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SELF-CONSISTENT THEORY OF EQUILIBRIUM AND ACCELERATION OF A HIGH-CURRENT ELECTRON RING IN A MODIFIED BETATRON

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A theoretical and computational scheme is developed to describe the self-consistent equilibrium and acceleration of a high-current electron ring in a modified-betatron configuration. The principal output of the calculation is the set of nested drift surfaces of the electron beam.

Recently, several laboratories have been engaged in studies that are aimed at assessing the feasibility of developing high-current accelerators. One particularly promising scheme is the modified betatron, which is a betatron with a strong toroidal field.[1–3] A schematic of the device is shown in Fig. 1. The toroidal angle is θ, and the rz plane is denoted the poloidal plane, as is the conventional notation in tokamaks and other high toroidal-field plasma devices. A conventional betatron, with no toroidal field, fails to confine particles as soon as the density gets so high that the outward self-forces exceed the inward focusing forces. In the presence of a strong toroidal field, the particle orbits do again remain confined. But, at high current the self-fields become so strong that the device cannot be thought of as a collection of individual particles, but rather as a fluid. Previous theoretical studies of this accelerator have generally either assumed a density and current profile and calculated the orbits of test particles in fixed fields,[1,2,4,5] or else have used particle simulations to determine the self-consistent motion.[5,6] The former technique is limited because of the assumed charge and current profile, while the latter is limited by considerations of computing cost. For instance, it is not easy to simulate the beam for one complete drift orbit. For the modified betatron planned at the Naval Research Laboratory, a drift orbit takes several hundred nanoseconds, but the time scale for acceleration, flux diffusion, etc. is very much longer. This paper attempts to extend these available theoretical techniques by developing a computational and analytical scheme for analyzing both the self-consistent equilibrium in the modified betatron, and its adiabatic development as external parameters are varied slowly compared with a drift period. The theory and numerical results are given only briefly here. More details will be published elsewhere.[7]

We utilize a cold-fluid theory to describe the self-consistent equilibrium and its adiabatic evolution. Since the modified betatron has cylindrical symmetry, the total magnetic field can be written as

$$\boldsymbol{B} = \boldsymbol{B}_\perp + B_\theta \boldsymbol{i}_\theta = \nabla\psi \times \frac{\boldsymbol{i}_\theta}{r} + g\frac{\boldsymbol{i}_\theta}{r} \tag{1}$$

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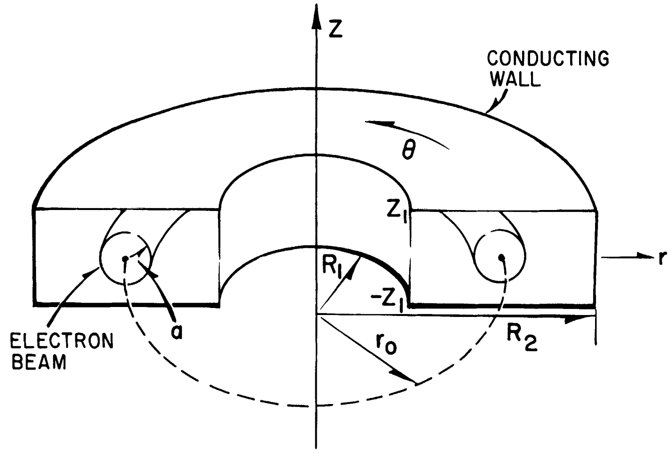

FIGURE 1 A schematic of the modified betatron.

where ψ is the poloidal flux, ($\psi = rA_\theta$) a function of r and z, and g is r times the toroidal magnetic field B_θ. If the electron beam has density n and velocity $\mathbf{v}$, then Amperes law is

$$-ne\mathbf{v} = \frac{c}{4\pi}\left(-\Delta^*\psi\frac{\mathbf{i}_\theta}{r} + \nabla g \times \frac{\mathbf{i}_\theta}{r}\right) \tag{2}$$

where $\Delta^*\psi = r^2\nabla \cdot r^{-2}\nabla\psi$. The toroidal component of the momentum equation in steady state ($\partial/\partial t = 0$) can be shown to reduce to

$$\mathrm{v} \cdot \nabla \bar{P}_\theta = \mathrm{v}_\perp \cdot \nabla \bar{P}_\theta = 0 \tag{3}$$

where

$$\bar{P}_\theta = \gamma m r \mathrm{v}_\theta - \frac{e\psi}{c}, \tag{4}$$

the fluid toroidal canonical angular momentum. Taking the dot product of Eq. (2) with ∇P_θ, one can arrive at the result that

$$g = g(\bar{P}_\theta). \tag{5}$$

Equations (3) and (5) imply that $\bar{P}_\theta$ labels the drift surfaces of the electrons in the poloidal plane and that g is constant on these drift surfaces. Clearly, equilibrium can only exist if these $\bar{P}_\theta$ surfaces are closed surfaces in the poloidal plane. Thus, this $\bar{P}_\theta$ plane must contain one 0 point around which the confined orbits drift. This 0 point will be called the reference orbit. For instance, for a very low density beam, the $\bar{P}_\theta$ surfaces are defined only by the external fields. If the reference orbit has position ($r = r_0$, $z = 0$) and energy γ, then the vertical field B_z at $(r_0, 0)$ is given by $eB_z r_0/\gamma mc = \mathrm{v}_\theta$, where v_θ is the velocity of the electron in the θ direction. Then if the confining vertical field has index η, one can easily calculate that

$$\bar{P}_\theta = K - \frac{eB_z}{c}\left[\frac{(1-\eta)(r-r_0)^2}{2} + \eta\frac{z^2}{2}\right], \tag{6}$$

where K is a constant which may be time dependent if other parameters (for instance B_z) vary in time. As long as the field index is between 0 and 1, P_θ has a 0 point at $r = r_0, z = 0$, and this 0 point corresponds to a maximum of $\bar{P}_\theta$. There are no other singular points in the $\bar{P}_\theta$ plane. As we will see shortly, when strong self-fields are included, the topology of the $\bar{P}_\theta$ surfaces can change considerably.

We now proceed with our fluid formulation. We assume that the toroidal magnetic field B_θ is much greater than the vertical field B_z and also that the density is sufficiently low as to be well below Brillouin flow.[8] This means $2\omega_{pe}^2/\gamma\Omega_{ce}^2 \ll 1$, where ω_{pe} is the electron plasma frequency, $\omega_{pe}^2 = 4\pi ne^2/m$ and Ω_{ce} is the nonrelativistic cyclotron frequency in the toroidal magnetic field. In this case, poloidal inertia and centrifugal force are small terms in the poloidal (i.e., r, z) momentum balance equation. Then, if the total energy is

$$E = \gamma mc^2 - e\phi, \tag{7}$$

where ϕ is the electrostatic potential, then after some manipulation one can show that E is also a function of $\bar{P}_\theta$. We also assume, consistent with the neglect of poloidal inertial, that $\gamma = (1 - (\mathrm{v}_\theta/c)^2)^{-1/2}$. Furthermore, one can show that

$$\frac{\mathrm{v}_\theta}{r} - \frac{g}{4\pi nr^2}\frac{dg}{d\bar{P}_\theta} = \frac{dE}{d\bar{P}_\theta} \equiv \Omega(\bar{P}_\theta), \tag{8}$$

so that the left hand side of Eq. (8) is also a function of P_θ. For the particular case that the energy is independent of $\bar{P}_\theta$, then Eq. (8) gives a particular simple expression for the density

$$n = \frac{g}{4\pi r\mathrm{v}_\theta}\frac{dg}{d\bar{P}_\theta}. \tag{9}$$

We now digress briefly on the implications of Eq. (9) for the drift surfaces ($\bar{P}_\theta$ surfaces) in the modified betatron. Clearly for positive density, $dg/d\bar{P}_\theta > 0$. Thus if the reference orbit corresponds to a maximum in $\bar{P}_\theta$ (as implied by Eq. (6)), then the toroidal field decreases away from the reference orbit. This means that the poloidal currents exert an outward force on the beam. In plasma parlance, this means the beam is paramagnetic. Therefore, force balance implies that the inward focusing force (from the field index) is equal to the sum of the outward self-forces and outward force from $\boldsymbol{J}_\perp \times \boldsymbol{B}_\theta$. Clearly, this implies a low-current beam. Self-consistent equilibria for low-current electron rings have been worked out using a Vlasov theory.[9] As the beam density increases, the outward self-forces ultimately exceed the inward focusing force. This means that for equilibrium the $\boldsymbol{J}_\perp \times \boldsymbol{B}_\theta$ force must be inwards, or the 0 point in the $\bar{P}_\theta$ plane corresponds to a relative minimum. Again, in plasma parlance, this means that the beam is diamagnetic. While these conclusions were derived from $\Omega(\bar{P}_\theta) = 0$, we find that they still generally apply even if $\Omega \neq 0$.

To reiterate, the reference orbit for a high-current beam corresponds to a relative minimum of $\bar{P}_\theta$. But once the beam density falls to zero, the self-fields decrease rapidly with distance from the reference orbit. Therefore, far from the beam, $\bar{P}_\theta$ is as given in Eq. (6), even for high-current beams. Hence, while the reference orbit corresponds to a relative minimum, it is not an absolute minimum. This complicates the topology of the $\bar{P}_\theta$ surfaces considerably. Somewhere near the 0 point is an $\times$ point, and a separatrix then distinguishes the diamagnetic region of the beam from neighboring paramagnetic regions. Outside this separatrix there may be still other 0 points and × points. Possible topologies of the $\bar{P}_\theta$ surfaces are shown in Fig. 2. In the calculations presented here, we assume that the beam lies entirely within the separatrix.

To continue with our fluid formulation, all that is needed is an equation for the self electric and magnetic fields ϕ and ψ. These are simply Poisson's and Ampere's law

$$\nabla^2 \phi = 4\pi\, ne \tag{10}$$

$$\Delta^* \psi = 4\pi\, nerv_\theta/c. \tag{11}$$

On the conducting boundary, ϕ must vanish. However, ψ must be specified on the conducting wall. For times short compared with a flux-diffusion time, ψ is specified externally (the vertical field with the proper index) and the self-field contribution to ψ vanishes on the wall. For times long compared with a flux-diffusion time, ψ on the walls is the sum of the imposed flux plus the flux of a current ring having specified current. It may also be that the diffused flux is partly compensated by external coils.[5,7]

Because the equations as written are highly implicit, an iterative scheme is necessary for their solution. The iteration scheme is as follows. First select a position for the reference orbit and the energy of the particle there. Further assume for purposes of illustration $\Omega = 0$, so all particles have the same total energy. Then pick a first guess for

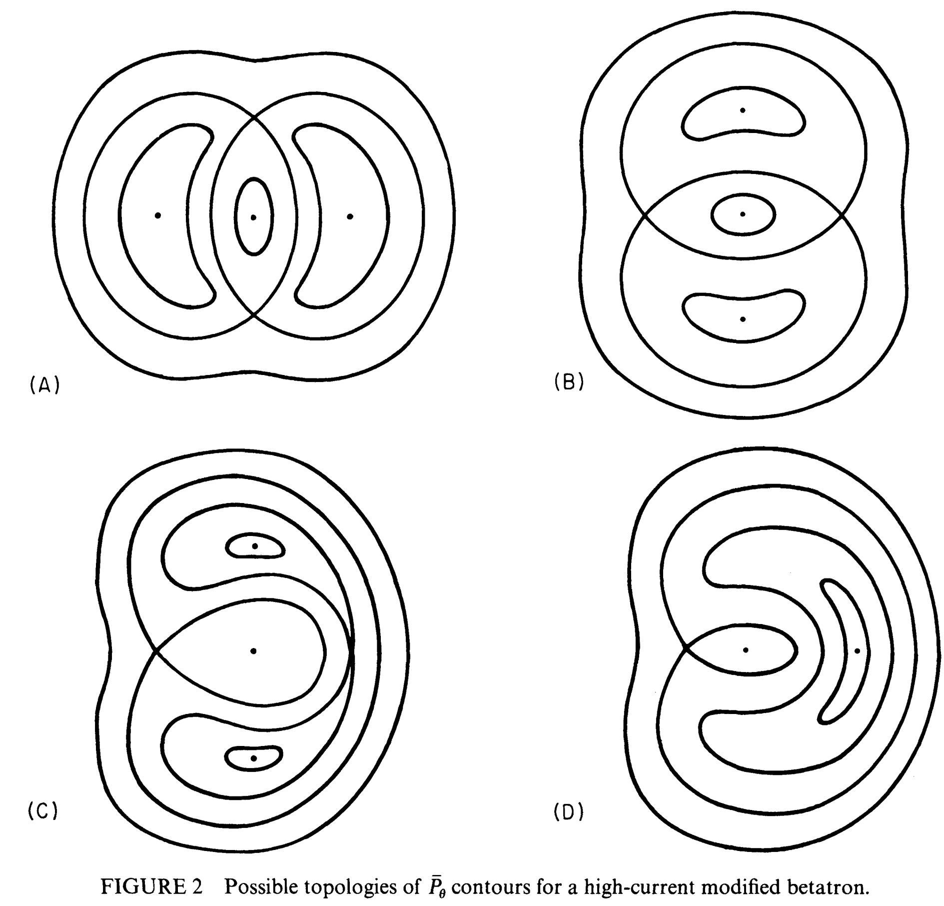

FIGURE 2 Possible topologies of $\bar{P}_\theta$ contours for a high-current modified betatron.

the density and current profile. We generally pick v_θ = constant and a density profile that is parabolic, centered at the reference orbit and going to zero a certain distance away so that the total number of particles and maximum density are specified. Then values of ϕ and ψ are calculated from Eqs. (10) and (11). From ϕ, the particle's velocity v_θ and kinetic energy are calculated from Eq. (7). Then $\bar{P}_\theta$ is calculated from Eq. (4) and the $\bar{P}_\theta$ surfaces are contoured. The 0 point will generally be different from the assigned value. The external vertical field is then adjusted so that the reference orbit is at the assumed position r_0. This then gives a first guess for $\bar{P}_\theta$ surfaces which have their 0 point at the correct radius.

To continue, we select the function $g(\bar{P}_\theta)$ which according to Eq. (9) (or Eq. (8) if $\Omega \neq 0$) is equivalent to selecting the density. We use

$$g\frac{dg}{d\bar{P}_\theta} = \begin{cases} Q\dfrac{\bar{P}_\theta - \bar{P}_{\theta 0}}{\bar{P}_{\theta c} - \bar{P}_{\theta 0}} & \text{for } \bar{P}_\theta < \bar{P}_{\theta c} \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

where $\bar{P}_{\theta 0}$ is the value of $\bar{P}_\theta$ at the 0 point and $\bar{P}_{\theta c}$ is the cutoff value of $\bar{P}_\theta$ beyond which the density vanishes. We assume that $\bar{P}_{\theta c}$ lies within the separatrix. The density given by Eq. (12) will be parabolic near the 0 point if the $\bar{P}_\theta$ surfaces are circular. Since we have the first guess for the $\bar{P}_\theta$ surfaces, the density and current as functions of $\bar{P}_\theta$ can now be calculated as functions of r and z. In doing so, values of Q and $\bar{P}_{\theta c}$ are selected so that both the maximum density and total number of particles have specified values. Then new values of ϕ and ψ are calculated, and so are new $\bar{P}_\theta$ surfaces, as well as values of B_z, Q and $\bar{P}_{\theta c}$. The entire process is then iterated until a self-consistent equilibrium is found. In practice, the iteration scheme converges very quickly; about 10 iterations usually give extremely accurate results. Figure 3 shows final plots of $\bar{P}_\theta$ surface for a 10-kA beam with $\gamma_0 = 7.9$, beam radius 2 cm having $B_{\theta 0} = 2$ kG. The boundary is a rectangular conducting wall at $z = \pm 7.5$ cm, $r = 87.5$ cm and 112.5 cm, and the field index is 0.5. In Fig. 3, the innermost dotted curve is the $\bar{P}_\theta$ surface at $\bar{P}_\theta = \bar{P}_{\theta c}$. The next dotted curve is the separatrix.

We continue by investigating how this equilibrium evolves if external parameters change on a time scale slow compared with a drift time. This could correspond to acceleration, flux diffusion, compression etc. We have shown that the equilibrium is specified by the flux ψ on the walls and two other functions of $\bar{P}_\theta$, $g(\bar{P}_\theta)$ and $\Omega(\bar{P}_\theta)$. (So far we have considered only $\Omega = 0$.) Therefore the problem of adiabatic evolution is

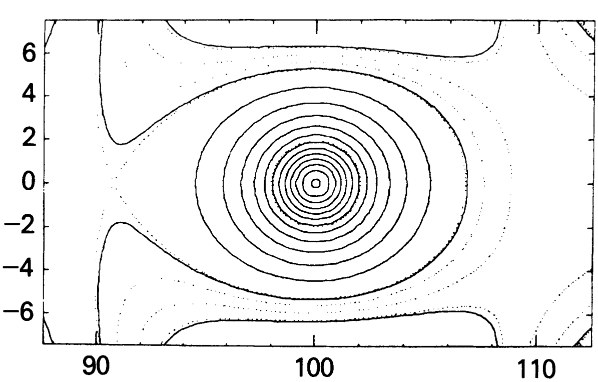

FIGURE 3 The $\bar{P}_\theta$ surfaces for a self consistent 10 kA beam equilibrium having $\gamma = 7.9$.

solved if at any time t, the functions $g(\bar{P}_\theta, t)$ and $\Omega(\bar{P}_\theta, t)$ can be specified from their values at $t = 0$.

To accomplish this, we employ two invariants of the evolution, the number of particles within a $\bar{P}_\theta$ drift surface and the toroidal flux within a drift surface. The first of these is trivially invariant because $\bar{P}_\theta$ is a constant of motion for each particle, due to the θ symmetry. Therefore the number of particles having each value of P_θ cannot change.

The proof that the toroidal flux within a drift surface is an adiabatic invariant is more difficult and is given elsewhere.[7] Our proof is based on calculating the fluid motion of a drift surface, and subtracting the change in poloidal flux from the change in total flux. Doing so, we arrive at the result that the difference of the two, the toroidal flux is conserved. From another viewpoint, this result is reasonable because the toroidal flux through the drift orbit corresponds to the third adiabatic invariant of a particle's motion.[10] In any case, we re-emphasize here that although $\Omega(P_\theta)$ may be zero at the start of the acceleration, it does not remain zero as the beam is accelerated.

Using the expression for n from Eq. (8), we find that the number of particles within a $\bar{P}_\theta$ surface is given by

$$N(\bar{P}_\theta) = \frac{1}{2}\int_{\bar{P}_{\theta 0}}^{\bar{P}_\theta} d\bar{P}_\theta g(\bar{P}_\theta)\frac{dg}{d\bar{P}_\theta}\oint\frac{dl}{|\nabla\bar{P}_\theta|}\frac{1}{\mathrm{v}_\theta - r\Omega(\bar{P}_\theta)}, \tag{13}$$

where l is the length along a $\bar{P}_\theta$ contour in the poloidal plane. If the toroidal flux is denoted Φ_t, the other adiabatic invariant is

$$\Phi_t(\bar{P}_\theta) = \int_{\bar{P}_{\theta 0}}^{\bar{P}_\theta} d\bar{P}_\theta g(\bar{P}_\theta)\int\frac{dl}{r|\nabla\bar{P}_\theta|}. \tag{14}$$

Thus, given $N(\bar{P}_\theta)$ and $\Phi_t(\bar{P}_\theta)$, the procedure is to solve Eqs. (13) and (14) for $g(\bar{P}_\theta)$ and $\Omega(\bar{P}_\theta)$.

In our numerical computations, we have simplified this procedure somewhat by assuming Ω is constant, and conserving toroidal flux only at the separatrix. A posteriori checks on the validity of this approximation show that it is generally good to a few percent.

To do the calculation, start with a reference equilibrium like that given in Fig. 3, but increase the reference orbit energy to γ_1 corresponding to acceleration of the beam. Then a guess for beam density and current give first guesses for $\bar{P}_\theta$ surfaces as before. Generally these $\bar{P}_\theta$ surfaces (a) will not have the 0 point at r_0, (b) will not have the correct value of $\bar{P}_\theta = \bar{P}_{\theta 0}$ at the 0 point, and (c) will not conserve flux within the separatrix. These can be corrected as follows (a) the position of the 0 point can be changed by changing the external vertical field, (b) the value of $\bar{P}_\theta$ at the 0 point can be changed by adding an arbitrary constant to ψ. This corresponds to the change in flux through the orbit. The relation between flux change and vertical field change then gives the betatron conditions corrected for self-fields, (c) finally, the toroidal flux through the separatrix can be changed by varying Ω. The details of the iterative scheme are as given in Ref. 7. Once the $\bar{P}_\theta$ surfaces are modified so that conditions a-c are accounted for, $g(P_\theta)$ is determined from Eq. (13).

$$\frac{d}{d\bar{P}_\theta}g^2 = 2\frac{dN}{dP_\theta}\left[\oint\frac{dl}{|\nabla\bar{P}_\theta|}\frac{1}{\mathrm{v}_\theta - r\Omega}\right]^{-1} \tag{15}$$

and Eq. (15) for g^2 is integrated from $\bar{P}_{\theta c}$, where g is known as the external applied field, inward in $\bar{P}_\theta$ to $\bar{P}_{\theta 0}$. From g, one has the density from Eq. (8) and the entire process is iterated until it converges. As was the case in calculating the reference equilibrium, the iteration scheme converges relatively quickly.

Figures 4 and 5 show two cases of beam acceleration. Figure 4 shows the $\bar{P}_\theta$ surfaces for a 10 kA beam with $B_\theta = 2kG$ and having a field index of $\eta = 0.5$, and Fig. 5 shows

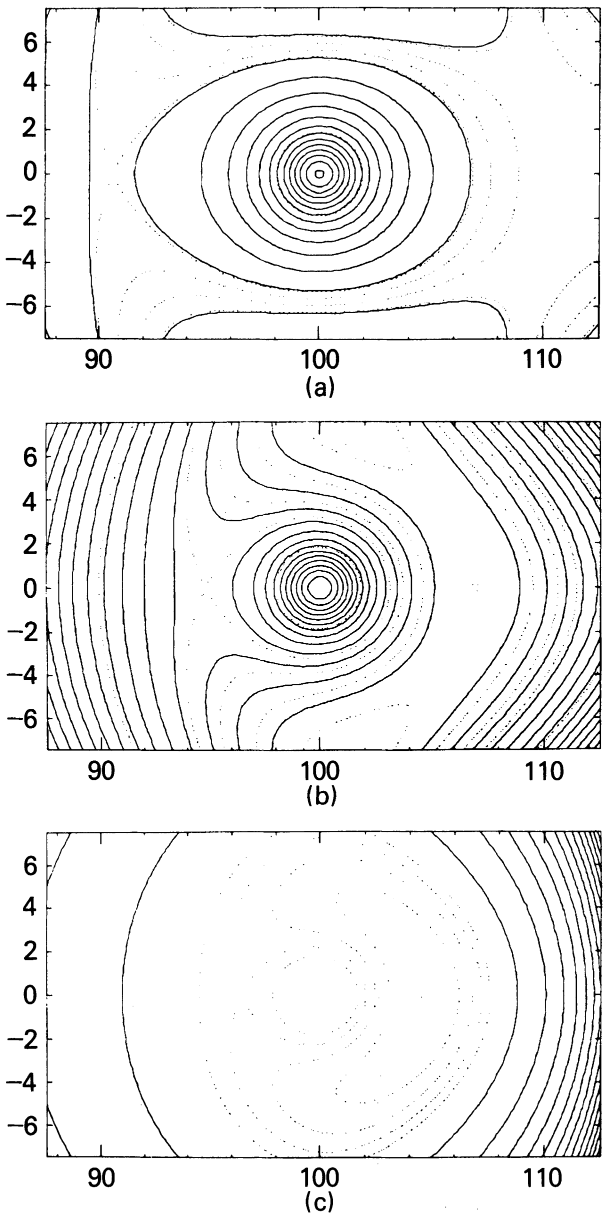

FIGURE 4 The $\bar{P}_\theta$ surfaces for a 10 kA beam having field index $\eta = 0.5$ as it accelerates from $\gamma = 3.85$ to 6.96 to 11.05. Notice that as the inner part of the separatrix approaches the beam edge, the outer part of the separatrix is well confined within the walls.

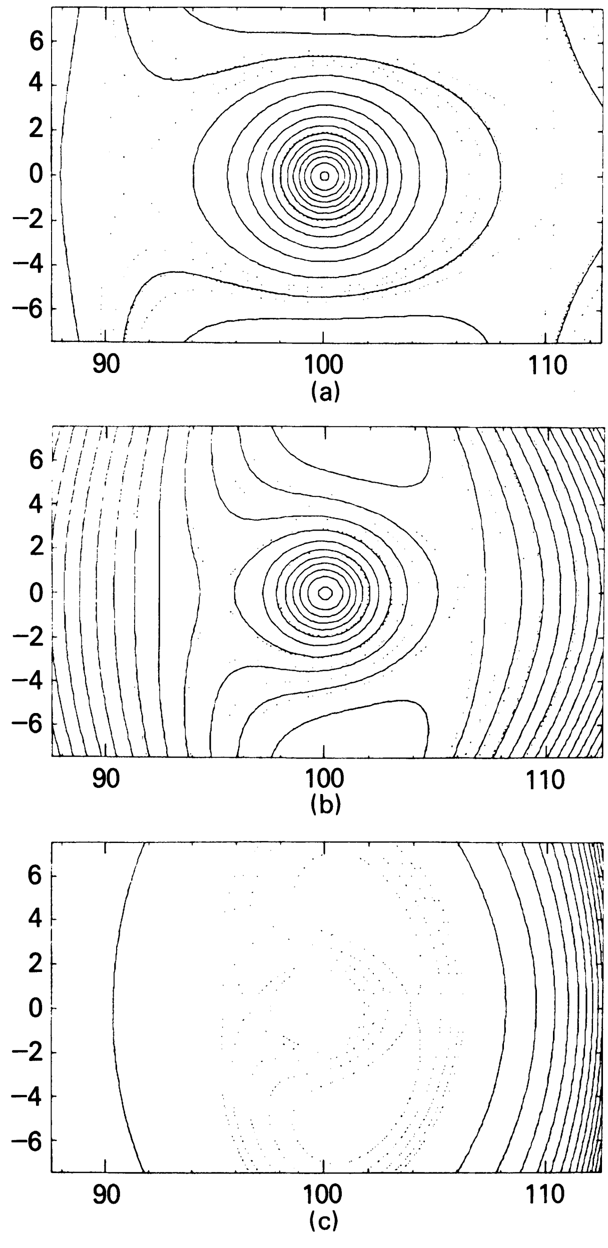

FIGURE 5 Same as Fig. 4 but for a field index $\eta = 0.4$. Notice that the outer part of the separatrix is not confined as the inner part scrapes the beam. This implies large beam losses as the beam accelerates further.

the same thing but for a field index of $\eta = 0.4$. In Fig. 4 the γ's are 3.85, 6.96 and 11.05, corresponding to an increase in vertical field from 114.5 G to 238.0 G. In Fig. 4 the steps in γ are 3.91, 7.02 and 10.07 corresponding to an increase in vertical field from 114.6 G to 220.6 G. In each case, the separatrix moves in as the beam accelerates. At some critical γ, the separatrix touches the outer edge of the beam, and these outer particles scrape off and begin to populate the paramagnetic region on the other side of the separatrix. Thus, in order not to lose the beam, the outer part of the separatrix must be well confined within the walls when the inner part scrapes the beam. This is so in Fig. 4, but not in Fig. 5. Thus for the shape walls we consider, the beam will be able to make the diamagnetic to paramagnetic transition (i.e., go through the region of orbit instability)[1,2,4] for a field index of 0.5, but not 0.4.

To conclude, we have shown that self-consistent equilibria and adiabatic evolution of these equilibria can be calculated in a high-current modified betatron. The simulation code developed can also provide important design information, such as how to maximize the chances of passing through the region of orbital instability without disruption.

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REFERENCES

1. P. Sprangle and C. A. Kapetanakos, *J. Appl. Phys.*, **49,** 1 (1978).
2. N. Rostoker, *Comm. Plasma Phys.*, **6,** 91 (1980).
3. C. A. Kapetanakos, P. Sprangle and S. J. Marsh, *Phys. Rev. Lett.*, **49,** 741 (1982).
4. D. P. Chernin and P. Sprangle, *Particle Accelerators*, **12,** 85 (1982).
5. C. A. Kapetanakos, P. Sprangle, D. Chernin, S. J. Marsh and I. Haber, *Particle Accelerators*, to be published. (Also see NRL Memo Report 4905.)
6. P. Sprangle, C. A. Kapetanakos and S. J. Marsh, Proc. of Intern. Topical Conf. on High-Power Electron and Ion Beam Research and Technology, Palaiseau, France, June 29–July 3, 1981, p. 803.
7. J. M. Finn and W. M. Manheimer, *Phys. Fluids*, to be published.
8. R. C. Davidson, *Theory of Non Neutral Plasmas* (W. A. Benjamin, Inc., 1974), p. 7.
9. *Ibid*, Sec. 3.5, and references therein.
10. T. G. Northrup, *The Adiabatic Motion of Charged Particles* (Wiley Interscience, New York, 1963).