles:

$$\nu = \sum_i (-1)^i \alpha_i. \tag{12}$$

The next section will show how the positions and precession angles of the snakes must be chosen for strong spin matching.

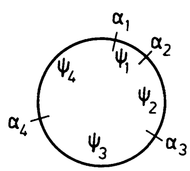


FIGURE 2 Sketch of a general ring with $2M$ snakes; here $2M = 4$.

2.3. *Strong Spin Matching for Vertical Betatron Oscillations in a Periodic Cell Lattice Equipped with Snakes*

Since we are first of all interested in the periodic lattice that makes up most of the ring, we will here ignore any insertions and assume a "circular" ring made of a periodic lattice, into which, between cells, M identical pairs of snakes are inserted, with constant intervals between individual snakes. The ring is thus broken down into M superperiods which each contain a pair of snakes.

Denoting by n the number of cells per half superperiod, i.e., between snakes, and by α_1 and α_2 the spin precession angles of the snakes at the end of the first and the second half superperiod, the spin-orbit coupling integral over the superperiod (SP) is, for vertical betatron oscillations, given by

$$I_{\mathrm{SP}\pm} = \int_{n\ \mathrm{cells}} e^{i\psi} \cdot k\sqrt{\beta_z} e^{\pm i\psi_z}\, ds + \int_{n\ \mathrm{cells}} e^{i(n\cdot 2\pi\nu_0 - \alpha_1 - \psi)} k\sqrt{\beta_z} e^{\pm i(n\cdot 2\pi Q_{z0} + \psi_z)}\, ds. \quad (13)$$

In this notation, ψ and ψ_z are chosen to be zero at the beginning of each half superperiod. In the first hald superperiod, the spin precession phase increases by $n \cdot 2\pi\nu_0$, and the first snake adds the precession angle $-\alpha_1$; in the second half superperiod, the phase decreases again by $-n \cdot 2\pi\nu_0$, spinning backwards, before the angle α_2 is added by the second snake. Therefore, the energy-dependent part $n \cdot 2\pi\nu_0$ cancels, and the spin precession phase advances by $\alpha_2 - \alpha_1 = 2\pi a$ per superperiod, independently of energy, while the betatron phase advances by $\pm 2n \cdot 2\pi Q_{z0}$. Over a string of m superperiods, the spin-orbit coupling integral may thus be written as

$$I_\pm = \sum_{k=0}^{m-1} e^{k\cdot 2\pi i(a \pm 2nQ_{z0})} \cdot I_{\mathrm{SP}\pm} = \frac{1 - e^{m\cdot 2\pi i(a\pm 2nQ_{z0})}}{1 - e^{2\pi i(a\pm 2nQ_{z0})}} \cdot I_{\mathrm{SP}\pm}. \quad (14)$$

By analogy to Section 1.1, we can now make the string of m superperiods spin-transparent at all energies by choosing for the "snake tune" $a = (\alpha_2 - \alpha_1)/2\pi$ one of those values that make the integral [Eq. (14)] vanish by fulfilling the conditions

$$m(a \pm 2nQ_{z0}) = p, \quad p = 1, 2, \ldots \quad (15a)$$

but

$$a \pm 2nQ_{z0} = \frac{p}{m} \neq q, \quad q = 1, 2, \ldots \quad (15b)$$

For maximum stability of polarization, one will here again apply strong spin matching by choosing a sufficiently small number $2n$ of cells per superperiod, i.e., installing a sufficient number of snake pairs in the ring, and by then making the number m of superperiods in the matched string as small as possible, or at least small, in accordance with Eqs. (15).

The combinations of parameters a, m, n eligible for strong spin matching are given in Table III for a phase advance of $\varphi = 90°$ per cell, and in Tables IV and V for phase advances of 72° and 60°, respectively. Selecting the snake tune

TABLE III

Snake tune $a = (\alpha_2 - \alpha_1)/2\pi$ per superperiod for energy-independent strong spin matching over a string of m superperiods, each composed of $2n$ cells with $\varphi = 90°$, plus one snake pair with snake precession angles α_1, α_2

no. of SP's \ no. of cells	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8	n = 9	n = 10
m = 2	a = ± 1/2	a = 0	a = ± 1/2	a = 0	a = ± 1/2	a = 0	a = ± 1/2	a = 0	a = ± 1/2
m = 3	a = { ± 1/3, ± 2/3	a = { ± 1/6, ± 5/6	a = { ± 1/3, ± 2/3	a = { ± 1/6, ± 5/6	a = { ± 1/3, ± 2/3	a = { ± 1/6, ± 5/6	a = { ± 1/3, ± 2/3	a = { ± 1/6, ± 5/6	a = { ± 1/3, ± 2/3
m = 4	a = { ± 1/4, ± 1/2, ± 3/4	a = { 0, ± 1/4, ± 3/4	a = { ± 1/4, ± 1/2, ± 3/4	a = { 0, ± 1/4, ± 3/4	a = { ± 1/4, ± 1/2, ± 3/4	a = { 0, ± 1/4, ± 3/4	a = { ± 1/4, ± 1/2, ± 3/4	a = { 0, ± 1/4, ± 3/4	a = { ± 1/4, ± 1/2, ± 3/4
m = 5	a = ± k/5, k = 1,2,3,4	a = { ± 1/10, ± 3/10	a = ± k/5, k = 1,2,3,4	a = { ± 1/10, ± 3/10	a = ± k/5, k = 1,2,3,4	a = { ± 1/10, ± 3/10	a = ± k/5, k = 1,2,3,4	a = { ± 1/10, ± 3/10	a = ± k/5, k = 1,2,3,4
m = 6	a = ± k/6, k = 1,...,5	a = { 0, ± k/6, k = 1,2,4,5	a = ± k/6, k = 1,...,5	a = { 0, ± k/6, k = 1,2,4,5	a = ± k/6, k = 1,...,5	a = { 0, ± k/6, k = 1,2,4,5	a = ± k/6, k = 1,...,5	a = { 0, ± k/6, k = 1,2,4,5	a = ± k/6, k = 1,...,5

TABLE IV

Snake tune $a = (\alpha_2 - \alpha_1)/2\pi$ per superperiod for energy-independent strong spin matching over a string of m superperiods, each composed of $2n$ cells with $\varphi = 72°$, plus one snake pair with snake precession angles α_1, α_2

no. of SP's \ no. of cells	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8	n = 9	n = 10
m = 2	—	—	—	a = ± 1/2	—	—	—	—	a = ± 1/2
m = 3	—	—	—	a = ± 1/3	—	—	—	—	a = ± 1/3
m = 4	—	—	—	a = { ± 1/4, ± 1/2, ± 3/4	—	—	—	—	a = { ± 1/4, ± 1/2, ± 3/4
m = 5	a = { 0, ± 2/5	a = { 0, ± 2/5	a = { 0, ± 1/5	a = ± k/5, k = 1,2,3,4	a = { 0, ± 1/5	a = { 0, ± 2/5	a = { 0, ± 2/5	a = { 0, ± 1/5	a = ± k/5, k = 1,2,3,4
m = 6	—	—	—	a = ± k/6, k = 1,...,5	—	—	—	—	a = ± k/6, k = 1,...,5

TABLE V

Snake tune $a = (\alpha_2 - \alpha_1)/2\pi$ per superperiod for energy-independent strong spin matching over a string of m superperiods, each composed of $2n$ cells with $\varphi = 60°$, plus one snake pair with snake precession angles α_1, α_2

no. of SP's \ no. of cells	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8	n = 9	n = 10
m = 2	—	a = ±1/2	—	—	a = ±1/2	—	—	a = ±1/2	—
m = 3	a = 0	a = ±1/3	a = 0	a = 0	a = ±1/3	a = 0	a = 0	a = ±1/3	a = 0
m = 4	—	a = {±1/4, ±1/2, ±3/4}	—	—	a = {±1/4, ±1/2, ±3/4}	—	—	a = {±1/4, ±1/2, ±3/4}	—
m = 5	—	a = ±k/5, k = 1,2,3,4	—	—	a = ±k/5, k = 1,2,3,4	—	—	a = ±k/5, k = 1,2,3,4	—
m = 6	a = {0, ±k/6,} k = 1,3,5	a = ±k/6, k = 1,...,5	a = {0, ±k/6,} k = 1,3,5	a = {0, ±k/6,} k = 1,3,5	a = ±k/6, k = 1,...,5	a = {0, ±k/6,} k = 1,3,5	a = {0, ±k/6,} k = 1,3,5	a = ±k/6, k = 1,...,5	a = {0, ±k/6,} k = 1,3,5

$a = (\alpha_2 - \alpha_1)/2\pi$ means, in practice, that the two types of snakes in the pair will be so designed that the difference of their intrinsic precession angles will assume a certain value. Since only the difference matters, one of the individual values may be freely chosen to get for example, a compact design and maximum stability of snake performance, as shown in Section 3. To ensure that the matching conditions [Eqs. (15)] are satisfied at all energies, it may be necessary to regulate or program the spin tune, since Q_{z0} generally wobbles a little during acceleration.[29]

After the polarized beam has been accelerated in the ring so equipped with snakes, the ultimate strong spin matching will then be achieved by choosing a particular working energy such that each half superperiod becomes spin-transparent in itself and the integrals over n cells in Eq. (13) will vanish. This means applying an additional, energy-dependent strong spin matching to the half superperiod by making the local spin tune ν_0 per cell satisfy the condition given in Table I for the chosen number n of cells, as explained in Section 1.1.

In the idealized "circular" ring considered so far, strong spin matching will shift the overall spin and betatron tunes to integral or half-integral values, which is of course not acceptable. However, in a large real ring, there will be insertions with matched special lattices that have less or no bending, and these can be used to adjust the overall spin and betatron tunes to be away from resonances. At the same time, these insertions must be equipped with snakes and thereby made spin-transparent, independent of energy. Depending on the insertion lattice, this may be a more complex problem that will not be treated in this paper and calls for further investigation.

Again, in the general ring with snakes, the value of the spin-orbit coupling integral $I_{\pm}(\sigma)$ can be determined in a step-by-step integration over the sections between snakes, as in Eq. (13), and then Eq. (6) permits one to assign an inverse quality factor q for comparing the benefit of different ring designs.

3. STRONG SPIN MATCHING WITH SNAKES IN THE SSC

3.1. Missing-Magnet Scheme and Insertion of Snakes in the Periodic Cell Lattice

The $2 \times 20\,\mathrm{TeV}$ proton storage ring projected in the U.S., the Superconducting Super Collider (SSC), has two (almost-) half rings, each composed of 143 FODO cells with a betatron phase advance of $\varphi = 90°$.[30] To make room for the snakes, it is suggested to leave out a small fraction of the bending magnets, in a regular pattern so chosen that the spurious dispersion generated will periodically cancel and thus remain small.[31,32a,33] Eligible missing-magnet patterns are shown schematically in Fig. 3 for phase advances of $\varphi = 90°$, 72°, and 60° per cell. In the SSC, it would be a good choice to leave out half a dipole magnet in every sixth cell. Since there are 12 dipole magnets per cell, this would amount to decreasing the average bending strength by 0.69%, and the remaining dipoles would have to be made 0.69% stronger for the same energy. The average tunnel radius could remain unchanged, but slight lateral magnet readjustments of up to ±2.5 cm

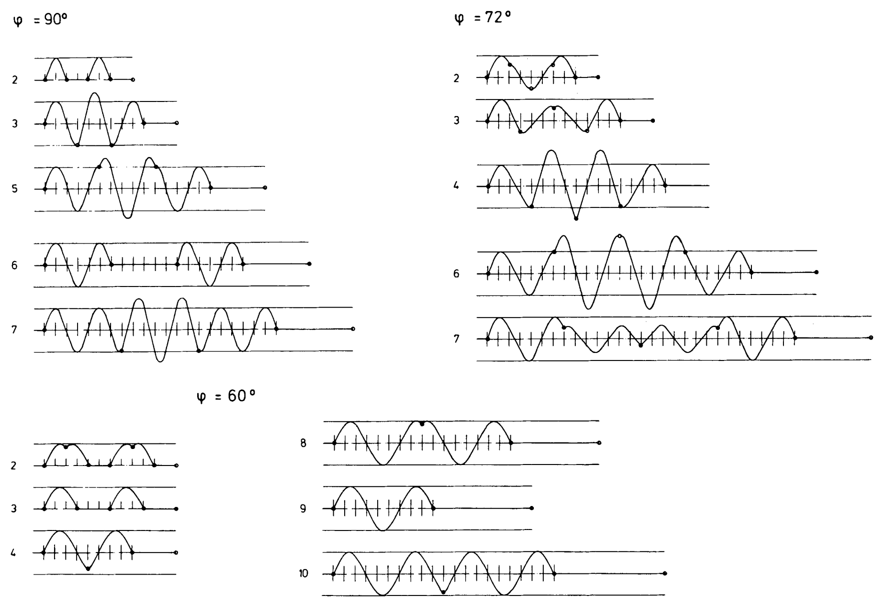


FIGURE 3 Spurious dispersion in regular missing magnet patterns, with φ = betatron phase advance per cell; 1 bin = 1 cell.

would be required. The dipole length is 16.54 m, and snakes can be built at half that length, as shown in the next chapter.

A total of about 48 snakes at a spacing of 1.37 km would then be installed in the arcs of the SSC, with $n = 6$ cells between them. The snakes would alternate between two types that have snake precession angles α_1 and α_2, respectively. An inspection of Table III tells that, with $n = 6$, strong spin matching is achieved over $m = 2$ superperiods, i.e., over 24 cells, if the snakes are designed to have precession angles with $\alpha_2 - \alpha_1 = 180°$. Practicable examples of snake pairs which satisfy this requirement are given in the next section.

Consulting Table I, we find those particular working energies at which, after acceleration, the ultimate strong spin matching is achieved, where not only pairs of superperiods are spin-transparent, but even every half superperiod by itself ($n = 6$ cells). With a deflection angle of $\Delta = 0.01967681$ rad per cell we have, with Eq. (10),

$$\nu_0 = \frac{p}{2} \pm \frac{1}{12} = \frac{E[\text{GeV}]}{0.52335} \cdot \frac{\Delta}{2\pi}, \quad p = 1, 2, \ldots,$$

and thus the particular energies

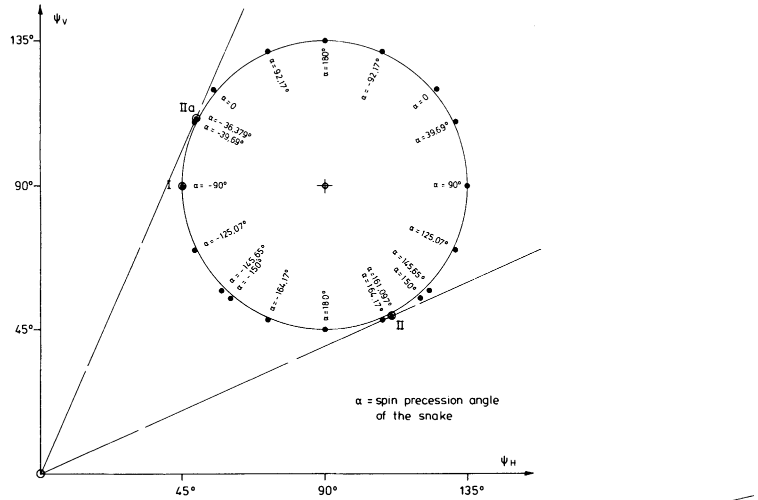
$$E_{\text{particular}} = 83.558(p \pm \tfrac{1}{6})\text{GeV}, \quad p = 1, 2, \ldots.$$

3.2. Snake Design

As an array of horizontal and vertical bends in an alternating sequence, the snake must rotate the spin by a given amount in each magnet, independently of energy. The magnetic fields, therefore, must stay constant during acceleration, and the snake, being part of a fixed ring geometry, must be straight. It will thus consist of matched horizontal and vertical beam bumps which are folded into each other and can be characterized by their initial horizontal and vertical spin rotation angles ψ_H and ψ_V. Several such topologies have been investigated, looking for small beam bump amplitudes, and it was discovered that one of them, for certain combinations (ψ_H, ψ_V), yields a continuous family of snakes with a continuously varying snake precession angle α.[17]

In this family of snakes, the magnet sequence is $(-H, -V, 2H, 2V, -2H, -V, H)$ and the locus of points (ψ_H, ψ_V) where the array acts as a snake is shown in Fig. 4. It looks almost like a circle and thus lends itself to an obvious clock notation. The snake precession angle α is indicated for a number of points; its sign can be inverted by inverting the sign of all horizontal bends. We have $\alpha = \pm 90°$ at the 9.00 h point, for example, and $\alpha = 180°$ at the 6.00 h point.

Since, in our SSC example, a snake pair with $\alpha_2 - \alpha_1 = 180°$ is required, one simple solution will be to employ everywhere the 9.00 h snake, with alternating sign of α. But it appears from Fig. 4 that there probably is an even better choice. At the, say, 10.00 h and 5.00 h points where the tangential lines from the origin touch the locus of snakes, the corresponding snakes have a singular and valuable property: They do, in linear approximation, maintain their snake action, i.e., give



				IIa		I							II								
ψ_V	135°	131,97°	120°	111°	110°	90°	70°	57,235°	54,736°	48,03°	45°	48,03°	49,237°	54,736°	57,235°	70°	90°	110°	120°	131,97°	135°
ψ_H	90°	72°	54,736°	49,237°	48,808°	45°	48,808°	57,235°	60°	72°	90°	108°	111°	120°	122,765°	131,192°	135°	131,192°	125,264°	108°	90°
α	180°	92,17°	0	-36,379°	-39,69°	-90°	-125,07°	-145,65°	-150°	-164,17°	180°	164,17°	161,097°	150°	145,65°	125,07°	90°	39,69°	0	-92,17°	180°

FIGURE 4 Vertical and horizontal spin rotation angles ψ_V and ψ_H in the novel-type Siberian snake. Versions I and IIa are the fixed snake designs recommended for SSC in this paper. Version II is the origin of the "unsymmetric" Version IIb that is described in the text and in Fig. 5. The straight lines indicate why versions II are less sensitive than version I to proportional variations of ψ_V and ψ_H.

an exact spin flip, even if the angles ψ_H, ψ_V deviate from the design value by an equal proportion. The precision required of the (common) magnet power supply will then be greatly reduced. In turn, if a very stable power supply is employed, it can be used for fine adjustment of the snake tune. The difference of snake precession angles between the two singular points, however, is close but not equal to 180°. But, by introducing in the 5.00 h snake slightly unsymmetric slopes in the horizontal beam bump, the difference $\alpha_2 - \alpha_1$ can be adjusted to be 180° exactly, with the added advantage of making the snake shorter. The result is shown in Fig. 5, where I is the 9.00 h snake pair with $\alpha \pm 90°$, IIa the 10.00 h member of the singular snake pair, and IIb the second member, the unsymmetric version of the 5.00 h snake. The beam excursions shown in Fig. 5 correspond to the injection energy of 1 TeV; they stay below 1.8 mm.

3.3. *A Note on Terrain-Following*

The often-held belief that polarization cannot survive in a terrain-following machine is not correct. It is possible to design vertical bends with a spin rotation equal to the rotation of beam direction such that, in the orbit-following coordinate system, the spin direction stays unchanged at all energies. Such a "spin-trailing" hinge is shown in Fig. 6. It is based on a proposal by J. Buon[32b] which has here been modified[33] in order to become, in linear approximation, independent of common errors in horizontal bending strength. For an error of

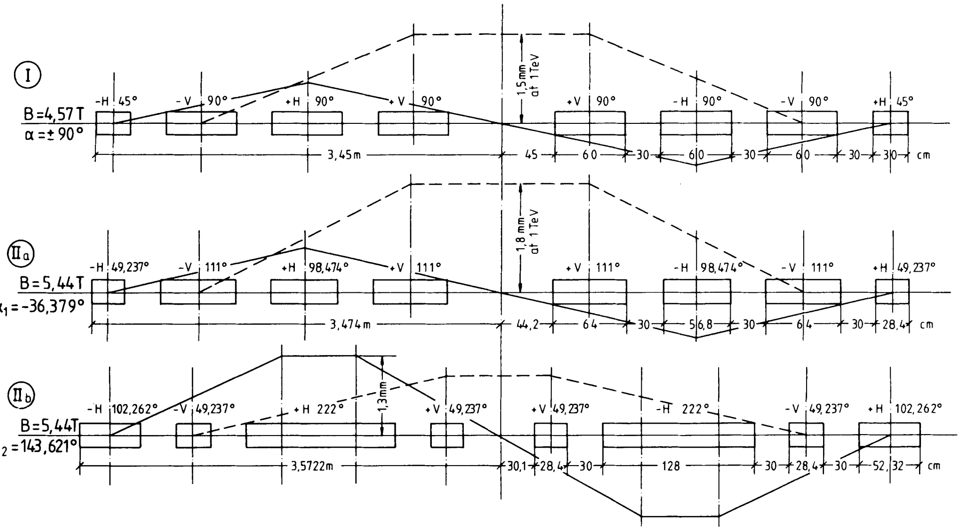


FIGURE 5 Snakes for SSC: (I) Left/right-pointed snake with precession angles $\alpha = \pm 90°$; (IIa,b) Snake pair with $\alpha_2 - \alpha_1 = 180°$ and singular insensitivity to current variations.

-1% in the strength of the horizontal magnets, the angular offset of the outgoing spin from the orthogonal direction to the beam will be less than 20 μrad (!) for all vertical spin rotation angles. This is shown in Fig. 7, which also shows that, at the same time, a small error in spin precession will result (less than 13 mrad).

Such vertical hinges can be inserted into the ring to make it follow the slopes of a shallow valley, for example, without distorting the spin motion.

3.4. Concluding Remark

Based on the foundations laid in this paper, and on the proposed snake scheme, the author believes that, at little expense, polarization in the SSC can be made very stable during acceleration, and even more so at the stationary working energy. Ways to incorporate snakes in the insertions, however, have not been devised and need further work.

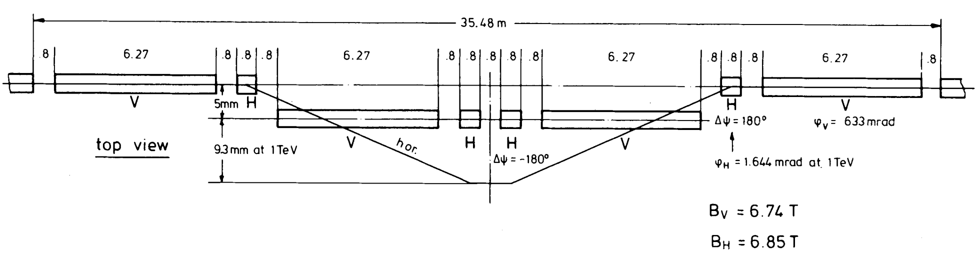


FIGURE 6 Spin-trailing vertical hinge replacing about 2 standard SSC dipoles; maximum vertical deflection: $4\varphi_V = 2.53$ mrad.

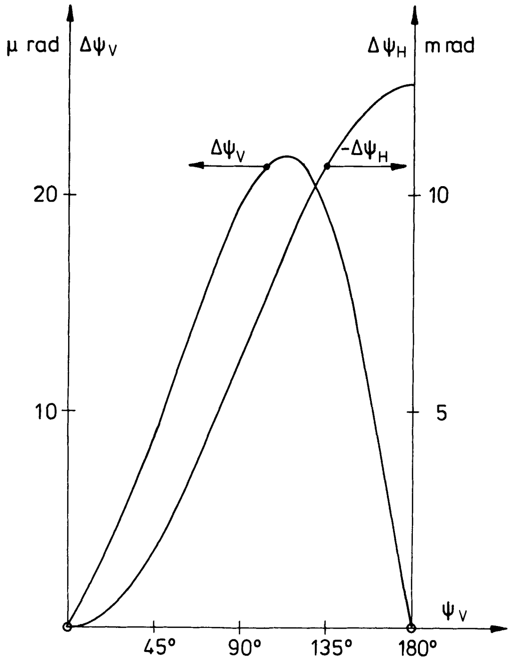


FIGURE 7 Spin-trailing error of vertical hinge, as a function of vertical spin rotation per magnet, for a common error of -1% in the strengths of the four horizontal bends. $\Delta\psi_V$ is the overall error in vertical spin rotation (in μrad), and $\Delta\psi_H$ the overall error in spin precession (in mrad).

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