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UNCERTAINTIES AND INSTABILITIES IN CELESTIAL MECHANICS

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Uncertainty and instability are closely associated properties of the dynamical systems of celestial mechanics. The short time predictions of celestial mechanics concerning the stellar universe when compared to the long-lived atomic universe seem to be simple problems according to our old-fashioned ideas. In fact, modern celestial mechanics is known to be nondeterministic due to uncertainties in modeling, due to unsatisfactory information concerning initial conditions, and because of the global nonintegrable nature of the equations. Instabilities, inherently embedded even in our simplest systems, make determinsitic predictions into unrealistic expectations.

I. INTRODUCTION

Computations in celestial mechanics may be compared to those in accelerators when the pertinent numbers of revolutions are evaluated. The earth makes one revolution around the sun in one year. Our predictions concerning the orbits of the planets are considered at present reliable for 10^6 years, and the age of the system is estimated to be approximately 10^9 years. In other words, our computational competence in celestial mechanics is limited by 10^6 revolutions. At the other extreme are the hydrogen atoms in the ground state with 3×10^{23} revolutions per year and the particles in accelerators with 6×10^{10} revolutions per year. Comparing the motion in accelerators with the solar system where 10^6 seems to be the computational limit, one arrives at a corresponding limit of approximately 8 minutes. It is even more interesting to observe that in an accelerator the solar system's lifetime corresponds to 6 days. The results are that our present prediction competence in celestial mechanics corresponds to 8 minutes' running time of an accelerator and that 6 days in the accelerator contain the existence of our solar system. These are certainly rather humbling discoveries for the students of Newton, Lagrange, and Poincaré.

Nevertheless, both analytical and numerical, and both quantitative and qualitative, techniques of celestial mechanics should provide a useful background for orbit mechanics in accelerators.

In this paper, two fundamental concepts of celestial mechanics are described: instabilities and uncertainties. The first subject is one of the most difficult problem in dynamics, and the following example will emphasize its complexity. The second subject is well-known in quantum mechanics, and the treatment here will show how formal analogies between celestial mechanics and accelerator physics might be conjectured.

II. INSTABILITY

The conventional instability associated with changes of initial conditions using Lyapunov's isochronous stability criterion is well established for our simplest dynamical system known as the problem of two bodies. This problem is selected since it is integrable, and it is the most fundamental motion in celestial mechanics. Note that we do not have *closed-form* analytical solution as a function of the time, even for this simple problem. Furthermore, in general, we do not have "exponential instability." Nevertheless, many of the problems in stability analysis will surface, and arbitrary small initial displacements can lead to large deviations.

Consider a planetary reference orbit around a star, neglecting all perturbations. If the mass of the planet is negligible compared to the mass of the central body, the problem is known as the restricted problem of two bodies. The reference orbit may be elliptic or circular with a semimajor axis or radius **a** which is connected with the period T according to Kepler's law:

$$a = KT^{2/3},$$

where $K = (MG/4\pi^2)^{1/3}$, with M the central mass and G the constant of gravity.

The classical definition of stability requires that at any given time the distance between the reference and the disturbed orbit be limited to a prescribed value.[1,2] If to such a prescribed value the change of initial condition can be established, then the motion is considered stable. In other words, if for given

$$|x(t) - y(t)| < \varepsilon$$

there exists

$$|x(t_0) - y(t_0)| \leq \delta,$$

then the motion is stable. Here x and y represent the disturbed and the original position vectors.

In the case of the problem of two bodies, if the disturbed orbit's semimajor axis is slightly different from the original value, then the periods of the two orbits will also be slightly different. As a consequence, the positions on the two orbits will differ, initially, by a small amount, but eventually by $2a$. Consequently, arbitrary small ε cannot be required since no δ value can be found to satisfy such an ε. The motion is unstable concerning isochronous correspondence. (Note that the investigation of the geometric or normal correspondence associated with Poincaré leads to stability).

The above example leads to the following remarks:

(1) Even the simplest problem in celestial mechanics leads to large isochronous deviations when initial conditions are changed by an arbitrary small amount.

(2) Results of stability investigations are strongly dependent on the definition used.

(3) Changes of the initial conditions might create instability, or might not, depending on the disturbance. In the above example, the semimajor axis will not change for any isoenergetic disturbance; consequently, the motion will be stable for such disturbances.

(4) Without any change of the initial conditions, instability might occur when the values of the constants entering the system change. If the mass or the gravitational constant changes, then for the same value of **a**, the period will change and isochronous instability will occur.

III. UNCERTAINTY

It is essential that we clarify the similarities between celestial mechanics and particle physics, as well as point out that some of the analogies to be discussed are purely formal. One way to approach uncertainty and related problems is through Laplace's demon. Poincaré refers to this as Laplace's fantasy, and workers in the field of celestial mechanics, of course, agree with the founding father. We do not know exactly the initial conditions, in spite of the fact that the condition of our world is not influenced by observations (see Brillouin.[3]). Our observations establish the initial conditions with errors. Using these, we attempt to solve the nonintegrable differential equations of motion, realizing that these equations are approximations of the actual physical world. Consequently, we cannot accept the "imaginative poetry" expressed by Laplace's demon, not only for reasons of uncertainty of the original conditions but also because of our ignorance concerning the correct models and because of our inability to solve nonlinear differential equations of interest in celestial mechanics.

This last point might need amplification, and the application of modern numerical and classical analytical techniques should be looked upon to obtain solutions. Looking at the fundamental problem first, we recall Poincaré's theorem according to which the equations of celestial mechanics are not integrable (as Russian literature prefers to refer to it, they are not integrated). Note that Poincaré does not exclude the possible existence of locally valid integrals, and the theorem refers to the non-existence of $2n - 1$ global integrals of our n-degrees-of-freedom dynamical system. The series solutions are consequently divergent, or at best, semiconvergent. The results of the numerical integrations would be highly questionable, especially for long periods of time, even if the initial conditions were exact. The effect of possible instabilities enter the picture here since numerical inaccuracies usually emphasize inherent instabilities. The presently popular mapping techniques might be mentioned at this point, which allow long-time investigations of an approximate mathematical formulation of the problem without sufficient evaluation of the reliability of the results.

There are analytical techniques which increase the accuracy or long-time validity of numerical integrations. These methods are known as regularizing transformations and were originally designed to meet the conditions required by theorems to establish the existence of solutions. The transformations eliminate globally or locally (depending on the model applied) the singularities occurring in the equations of motion. The proper use of such reformulations of the basic equations allows extensions of the range of validity of the results. The extension is once again limited; consequently, the basic problem is not solved.

In summary, we might state that some of the reasons for uncertainty are different in celestial mechanics and in particle physics; nevertheless, both fields in effect are nondeterministic (see Born[4]).

The few qualitative results available in celestial mechanics, therefore, are extremely useful. In several cases, for simple models, ranges of the phase space can be established, even for uncertain initial conditions, without numerical integrations or analytical approximations. The method is known as Hill's surfaces or energy method, and it allows the computation of allowable regions of motion. Unfortunately, the technique often gives model-sensitive results; or, in other words, it is unstable. The classical example is the motion of the moon as influenced by the sun and the earth. Hill's original approach results in strong stability of the lunar orbit, i.e., it predicts that the moon will remain a satellite of the earth. This strong stability becomes weaker when Hill's model is replaced by the circular restricted problem of three bodies, i.e., the lunar orbit becomes more sensitive to outside perturbations. If the general problem of three bodies is used as a model, the stability turns into instability, i.e., the moon may leave the earth, and it may become an independent planet of the sun. This example demonstrates the instability of the energy method when different physical models are used.

Formal Analogy of the Uncertainty Principle

As a preliminary result, the behavior of the product $\Delta q \cdot \Delta p$ is described in an example connected with a well-known and important problem in celestial mechanics. Consider the triangular libration points in the restricted problem of three bodies (L_4, L_5). These are stable equilibrium points, provided the value of the mass parameter satisfies the inequality

$$\mu \leq .0385 = \mu_c.$$

Here $\mu = m_2/(m_1 + m_2)$, and m_1 and m_2 are the masses of the primaries (see Szebehely[5]).

The following nonlinear stability problem is of theoretical and practical interest. When a particle of small mass m_3 is placed at L_4 or L_5 with zero relative velocity, it will stay there. It will perform librational motion around $L_{4,5}$ if its initial conditions are not exact. Consequently, if a space probe, space station, asteroid, or the like, reaches the vicinity of these points with a velocity error of Δp and a position error of Δq, it will stay in the neighborhood of $L_{4,5}$ provided the above errors are sufficiently small. For large errors, the motion will not be librational and m_3 will leave the vicinity of $L_{4,5}$. It might be captured by m_1 or m_2, it might perform circulation, it might become chaotic or leave the system. The stability problem mentioned above is to determine the limiting values of Δq and Δp which will allow librational motion. All other motions will be termed unstable.

Preliminary results seem to verify the conjecture that large Δq values are associated with small Δp and vice versa, or that

$$\Delta q \cdot \Delta p \simeq C,$$

where the constant C will show some dependence on the value of μ.

Because of symmetry, only motion around L_4 will be discussed in the following. Assuming first that m_3 has no velocity error, we determine the maximum position error which results in librational motion as a function of the orientation $\Delta q(\phi)$. The result[6] is a region of the plane around L_4. Then we place m_3 at L_4 and determine the maximum allowable velocity errors, again as a function of the direction of the initial velocity $\Delta p(\phi)$. The result is a region around L_4 which point has zero velocity.[7]

The product $\Delta q(\phi) \cdot \Delta p(\phi)$ might, therefore, be a constant if a large maximum allowable position error in the direction of ϕ is associated with a small maximum allowable velocity error in the same direction.

Preliminary numerical results indicate that this conjecture is correct in the range of $10^{-3} \leq \mu \leq 10^{-2}$, and the value of the product becomes $C = 0(10^{-3})$. As μ approaches the critical value (μ_c), the value of C should decrease.

The above results indicate that the behavior of the phase space when centered at L_4 satisfies our uncertainty formula. Research presently in progress includes investigation concerning the behavior at points other than L_4 as well as the extension of the range of μ.

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