Particle Accelerators, 1986, Vol. 20, pp. 133–141
0031-2460/86/2002-0133$15.00/0

THE ROOT-MEAN-SQUARE EMITTANCE OF AN AXISYMMETRIC BEAM WITH A MAXWELLIAN VELOCITY DISTRIBUTION

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(Received June 17, 1985; in final form January 3, 1986)

A method of analyzing the rms emittance for an axisymmetric and nonrotating beam with a Maxwellian velocity distribution is described. The resulting emittance is expressed in terms of equations for the beam parameters, which may be easily constructed from the experimental measurements. It is also shown that for beam parameters of simple form, the emittance can also be calculated analytically.

1. INTRODUCTION

The root-mean-square (rms) emittance, proposed by Lapostolle[1] and Sacherer,[2] has been widely used as an alternative means of defining beam quality.[3,4] If a beam has no sharp envelope, as is often the case in practical situations, the rms emittance provides a useful means of describing the rms envelope of the beam, whereas use of the emittance (the area in two-dimensional trace space) is ambiguous, since it cannot be clearly defined. In practice, the rms emittance also accounts for the effective increase due to the filamentation phenomenon in the presence of nonlinear systems, in contrast to the emittance, which is invariant.[5]

The rapid development of new pulsed power technology in the past two decades has led to the production of high-current electron beams and ion beams for a variety of applications.[6] Some applications, such as heavy-ion inertial confinement fusion[7] and free-electron lasers,[8] require the use of high-quality beams. The evaluation of rms emittance, however, often requires an elaborate effort of data taking and analysis. It is thus imperative to find a simpler method of analyzing the rms emittance of such beams, which are mostly of a single-pulse nature.

In this paper, we describe a method of finding the rms emittance from experimental data obtained by using a simple emittance meter for an axisymmetric and nonrotating beam with a Maxwellian transverse velocity distribution. In the following sections, the definition of rms emittance is given, and a few practical assumptions are imposed on the distribution function, so as to utilize results of simple emittance meters. Subsequently, a method of analyzing data obtained by a simple emittance meter is described in detail. The resultant rms

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emittance is expressed in terms of integrals of empirical equations deduced from the experimental data. Also demonstrated are simple examples in which the rms emittance can be calculated analytically.

2. DEFINITION AND ASSUMPTION

The rms emittance is defined as[1,2]

$$\varepsilon_{\text{rms}} = 4(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2)^{1/2}, \tag{1}$$

where x' and y' denote the gradients of the particle trajectories, given by $x' = dx/dz$ and $y' = dy/dz$. The coordinates (x, y, x', y') in the "trace space"[9] are used throughout. The brackets ($\langle\ \rangle$) denote values averaged over the four-dimensional trace space, defined as

$$\langle \phi \rangle = \frac{1}{N} \int \phi \rho_4(x, y, x', y')\, dx\, dy\, dx'\, dy', \tag{2}$$

where N is the total number of particles, given by $N = \int \rho_4\, dx\, dy\, dx'\, dy'$, and ρ_4 is the density distribution in four-dimensional trace space, also known as the microscopic brightness. Since the quantity of interest, ϕ, in Eq. (1) is a second moment of $x - x'$ coordinates only, Eq. (2) may be reduced to a more convenient form:

$$\langle \phi \rangle = \frac{1}{N} \int \phi \rho_2(x, x')\, dx\, dx', \tag{3}$$

where $\rho_2(x, x') = \int \rho_4\, dy\, dy'$, the projected density on two-dimensional trace space. It is necessary for rms emittance evaluation that either ρ_4 or ρ_2 be determined by experimental measurement. As will be described in the next section, both can be directly measured by certain types of emittance meters. However, such emittance meters may not be practical; in particular, for beams of the single-pulse type. In order to utilize a simple emittance meter which can complete the measurement in a single pulse, it appears inevitable to make some assumptions regarding the form of the distribution ρ_4, such that ρ_4 can be uniquely determined from the experimental measurements.

We assume a Maxwellian distribution of ρ_4 in $x' - y'$ space. This may be a practical assumption,[3] since the Maxwellian distribution is that of a thermal equilibrium state. Actual beams, however, are not exactly in a thermal equilibrium state; nevertheless, they have a similar distribution, which inherently originates from their sources, such as thermionic cathodes and plasma ion sources. With this assumption, the distribution is then written as

$$\rho_4(x, y, x', y') = g(x, y) \exp\left[-(X'^2 + Y'^2)/2\sigma^2\right], \tag{4}$$

where the components of the random angle (velocity) are given by $X' = x' - \bar{x}'$ and $Y' = y' - \bar{y}'$, with $\bar{x}'$ and $\bar{y}'$ being mean values of angles (velocities) over $x' - y'$ space due to diverging or converging of the beam; i.e., $\bar{x}'(x, y) = n^{-1} \int x' \rho_4\, dx'\, dy'$ and $\bar{y}'(x, y) = n^{-1} \int y' \rho_4\, dx'\, dy'$, where $n(x, y) = \int \rho_4\, dx'\, dy'$.

In Eq. (4) σ is the rms value of angle (velocity) over $x' - y'$ space, $\sigma(x, y) = [n^{-1} \int (X'^2 + Y'^2)\rho_4 \, dx' \, dy']^{1/2}$, and can be related to an equivalent thermal temperature as $\sigma^2 = kT/(m_0\beta^2c^2)$. We also assume that the beam is axisymmetric and nonrotational, i.e., $g(x, y)$, $\alpha(x, y) = (\bar{x}'^2 + \bar{y}'^2)^{1/2}$, and $\sigma(x, y)$ are functions of $r = (x^2 + y^2)^{1/2}$ only, and the mean angle α has only the radial component, i.e., $\bar{x}'/\bar{y}' = x/y$, or $\bar{x}' = \alpha x/r$ and $\bar{y}' = \alpha y/r$. With these assumptions, the four-dimensional trace-space distribution function can be rewritten as

$$\rho_4(x, y, x', y') = g(r) \exp(-V^2/2\sigma^2), \tag{5}$$

where $V^2 = (x' - \alpha x/r)^2 + (y' - \alpha y/r)^2$.

3. EMITTANCE METERS

Although a number of different principles are employed in various emittance meters, the devices commonly used generally consist of two aperture plates and a detector (see Fig. 1). The upstream and downstream plates may have one or more holes or slits. The detector, placed downstream of both plates, measures the fraction of beam passed through both apertures. The emittance meters may be classified into four categories, named after the combination of the aperture types, as two-slit, two-hole, hole-slit, and slit-hole meters. Suppose a hole is placed on the upstream or downstream plates. Then each hole defines a pair of coordinates (x, y) or (ξ, η) on the upstream or downstream plate, respectively. The upstream aperture defines the spatial coordinates of a beamlet formed through it and allows an angular dispersion due to the transverse velocity distribution of the beam as it moves towards the downstream plane. The downstream aperture then analyzes the angular (velocity) distribution by further sampling the beam. One can relate these coordinates to that of the four-dimensional trace space by

$$\begin{aligned} x' &= (\xi - x)/L, \\ y' &= (\eta - y)/L, \end{aligned} \tag{6}$$

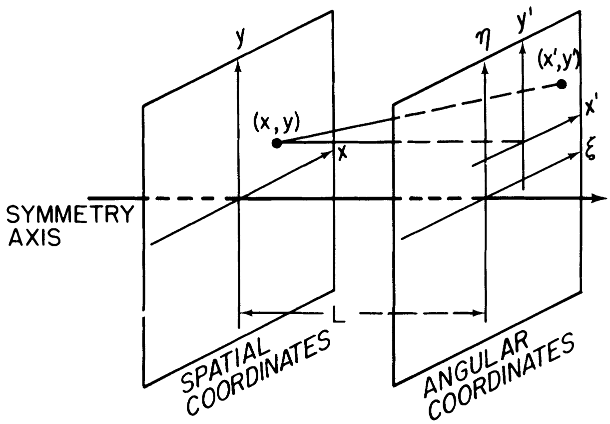

FIGURE 1 Trace-space coordinates in relation to a typical emittance meter.

where L is the distance between the upstream and downstream plates. With these relations it can be readily shown[4,10] that the intensity distribution measured by three methods (two-hole, two-slit, and hole-slit) can be expressed in terms of $\rho_4(x, y, x', y')$ or its integral. The two-hole method directly measures $\rho_4(x, y, x', y')$, the two-slit method measures $\rho_2(x, x') = \int \rho_4 \, dy \, dy'$ [or $\rho_2(y, y')$], and the hole-slit method can measure $\rho_3(x, y, x') = \int \rho_4 \, dy'$. Here, we have made use of the fact that a slit provides an effective integral of intensity distribution in the direction of the slit ($\int dy$ or $\int dy'$). In this analysis, we assume an ideal hole or slit whose size is infinitesimally small. In practice, however, one can choose any finite size, so long as the intensity distribution measured is proportional to the slit width or hole area, i.e., $I_4 = \rho_4 \, \Delta x \, \Delta y \, \Delta x' \, \Delta y'$, where I_4 is the measured intensity distribution, and Δx, Δy, $\Delta x'$, and $\Delta y'$ refer to the size of aperture. Since the emittance depends only on the form of distribution but not on the magnitude of the distribution, one can use any relative intensity distribution I for the ρ in this analysis without knowing the magnitude.

It is emphasized here that the slit-hole method can be utilized conveniently for the single-pulse beam,[11] although the measured intensity distribution appears not to be related in a straightforward way to the distribution ρ_4. Suppose the detector measures particles passing through a slit (parallel to y, say) and a pinhole at $\xi, \eta = 0$. The measured intensity distribution is $\int \rho_4(x, y, x', y') \, dy$. Noting that y and y' are related to each other by Eq. (6) as $y = -Ly'$, since $\eta = 0$, and thus $dy = -L \, dy'$, the measured distribution may be rewritten as $L \int \rho_4(x, Ly', x', y') \, dy'$. If ρ_4 is a slow function of y and is nearly constant within a range $y = \pm L\sigma$, where the σ is the rms value of y', then the measured distribution $L \int \rho_4 \, dy'$ is considered to be approximately the same as that obtained by the hole-slit method, excepting the factor L. This condition can easily be met when the beam has a small diverging angle ($\sigma \ll 1$) or by choosing the distance between the aperture plates very small, such that $(\partial \rho_4 / \partial y) L\sigma \ll 1$.

4. EMITTANCE ANALYSIS

Now we describe a method of analyzing experimental data obtained by the slit-hole type emittance meter. This slit-hole method provides relatively accurate information on emittance. A typical configuration of the system consists of an array of slits and a detector plane as depicted in Fig. 2. The slits at x_i sample the beam and produce sheet beamlets. These beamlets on the detector plane reveal angular distribution. This distribution is scanned along the line of $\eta = 0$ by a small pinhole and a detector (e.g., a Faraday cup, or film subsequently scanned by a microdensitometer), allowing construction of the density distributions $\beta(x_i')$ corresponding to the slit positions x_i, where $x_i' = (\xi_i - x_i)/L$.

Each $\beta(x_i')$ is assumed to be Maxwellian and thus characterized by the peak height β_i, rms width σ_i, and mean angle (velocity) $\bar{x}_i$. From these discrete sets of experimental data, one can construct empirical equations for $\beta(x)$, $\sigma(x)$, and $\bar{x}'(x)$ as functions of x, which smoothly connect (or curve fit) data points β_i, σ_i, and $\bar{x}_i$, respectively, as shown in Fig. 3.

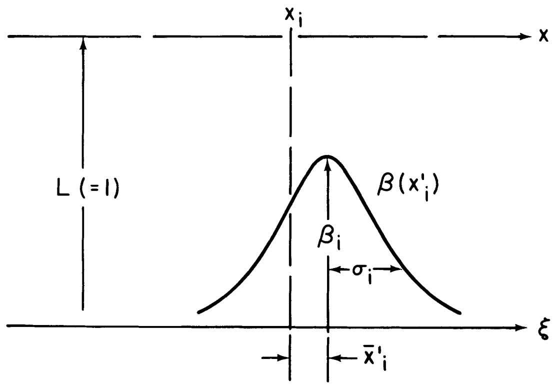

FIGURE 2 Schematic representation of the slit-hole method. Shown is an intensity distribution $\beta(x_i')$ after slit x_i, whose mean angle, peak height, and rms width are $\bar{x}_i'$, β_i, and σ_i, respectively. (For simplicity the distance L between the two plates has been chosen to be unity.)

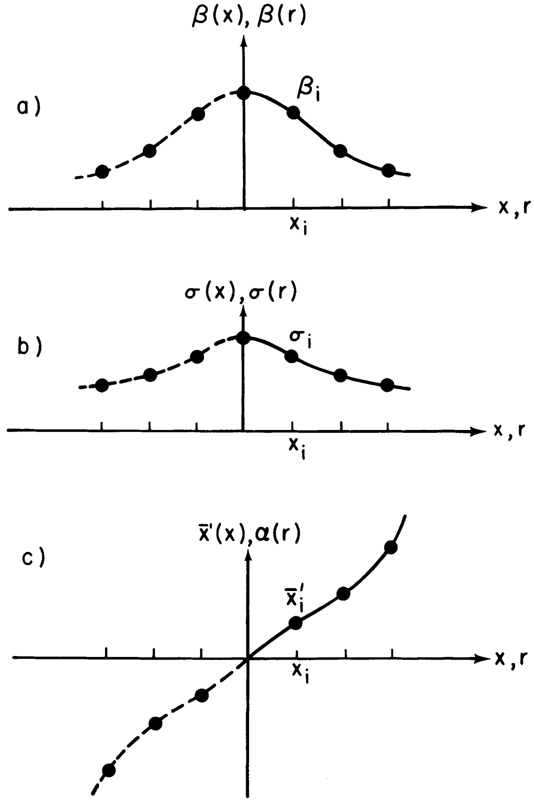

FIGURE 3 Construction of empirical equations of beam parameters: (a) The peak value of an individual distribution as a function of x or r, (b) the rms width as a function of x or r, and (c) the mean (diverging) angle as a function of x or r.

Noting (i) that the peak value of each distribution β_i is the measure of particles of velocity equal to $\bar{x}'$ passed through the ith slit located at x_i and arriving at $\eta = 0$, and (ii) that the slit in the Y direction effectively provides the integral $\int dy'$, we may relate the empirical functions to each other by the four-dimensional distribution function

$$\beta(x) = \int \rho_4(x, y=0, x'=\bar{x}', y')\, dy'. \tag{7}$$

By virtue of the Gaussian form, the integral $\int dy'$ in Eq. (7) is replaced by $\sqrt{2\pi}\,\sigma(x)$, yielding a simple relation,

$$\beta(x) = \sqrt{2\pi}\,\sigma(x) g(x). \tag{8}$$

Thus, the four-dimensional distribution in terms of measured empirical parameters may be written as

$$\rho_4(x, y, x', y') = \frac{\beta(r)}{\sqrt{2\pi}\,\sigma(r)} \exp\{-[(x'-\bar{x}')^2 + (y'-\bar{y}')^2]/2\sigma^2\}, \tag{9}$$

where Eq. (8) has been used with the argument $r = (x^2+y^2)^{1/2}$ in place of x because of axisymmetry. One can easily recognize the spatial density distribution in this case as

$$n(x, y) = \int \rho_4\, dx'\, dy' = \sqrt{2\pi}\,\beta(r)\sigma(r). \tag{10}$$

Finally, the two-dimensional projected distribution on the x–x' trace space is found to be

$$\rho_2(x, x') = \int \beta(r) \exp[-(x' - \alpha x/r)^2/2\sigma^2]\, dy, \tag{11}$$

where $\beta(r)$, $\sigma(r)$, and $\alpha(r)$ are empirical parameters which are functions of $r = (x^2+y^2)^{1/2}$. The isodensity contours of $\rho_2(x, x')$ in x–x' space, known as the emittance plot, provide very useful information about the beam.

For evaluation of rms emittance, three average values of second moments, $\langle x^2 \rangle$, $\langle x'^2 \rangle$, and $\langle xx' \rangle$, are needed. Upon substituting the two-dimensional distribution function given by Eq. (11) in Eq. (3), the averaging integral in x' space is replaced by $\sqrt{2\pi}\,\sigma$ because of the Gaussian distribution, and the expression for $\langle x^2 \rangle$ is obtained as

$$\langle x^2 \rangle = \frac{\sqrt{2\pi}}{N} \int_{-\infty}^{\infty} x^2 \beta(r)\sigma(r)\, dx\, dy. \tag{12}$$

Similarly $\langle x'^2 \rangle$ and $\langle xx' \rangle$ are given by

$$\langle x'^2 \rangle = \frac{\sqrt{2\pi}}{N} \int_{-\infty}^{\infty} \left\{[\sigma(r)]^2 + \left(\frac{\alpha x}{r}\right)^2\right\} \beta(r)\sigma(r)\, dx\, dy, \tag{13}$$

$$\langle xx' \rangle = \frac{\sqrt{2\pi}}{N} \int_{-\infty}^{\infty} \frac{x^2 \alpha(r)\beta(r)\sigma(r)}{r}\, dx\, dy, \tag{14}$$

and the total number of particles N is

$$N = \sqrt{2\pi}\int_{-\infty}^{\infty} \beta(r)\sigma(r)\, dx\, dy. \tag{15}$$

It is of interest to note that Eqs. (11)–(15) are of the Abel type,[12] i.e.,

$$I(x) = 2\int_0^{\infty} R(r)\, dy = 2\int_x^{\infty} \frac{R(r)}{\sqrt{(r^2 - x^2)}}\, r\, dr.$$

Thus, these equations can be rewritten in a more useful form:

$$\rho_2(x, x') = 2\int_x^{\infty} \frac{\beta(r)\exp[-(x' - \alpha x/r)^2/2\sigma^2]}{\sqrt{r^2 - x^2}}\, r\, dr, \tag{16}$$

$$\langle x^2\rangle = \frac{4\sqrt{2\pi}}{N}\int_0^{\infty} x^2\int_x^{\infty} \frac{\beta\sigma r\, dr}{\sqrt{r^2 - x^2}}\, dx, \tag{17}$$

$$\langle x'^2\rangle = \frac{4\sqrt{2\pi}}{N}\int_0^{\infty}\left[\int_x^{\infty} \frac{\beta\sigma^3 r\, dr}{\sqrt{r^2 - x^2}} + x^2\int_x^{\infty} \frac{\alpha^2\beta\sigma\, dr}{r\sqrt{r^2 - x^2}}\right] dx, \tag{18}$$

$$\langle xx'\rangle = \frac{4\sqrt{2\pi}}{N}\int_0^{\infty} x^2\int_x^{\infty} \frac{\alpha\beta\sigma\, dr}{\sqrt{r^2 - x^2}}\, dx, \tag{19}$$

$$N = 4\sqrt{2\pi}\int_0^{\infty}\int_x^{\infty} \frac{\beta\sigma r\, dr}{\sqrt{r^2 - x^2}}\, dx. \tag{20}$$

The beam parameters $\alpha(r)$, $\beta(r)$, and $\sigma(r)$, since they are determined by experimental measurements, could be any functions, including empirical curves. But a simple numerical method may be employed for integration of either Eqs. (11)–(15) or Eqs. (16)–(20) to evaluate the emittance.

It is important to note that if $\alpha(r)$ is proportional to r, i.e., for linear focusing, then Eq. (17) times the second term of Eq. (18) is equal to the square of Eq. (19), regardless of the forms of functions $\beta(r)$ and $\sigma(r)$, thus canceling the beam-diverging effect in Eq. (1). Thus, the rms emittance is independent of linear focusing. When the beam is passing through a linear lens system, the spatial coordinates of beam particles remain unchanged, but the angle (velocity) is changed in proportion to the radial position of each particle. Thus, the rms emittance may be regarded as an invariant of motion through a linear lens system.

It should be mentioned that the analysis in this section is equally applicable to the experimental data obtained by the hole-slit type of emittance meter.

5. ANALYTICAL EXAMPLES

For a certain type of integrand in Eqs. (16)–(20), the integration can be performed analytically. We take three simple examples which are of practical importance. First, suppose the following beam parameter equations: $\alpha(r) = r/a$, $\beta(r) = \beta_0 H(b^2 - r^2)$, and $\sigma(r) = \sigma_0$, where H is a Heaviside unit step function.

This set of functions may approximately describe a beam which is emitted by a uniform hot cathode, measured immediately downstream of an anode with a hole of radius b. One can easily obtain $\rho_2(x, x')$ from Eq. (16) as

$$\rho_2(x, x') = 2\beta_0\sqrt{b^2 - x^2}\exp[-(x' - x/a)^2/2\sigma_0^2]. \tag{21}$$

It is also straightforward to find the integrals in Eqs. (17)–(20), obtaining as mean values $N = \sqrt{2}\,\pi^{3/2}\beta\sigma_0 b^2$, $\langle x^2\rangle = b^2/4$, $\langle x'^2\rangle = \sigma^2 + (b/2a)^2$, and $\langle xx'\rangle = b^2/4a$. The rms emittance is then given by

$$\varepsilon_{\rm rms} = 2b\sigma_0. \tag{22}$$

As mentioned earlier, this emittance is independent of $\alpha(r) = r/a$. The same result can be expressed in terms of the thermal temperature of the beam[3] as $\varepsilon_{\rm rms} = 2b(kT/m_0\beta^2c^2)^{1/2}$, which is evidently true, even if the beam is focused (as long as the focusing is of linear form).

The second example is for a beam with a Gaussian spatial distribution. The beam parameters are given by $\alpha(r) = r/a$, $\beta(r) = \beta_0\exp(-r^2/2b^2)$, and $\sigma(r) = \sigma_0$. This case may approximately represent the same beam as the previous example, but measured farther downstream. We obtain

$$\rho_2(x, x') = \sqrt{2\pi}\,\beta_0\sigma_0\exp[-x^2/2b^2 - (x' - x/a)^2/2\sigma_0^2]. \tag{23}$$

The mean values given by Eqs. (17)–(20) are easily found as $N = 8\sqrt{2\pi}\,\pi\beta_0\sigma_0 b$, $\langle x^2\rangle = b^2$, $\langle x'^2\rangle = \sigma_0^2 + (b/a)^2$, and $\langle xx'\rangle = b^2/a$. The rms emittance from Eq. (1) is

$$\varepsilon_{\rm rms} = 4b\sigma_0. \tag{24}$$

Notice that this result is also independent of r/a.

Our third example is a beam with a linear focusing, a Gaussian spatial profile, and a Gaussian spatial dependence of rms angle (transverse velocity). This beam is described by the parameter equations $\alpha(r) = r/a$, $\beta(r) = \beta_0\exp(-r^2/2b^2)$, and $\sigma = \sigma_0\exp(-r^2/2c^2)$. If we make the substitutions $1/p^2 = 1/b^2 + 1/c^2$ and $1/q^2 = 1/b^2 + 3/c^2$, then Eqs. (17)–(20) may be easily solved to give $\langle x^2\rangle = p^2$, $\langle x'^2\rangle = (\sigma_0 q/p)^2 + (p/a)^2$, and $\langle xx'\rangle = p^2/a$. Hence,

$$\varepsilon_{\rm rms} = 4q\sigma_0. \tag{25}$$

It can easily be seen that in the limit $c \to \infty$ this is identical to the previous example.

In all examples, $\langle x'^2\rangle$ increases, since the transverse velocity increases due to the linear focusing $\alpha(r) = r/a$. However, this velocity is not a random one; it is linearly correlated with x. As defined by Eq. (1), this effect is reduced by the linear correlation term $\langle xx'\rangle$, resulting in the rms emittance being independent of the linear focusing, as mentioned earlier.

6. CONCLUSION

For an axisymmetric and nonrotating beam with a Maxwellian velocity distribution, we have described a method of determining the rms emittance from experimental data which can be easily obtained by using simple emittance meters. The resultant emittance and two-dimensional projected distribution in x–x' trace space are expressed in terms of empirical equations of beam parameters. We have also demonstrated that for simple types of beam parameter equations, the rms emittance can be calculated analytically.

ACKNOWLEDGMENTS

The authors are grateful to W. Namkung for useful discussions. This work was supported by the Air Force Office of Scientific Research and the U.S. Department of Energy.

REFERENCES

1. P. M. Lapostolle, *IEEE Trans. Nucl. Sci.* **NS-18,** 1101 (1971).
2. F. J. Sacherer, *IEEE Trans. Nucl. Sci.* **NS-18,** 1105 (1971).
3. J. D. Lawson, The Physics of Charged Particle Beams (Clarendon Press, Oxford, 1977), 172–215.
4. C. Lejeune and J. Aubert, in Applied Charged Particle Optics, A. Septier, Ed. (Academic Press, New York, 1980), pp. 159–259.
5. J. D. Lawson, P. M. Lapostolle, and R. C. Gluckstern, *Particle Accelerators* **5,** 61 (1973).
6. J. Nation, *Particle Accelerators* **10,** 1 (1979).
7. W. Chupp, A. Faltens, E. Hartwig, E. Hoyer, D. Keefe, C. Kim, M. Lampel, E. Lofgren, R. Nemetz, S. S. Rosenblum, J. Shiloh, M. Tiefenback, D. Vanecek, and W. Herrmannsfeldt, *IEEE Trans. Nucl. Sci.* **NS-28,** 3389 (1981).
8. P. A. Sprangle, R. A. Smith, and V. L. Granatstein, in Infrared and Submillimeter Waves, K. Button, Ed. (Academic Press, New York, 1980), Vol. 1, p. 279 and references.
9. J. Fronteau, in Focusing of Charged Particles, A. Septier, Ed. (Academic Press, New York, 1967), Vol. 2, p. 421.
10. A. Van Steenbergen, *Nucl. Instrum. Methods* **51,** 245 (1967).
11. M. J. Rhee, G. T. Zorn, R. C. Placious, and J. H. Sparrow, *IEEE Trans. Nucl. Sci.* **NS-18,** 468 (1971).
12. W. Lochte-Holtgreven, in Plasma Diagnostics, W. Lochte-Holtgreven, Ed. (American Elsevier, New York, 1968), p. 184.