XI. PHYSICAL ACOUSTICS*

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RESEARCH OBJECTIVES

In physical acoustics the problems involved concern the emission, propagation, and absorption of sound and vibrations in matter. Specific problems include relaxation phenomena in gases and solids, problems in nonlinear acoustics, and the interaction of sound and turbulence.

In plasma physics our present main interest is in the area of instabilities in plasmas and liquid conductors, and of wave propagation in plasmas.

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A. INSTABILITY OF LIQUID CONDUCTORS IN A MAGNETIC FIELD

The study of instability of liquid conductors in a magnetic field has been conducted with an analysis of the experimental results obtained earlier. Some of these results have been mentioned briefly in a previous progress report.[1] The rate of change of the extreme diameter in a pinch instability has been evaluated as a function of the wavelength of the perturbation on the stream for several different values of the current through the stream.

Similarly, the rate of growth of the spiral diameter in an $m = 1$ type of instability in a longitudinal magnetic field has been determined. This last rate is found empirically to be well described by the expression

$$\beta \approx C\,(IB/\rho d^3)^{1/2}\ \mathrm{sec}^{-1},$$

where I is the current through the stream, B the longitudinal magnetic field, ρ the density, and d the diameter of the liquid conductor. The dimensionless constant C is 4.5. A more detailed account of this investigation will appear in the December issue of The Physics of Fluids.

U. Ingard, D. S. Wiley

References

1. D. S. Wiley and U. Ingard, Bifurcation instability of a liquid conductor in an axial magnetic field, Quarterly Progress Report No. 63, Research Laboratory of Electronics, M.I.T., October 15, 1961, p. 133.

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B. SOUND PROPAGATION OVER A PLANE BOUNDARY

The analysis of the influence of turbulence on sound propagated over a plane boundary has been completed. The new analysis of the problem differs from that presented in Quarterly Progress Report No. 55 (pages 141-149) in two respects. First, both amplitude and phase fluctuations have been included in the calculations; and, second, the amplitude and phase fluctuations are assumed to be normally distributed. The experimental results discussed in Quarterly Progress Report No. 66 (pages 69-70) have been

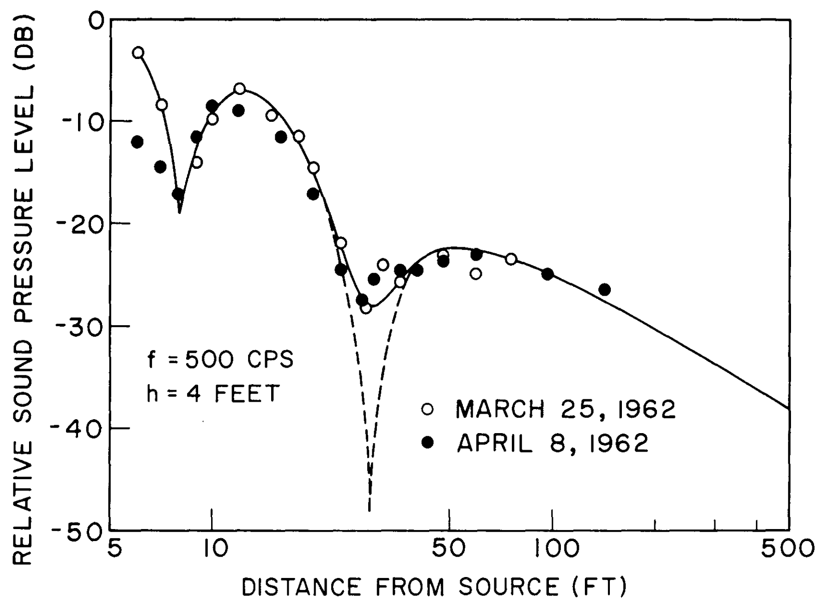

Fig. XI-1. Mean-square sound-pressure level in a turbulent atmosphere over a plane reflecting boundary as a function of distance from the source (frequency, 500 cps). Dashed curve refers to the calculated distribution in a homogeneous atmosphere; solid curve, the calculated distribution for a turbulent atmosphere.

analyzed in detail, and the fractional standard deviation of the mean-square pressure fluctuations (Std. Dev./p^2_{rms}) has been calculated as a function of distance from the source, and compared with the analysis that follows.

If it is assumed that both the amplitude fluctuations (a) and phase fluctuations (δ) are normally distributed, with variance given by

$$\sigma^2 = \langle\delta^2\rangle = \frac{\sqrt{\pi}}{2}\,\mu_o^2(k_ox)(k_oL)\left(1+\frac{\sqrt{x^2+4h^2}}{x}\right)$$

$$\langle(\ln(1+a))^2\rangle = \frac{\sqrt{\pi}}{2}\,\mu_o^2(k_ox)(k_oL),$$

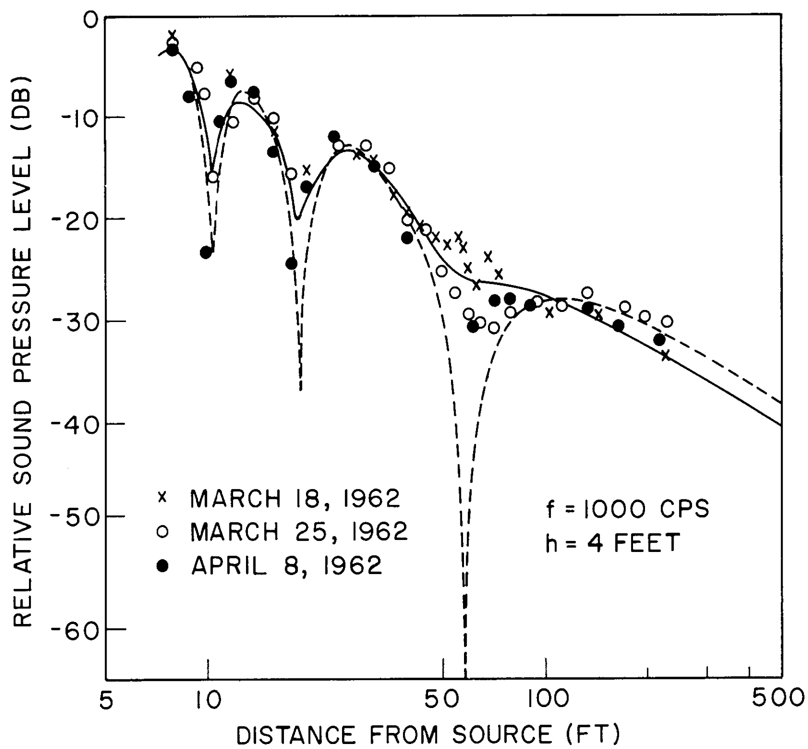

Fig. XI-2. Mean-square sound-pressure level in a turbulent atmosphere over a plane reflecting boundary as a function of distance from the source (frequency, 1000 cps). Dashed curve refers to the calculated distribution in a homogeneous atmosphere; solid curve, the calculated distribution for a turbulent atmosphere.

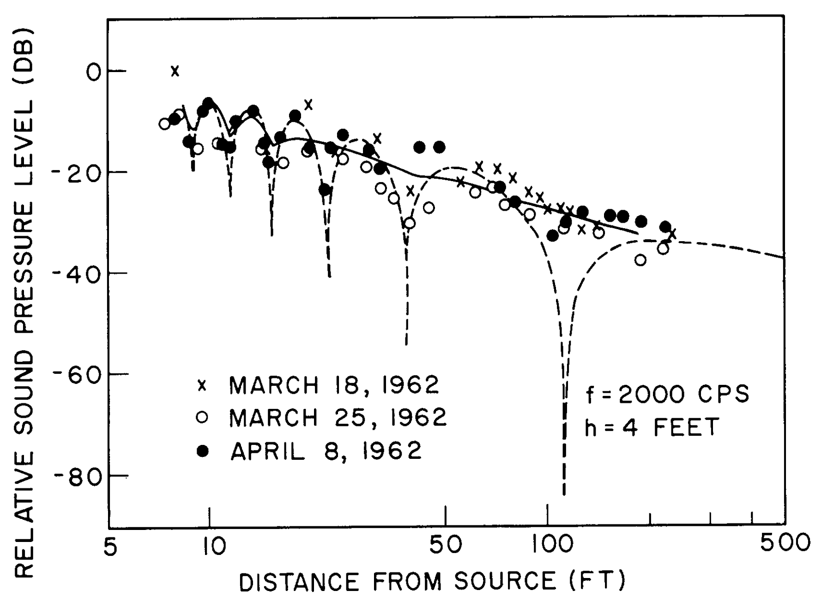

Fig. XI-3. Mean-square sound-pressure level in a turbulent atmosphere over a plane reflecting boundary as a function of distance from the source (frequency, 2000 cps). Dashed curve refers to the calculated distribution in a homogeneous atmosphere; solid curve, the calculated distribution for a turbulent atmosphere.

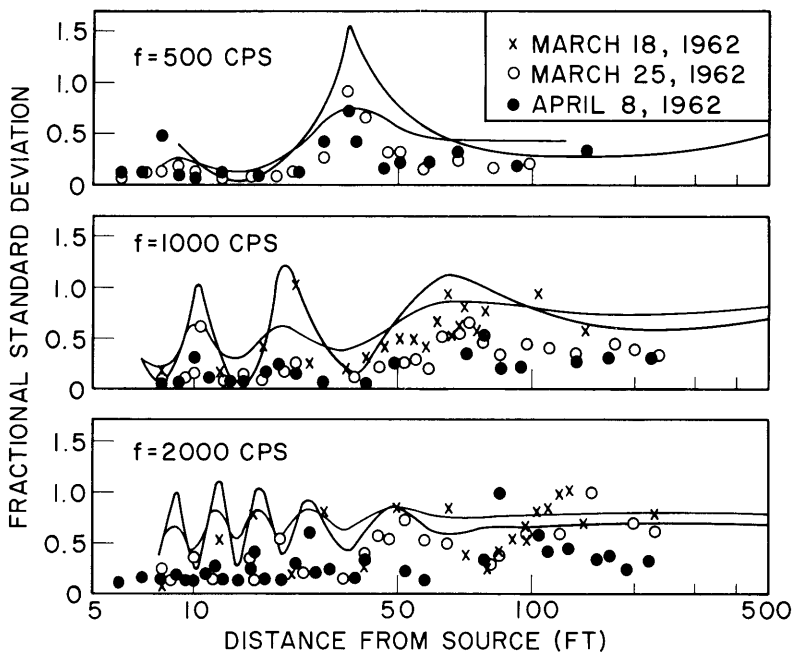

Fig. XI-4. Measured and calculated fluctuations of the sound-pressure amplitude as a function of distance from the source.

then it can be shown that the mean-square sound-pressure level as a function of distance from the source is given approximately by

$$m = \langle \overline{p^2} \rangle \doteq \frac{1}{xr}\left(\langle a^2\rangle + \frac{r}{2x}\left(1-\frac{x}{r}\right)^2 + \left(1+\cos\beta_o e^{-\sigma^2/2}\right)\right),$$

and the variance of the fluctuations in mean-square pressure, $V = \langle (\overline{p^2})^2 \rangle - \langle (\overline{p^2}) \rangle^2$, may be approximated as

$$V = \frac{1}{2}\left(1+2\langle a^2\rangle\right)\left(1-e^{-\sigma^2}\right)\left(1-e^{-\sigma^2}\cos 2\beta_o\right) + 2\langle a^2\rangle\left(1+\cos\beta_o e^{-\sigma^2/2}\right).$$

If the value of the parameter $\mu_o^2 L$ is chosen so that the calculated value of m is in good agreement with experiment at 500 cps, and x = 28.5 ft, the correlation at other frequencies and at other distances is as illustrated in Figs. XI-1, XI-2, and XI-3. The dashed curves are the calculated sound pressures for a quiescent atmosphere. Plots of $\sqrt{V/m}$ vs distance are presented in Fig. XI-4. The sharply peaked curves were calculated under the assumption that amplitude fluctuations could be neglected. It is seen that somewhat better agreement with experiment can be obtained if the effect of amplitude fluctuations is included.

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References

1. L. A. Chernov, Wave Propagation in a Random Medium (McGraw-Hill Book Company, Inc., New York, 1961).

C. STABILITY OF PARALLEL FLOWS

1. Introduction

If $(0, 0, W(x_1))$ is a parallel flow solution for the Navier-Stokes equations and $(u_1(\bar{x}, 0),\ u_2(\bar{x}, 0),\ W(x_1) + u_3(\bar{x}, 0))$ represents an infinitesimally disturbed flow condition at $t = 0$, then the disturbed system will evolve according to

$$\frac{\partial u_i}{\partial t} = \nu\nabla^2 u_i - W\frac{\partial u_i}{\partial x_3} - \frac{\partial p}{\partial x_1'} - \delta_{i3}\frac{dW}{dx_1}u_1 \tag{1a}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{1b}$$

with nonlinear terms neglected.

Taking the divergence of (1a), and using (1b), we find that

$$\nabla^2 p = -2\,\frac{dW}{dx_1}\frac{\partial u_1}{\partial x_3}, \tag{1c}$$

so that, with proper boundary conditions, p can be expressed in terms of the u_i and eliminated from (1a). In fact, taking ∇^2 in the first of Eqs. 1a and substituting $\nabla^2 p$ from (1c), we have

$$\frac{\partial}{\partial t}\nabla^2 u_1 = \nu\nabla^4 u_1 - W\frac{\partial}{\partial x_3}\nabla^2 u_1 + \frac{d^2W}{dx_1^2}\frac{\partial u_1}{\partial x_3}. \tag{2}$$

We have used the fact that W is, at most, quadratic in x_1.

The case $u_i(\bar{x}, t) = e^{\lambda t}u_i(\bar{x})$ has been studied extensively,[1] since for this case the stability analysis reduces to the (difficult) problem of discovering whether (2) has solutions for u_1 with $\mathrm{Re}\,(\lambda) \geq 0$. For many cases of interest, however, there exist no solutions, or only a restricted class of solutions of this simple type, so that the results under such an assumption are not entirely conclusive. We shall discuss a weak instability, not of exponential type, which is common to all inviscid parallel flows.

2. A Weak Instability

The equations of motion (1) and (2) are homogeneous in x_3, so that the u_i will remain independent of x_3 if they are so at $t = 0$. We consider, first, the case $\nu = 0$, for which (1) becomes

$$\frac{\partial u_1}{\partial t} = -\frac{\partial p}{\partial x_1} \tag{3a}$$

$$\frac{\partial u_2}{\partial t} = -\frac{\partial p}{\partial x_2} \tag{3b}$$

$$\frac{\partial u_3}{\partial t} = -\frac{dW(x_1, x_2)}{dx_1} u_1 \quad \frac{dW(x_1, x_2)}{dx_2} u_2 \tag{3c}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \tag{3d}$$

in which we now permit W to depend on x_2, as well as on x_1. We suppose that the flow is contained by a boundary $b(x_1, x_2)$, so that the component of u normal to B must vanish along B.

According to (3), $\partial^2 p/\partial x_1^2 + \partial^2 p/\partial x_2^2 = 0$, and the normal derivative of p vanishes at B. The only harmonic function satisfying the boundary conditions (and finite everywhere, if B is not closed) is p = constant. According to (3a) and (3b), then, u_1 and u_2 are time-independent. Equation 3c can be immediately integrated to give

$$u_3(\bar{x}, t) = u_3(\bar{x}, 0) - t\left[\frac{dW}{dx_1} u_1(\bar{x}) + \frac{dW}{dx_2} u_2(\bar{x})\right]. \tag{4}$$

This simple linear analysis thus yields a weak instability, linear in time, which is common to all inhomogeneous inviscid parallel flows. The instability arises from a mixing of parallel streamlines of varied flow strengths by a weak persistent convective motion.

We should expect that in an exact analysis u_3 should grow at first according to (4), and then flatten out or turn back when $u_3 \ll W$ ceases to be valid. We can indeed construct exact solutions of the Navier-Stokes equations which exhibit this behavior. In an exact analysis (4) becomes

$$\left.\begin{aligned} &\frac{\partial u_i}{\partial t} = -u_1 \frac{\partial u_i}{\partial x_1} - u_2 \frac{\partial u_i}{\partial x_2} - \frac{\partial p}{\partial x_1} \\ &u_n\big|_B = 0 \qquad\qquad \frac{\partial u_i}{dx_i} = 0 \end{aligned}\right\} \quad i = 1, 2 \tag{5a}$$

$$\left.\begin{aligned}&\frac{\partial u_3}{\partial t} = -\left[u_1 \frac{\partial}{dx_1} + u_2 \frac{\partial}{\partial x_2} \quad u_3\right]\\&u_3(\vec{x}, t=0) = W(x_1, x_2)\end{aligned}\right\} \tag{5b}$$

Equations 5a are the two-dimensional Navier-Stokes equations, which admit of time-stationary vortical motions $(u_1(\vec{x}),\ u_2(\vec{x}))$. Equation 5b simply expresses the continuity of $u_3(x_1, x_2)$ in the two-dimensional convective field (u_1, u_2). For small t, u_3 clearly evolves according to (4).

The second case, $\nu \neq 0$, is less transparent, and we consider only plane parallel flow between parallel plates at $x = (\pm\pi)$. Equation 2 in this case reduces to

$$\begin{aligned}&\frac{d}{dt} \nabla^2 u_1 = \nu\nabla^4 u_1\\&\frac{du_1}{dx_1}(\pm\pi) = u_1(\pm\pi) = 0.\end{aligned} \tag{6}$$

Equation 6 can also be written

$$\begin{aligned}&\frac{d}{dt} w = \nu\nabla^2 w\\&w(\pm\pi) = f(\pm\pi)\\&\frac{dw}{dx}(\pm\pi) = \frac{df}{dx}(\pm\pi),\end{aligned} \tag{7}$$

where $f = f(x_1x_2t)$, and $\nabla^2 f = 0$, so that $u_1 = w - f$ clearly satisfies (1). We expand

$u_1 = \int_{-\infty}^{\infty} \int_C \tilde{u}_1\left(x_1, k_2, \lambda^{(o)}\right) \exp(\lambda^{(o)}t) \exp(ik_2x_2)\, d\lambda^{(o)}dk_2$, etc., so that (2) becomes

$$\begin{aligned}&\lambda^{(o)}\tilde{w} = \nu\left(\frac{d^2}{dx_1^2} - k_2^2\right)\tilde{w}\\&\frac{d\tilde{w}}{dx}(\pm\pi) = \frac{d\tilde{f}}{dx}(\pm\pi)\\&\tilde{w}(\pm\pi) = \tilde{f}(\pm\pi).\end{aligned} \tag{8a}$$

Here, $\left(\frac{d^2}{dx_1^2} - k_2^2\right)\tilde{f} = 0$, so that $\tilde{f} = a \cosh k_2x_1 + b \sinh k_2x_1,\ k_2 \neq 0.$

With $\lambda = \frac{\lambda^{(o)}}{\nu} + k_2^2$, we have

$$\lambda\tilde{w} = \frac{d^2}{dx_1^2}\tilde{w}. \tag{8b}$$

Note that if $\tilde{w}_\lambda(x)$ is a solution of (8a) for some (λ, a, b), then $\tilde{w}_\lambda(-x)$ will be a solution for $(\lambda, a, -b)$. It follows that the even and odd parts of $\tilde{w}_\lambda$ separately will be solutions for $(\lambda, a, 0)$ and $(\lambda, 0, b)$, respectively. Thus we seek $\tilde{w}^{(e)}$ even, and $\tilde{w}^{(o)}$odd, satisfying

$$\begin{aligned}
&w^{(e)}(\pm\pi) = a \cosh k_2\pi \\
&\frac{dw^{(e)}(\pm\pi)}{dx} = \pm k_2 a \sinh k_2\pi \\
&w^{(o)}(\pm\pi) = \pm b \sinh k_2\pi \\
&\frac{dw^{(o)}(\pm\pi)}{dx} = k_2 b \cosh k_2\pi \\
&\lambda w = \frac{d^2}{dx_1^2} w
\end{aligned} \tag{9a}$$

or, more concisely,

$$\frac{1}{w^{(e)}}\frac{dw^{(e)}}{dx}(\pm\pi) = \pm k_2 \tanh k_2\pi$$

and (9b)

$$\frac{1}{w^{(o)}}\frac{dw^{(o)}}{dx}(\pm\pi) = \pm k_2 \coth k_2\pi.$$

With the given boundary conditions, we have

$$\begin{aligned}
\left(\frac{d^2u}{dx_1^2}, v\right) &= \int_{-\pi}^{\pi} \frac{d^2u}{dx_1^2}\bar{v}\, dx_1 = \frac{du}{dx_1}\bar{v}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{du}{dx_1}\frac{d\bar{v}}{dx_1} dx_1 \\
&= \frac{du}{dx_1}\bar{v}\Big|_{-\pi}^{\pi} - u\frac{d\bar{v}}{dx_1}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} u\frac{d^2\bar{v}}{dx_1^2} dx_1 \\
&= u\bar{v}\left(\frac{1}{u}\frac{du}{dx_1} - \overline{\frac{1}{v}\frac{dv}{dx_1}}\right)\Big|_{-\pi}^{\pi} + \left(u, \frac{d^2v}{dx_1^2}\right).
\end{aligned} \tag{10}$$

The conditions (9b) ensure the vanishing of the iterated term in (10).

The system is thus a regular self-adjoint one, so that the eigenfunctions of (9a) will be complete for the purposes of expansion. Even solutions of (9a) have the form

$$w^{(e)} = A \cosh \sqrt{\lambda}\, x_1$$

for which λ must be chosen so that (11)

$$\frac{1}{w^{(e)}} \frac{dw^{(e)}}{dx}(\pi) = \sqrt{\lambda} \tanh \sqrt{\lambda}\, \pi = k_2 \tanh k_2\pi.$$

The only solution with $\lambda \geq 0$ is $w^{(e)} = A \cosh k_2x_1$, for $\lambda = k_2^2$. In case $\lambda < 0$, we write $\sqrt{\lambda} \tanh \pi\sqrt{\lambda} = -p \tan p\pi = k_2 \tanh k_2\pi$, so that p is real, $(-p^2=\lambda)$. We get eigenvalues p_n with $n-\frac{1}{2} < p_n < n$, $n = 1, 2 \ldots$, so that $\lambda_n = -p_n^2$ and $\lambda_n^{(o)} = -\nu\left(p_n^2+k_2^2\right)$. The only odd solution of (9a) with $\lambda \geq 0$ is $w_o^{(o)} = A \sinh k_2x_1$, for $\lambda = k_2^2$.

For $\lambda < 0$, we set $\lambda = -q^2$ and write

$$w^{(o)} = B \sin qx_1, \tag{12}$$

where

$$\frac{1}{w^{(o)}} \frac{dw^{(o)}}{dx}(\pi) = q \cot q\pi = k_2 \coth k_2\pi,$$

or

$$\tan q\pi = \frac{q}{k_2} \tanh k_2\pi.$$

We get eigenvalues q_n with $n < q_n < n+\frac{1}{2}$, $n = 1, 2 \ldots$, ($q_n = 0$ is spurious). Thus we have solutions

$$\begin{aligned} u_i^{(e)}(x_1k_2p_n) &\propto \frac{\cos p_nx_1}{\cos p_n\pi} - \frac{\cosh k_2x_1}{\cosh k_2\pi} \qquad \lambda_n^{(o)} = -\nu\left(p_n^2+k_2^2\right) \\ u_1^{(o)}(x_1k_2q_n) &\propto \frac{\sin q_nx_1}{\sin q_n\pi} - \frac{\sinh k_2x_1}{\sinh k_2\pi} \qquad \lambda_n^{(o)} = -\nu\left(q_n^2+k_2^2\right) \end{aligned} \tag{13}$$

Finally, if $u_1(x_1, x_2, 0)$ is any perturbation that is continuous and piecewise smooth in x_1 and x_2, satisfies the boundary conditions on x_1, and $\epsilon\ L_1(X_2)$, we obtain

$$\begin{aligned} u_1(x_1x_2t) = &\int_{-\infty}^{\infty} \sum_{p_n(k_2)} \exp\left[-\nu\left(p_n^2+k_2^2\right)t\right] u_1^{(e)}(p_n,k_2) \left[\frac{\cos p_nx_1}{\cos p_n\pi} - \frac{\cosh k_2x_1}{\cosh k_2\pi}\right] \cdot \exp(ik_2x_2)\, dk_2 \\ &+ \int_{-\infty}^{\infty} \sum_{q_n(k_2)} \exp\left[-\nu\left(q_n^2+k_2^2\right)t\right] u_1^{(o)}(q_n,k_2) \left[\frac{\sin q_nx_1}{\sin q_2\pi} - \frac{\sinh k_2x_1}{\sinh k_2\pi}\right] \cdot \exp(ik_2x_2)\, dk_2. \end{aligned} \tag{14a}$$

Here,

$$u_1^{(e)}(p_u, k_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ik_2x_2) \frac{\int_{-\pi}^{\pi} u_1(x_1x_2, o) \cos p_nx_1 \, dx_1}{\cos p_n\pi\left(\pi + \frac{\sin 2p_n\pi}{2p_n}\right)} dx_2$$

$$u_1^{(o)}(q_n, k_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ik_2x_2) \frac{\int_{-\pi}^{\pi} u(x_1, x_2, o) \sin q_nx_1 \, dx_1}{\sin q_nx_1\left(\pi - \frac{\sin 2q_n\pi}{2q_n}\right)} dx_2. \tag{14b}$$

This expansion is of a type that is well known from the theory of heat conduction, and is shown to converge uniformly for all x_1 and $t \geq t_o$ ($t_o > 0$) and thus to satisfy the boundary conditions for all $t > 0$, and to converge to the series expansion for $u_1(x_1, x_2, 0)$ for $t \to 0$. This solution is unique under the given conditions.

If $u_1(x_1, x_2, 0)$ is taken to be periodic in x_2 instead of being $\epsilon\ L_1(X_2)$, the solution above remains valid if we replace the integrals in (14a) by summations on k_2, and the symbols $\frac{1}{2\pi}\int_{-\infty}^{\infty}$ in (14b) by $\frac{1}{2L}\int_{-L}^{L}$.

To complete the problem, we compute u_3. According to (1a),

$$\frac{\partial u_3}{\partial t} = \nu\nabla^2 u_3 - W'u_1 \qquad u_3\Big|_{x_1=\pm\pi} = 0, \tag{15}$$

where $W' = \frac{dW}{dx_1} = \frac{d}{dx_1}(V_o+V_1x+V_2x^2)$.

Equation 15 is the inhomogeneous heat equation in two dimensions. If u_3 and u_1 are continuous, piecewise smooth and periodic in x_2 of period 2L, we have the unique uniformly convergent series expansion

$$u_3(x_1, x_2, t) = \sum_{\substack{k_2=\pm\frac{\pi}{L}, \pm\frac{2\pi}{L}\dots \\ n=1, 2\dots}} \exp\left(-\nu\left[k_2^2+\left(n+\tfrac{1}{2}\right)^2\right]t\right) \cos\left(n+\tfrac{1}{2}\right)x_1$$

$$\cdot\left\{f_{k_2n}^e - \int_0^t \exp\left(\nu\left[k_2^2+\left(n+\tfrac{1}{2}\right)^2\right]s\right) g_{k_2n}^e(s)\, ds\right\} e^{ik_2x_2}$$

$$+ \sum_{k_2, n} \exp\left(-\nu[k_2^2+n^2]t\right) \sin nx_1$$

$$\cdot \left\{ f^{o}_{k_2 n} - \int_0^t \exp\left(\nu \left[k_2^2+n^2\right] s\right) g^{o}_{k_2 n}(s)\, ds \right\} e^{ik_2 x_2}. \tag{16a}$$

Here,

$$f^{(e)}_{k_2 n} = \frac{1}{2\pi L} \int_{-L}^{L} \int_{-\pi}^{\pi} u_3(x_1, x_2, 0) \cos\left(n+\frac{1}{2}\right) x_1\, e^{ik_2 x_2}\, dx_1 dx_2$$

$$f^{(o)}_{k_2 n} = \frac{1}{2\pi L} \int_{-L}^{L} \int_{-\pi}^{\pi} u_3(x_1, x_2, 0) \sin nx_1\, e^{-ik_2 x_2}\, dx_1 dx_2 \tag{16b}$$

and $g_{k_2 n}(s)$ corresponds to $f_{k_2 n}$ with $W'u_1(x_1, x_2, s)$ replacing $u_3(x_1, x_2, 0)$ in (16b).

There are clearly no true instabilities for $\nu \neq 0$; however, certain of the inhomogeneous terms in (16a) attain large values for finite times before decaying exponentially as $t \to \infty$. Substituting $u_1(x_1, x_2, s)$ from (14a) in (16a), we find terms of the typical magnitude

$$\exp\left(-\nu\left(k_2^2+n^2\right)t\right) \int_0^t \exp\left(\nu\left(n^2-p_n^2\right)s\right) ds\; u_1(p_m, k_2)(c_1V_1+c_2 2V_2) \sin nx_1\; e^{ik_2 x_2}, \tag{17}$$

where c_1 and c_2 are constants arising from the integrations in (16a), which are of the order of unity for $m \approx n$, except where they vanish by symmetry. For $n = m$, the argument $\left(n^2-p_n^2\right)$ is always positive, so that

$$\int_0^t \exp\left(\nu\left(n^2-p_m\right)^2\right) ds \geq t.$$

Choosing $n = 1$ and suppressing the spatial dependence in (17), we get the estimate

$$u_3 \approx t \exp\left(-\nu\left(k_2^2+1\right)t\right) Vu_1(0). \tag{18}$$

As $\nu \to 0$, the estimate (18) approaches the form of (4), the solution for $\nu = 0$.

Equation 18 attains a maximum for $t_o = \dfrac{1}{\nu\left(1+k_2^2\right)}$, so that

$$u_{3_{max}} \approx \frac{V}{\nu\left(1+k_2^2\right)}\, u_1 \approx \frac{R}{\pi^2} u_1(0), \tag{19}$$

showing that after a finite time the initial perturbation in u_1 appears as a term in u_3 and is amplified by a factor of the order of the Reynolds number $R = \dfrac{\pi^2 V}{\nu}$.

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References

1. C. C. Lin, The Theory of Hydrodynamic Stability (Cambridge University Press, London, 1955).