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MODE COUPLING IN THE MODIFIED BETATRON†

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The effects of quadratic nonlinearities in the single-particle equations of motion on electron orbits in the modified betatron are studied. Strong coupling of the two modes of betatron oscillation is found to occur for a particular value of the ratio B_θ/B_z unless the gradient in the field index takes a certain value. In general, the mode coupling appears to be quite harmless in the azimuthally symmetric case. When field or focusing errors are present the mode coupling reduces the effect on the orbit of the $l = 1$ orbital resonance but does *not* do so sufficiently so that the $l = 1$ resonance could be safely passed through in a practical device.

I. INTRODUCTION

For small displacements about a planar reference orbit, particle motion transverse to the toroidal field in the modified betatron may be represented as a linear superposition of two eigenmodes of motion: a "fast" mode, corresponding to gyration about the toroidal field lines and a "slow" mode, corresponding to an $\mathbf{F} \times \mathbf{B}$ drift motion, where here the force $\mathbf{F}$ is due to the weak-focusing betatron fields, space-charge forces, and induced (wall-image) fields. The linear theory of orbits in the modified betatron has been worked out in some detail[1–5] and will only be reviewed as needed here. In the present paper, we will mainly discuss the effect of quadratic nonlinearities on the motion.

Nonlinear terms in the equations of motion become important to consider if (1) displacements from the reference orbit become large, due, say, to the method of injection used or to the operation of an instability of some kind; (2) strong nonlinearities (e.g., large values of $\partial n/\partial r$) are present in the magnet design; or (3) the nonlinear term itself contains a resonant part. In the following we will illustrate two effects of quadratic nonlinearities on single-particle motion, viz. the amplitude dependence of the betatron frequencies and the exchange of energy between the oscillation modes under certain conditions. These conditions turn out to be analogous to the so-called Walkinshaw resonance[6] in accelerators without a toroidal magnetic field.[7–11] We will limit ourselves here to consideration of single-particle motion only, neglecting the effects of self-fields; the treatment here then will only be valid for fairly large values of γ in high-current devices, such that $\nu/\gamma \ll 1$, where ν is Budker's parameter.

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Four sections follow. In the first, we introduce some notation and sketch the derivation of the equations of motion to second order in displacements from and transverse velocities about the reference orbit, taken to be a circle in the symmetry plane. In the second section, the equations of motion are solved perturbatively and a condition for the generalized Walkinshaw resonance is obtained. Under this condition, we study the behavior of the betatron oscillations, giving a numerical example as illustration. As an interesting result, we find a particular value of field-gradient index for which the resonance ceases to have any effect on particle motion.

In these first two sections, the discussion will assume that all applied fields are azimuthally symmetric. In the presence of field perturbations, other orbital resonances may occur and it is interesting to ask whether the amplitude dependence of the betatron frequencies induced by the nonlinearities is sufficient to keep the oscillation amplitudes at finite, but tolerably small values. Although in general this is a difficult question, in the third section of this paper we will discuss a special simple case in which the Walkinshaw resonance coincides with both an integer and half-integer orbital resonance.

A final section summarizes these results and states some conclusions and conjectures.

II. THE EQUATIONS OF MOTION

The geometry of the modified betatron is shown in Fig. 1. We employ standard (r, θ, z) cylindrical coordinates. The exact equations of motion, using θ in favor of time for our

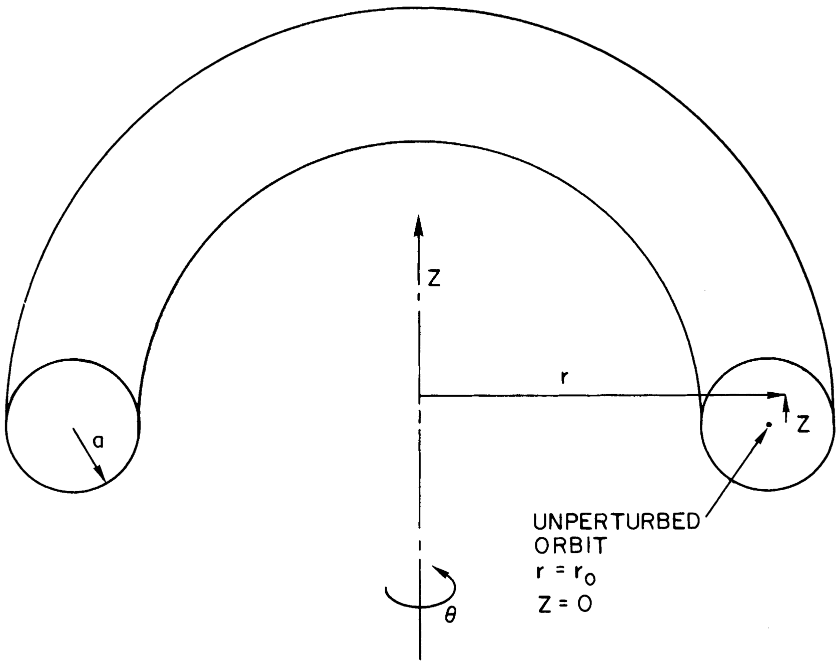

FIGURE 1 Geometry of the modified betatron.

independent variable, are written

$$\frac{d}{d\theta}[r'(r'^2 + r^2 + z'^2)^{-1/2}] = r(r'^2 + r^2 + z'^2)^{-1/2} + \lambda^{-1}(rB_z - z'B_\theta) \quad (1)$$

$$\frac{d}{d\theta}[z'(r'^2 + r^2 + z'^2)^{-1/2}] = \lambda^{-1}(r'B_\theta - rB_r), \quad (2)$$

where $\mathbf{B}$ is any suitable function of position (r, θ, z), $\lambda \equiv -mc^2\beta\gamma/e$, e and m are the magnitudes of the electron charge $(e > 0)$ and mass, β and γ are the usual relativistic factors, c is the speed of light and a prime $(')$ denotes $d/d\theta$.

We shall assume that B_r vanishes on the plane $z = 0$ and take all fields to be independent of θ. (The assumption of azimuthal symmetry will be relaxed in Section IV below.) We take the equilibrium orbit of a particle of relativistic factor γ_0 to be at $r = r_0$, $z = 0$, so

$$\lambda = -r_0 B_z(r_0, 0). \quad (3)$$

Let us now define the normalized coordinates $x \equiv (r - r_0)/r_0$ and $y \equiv z/r_0$. The vector potential is given correctly to third order by

$$A_\theta \cong r_0 B_{z0}\left[1 + \frac{1-n}{2}x^2 + \frac{n}{2}y^2 - \frac{n_2}{2}xy^2 + \frac{1}{6}(n + n_2 - 3)x^3\right], \quad (4)$$

where B_{z0}, n, and n_2 are constants. The corresponding fields are

$$B_r \cong -B_{z0}[n - n_2 x]y \quad (5)$$

$$B_z \cong B_{z0}\left[1 - nx + \frac{n_2}{2}x^2 + \frac{n - n_2}{2}y^2\right], \quad (6)$$

from which n_2 is identified as the second radial logarithmic derivative of B_z. The toroidal field is assumed to be given by

$$B_\theta = B_{\theta 0}/(1 + x) \cong B_{\theta 0}(1 - x + \cdots), \quad (7)$$

where $B_{\theta 0}$ is the value of the toroidal field at the reference orbit, $x = y = 0$.

Using the fields of Eqs. (5), (6), and (7) in the equations of motion (1) and (2) and keeping terms only of quadratic order gives the coupled equations

$$x'' + (1 - n)x = by' + \left(2n - 1 - \frac{n_2}{2}\right)x^2 - \left(\frac{n - n_2}{2}\right)y^2 + \frac{1}{2}(x'^2 - y'^2) \quad (8)$$

$$y'' + ny = -bx' - (2n - n_2)xy + x'y', \quad (9)$$

where $b \equiv B_{\theta 0}/B_{z0}$. These equations (8) and (9), are our starting points. In the following section we examine the behavior of an approximate solution to Eqs. (8) and (9) for various values of n, n_2, and b.

III. PERTURBATIVE SOLUTION OF EQUATIONS OF MOTION

In general, the quadratic terms in Eqs. (8) and (9) will be small, so we attempt to treat the equations perturbatively. Neglecting the nonlinear terms altogether one has the solution to the linear equations

$$\begin{pmatrix} x \\ \\ y \end{pmatrix} = A_f \begin{pmatrix} 1 \\ \dfrac{ib\nu_f}{\nu_f^{\,2} - n} \end{pmatrix} e^{i\nu_f\theta} + A_s \begin{pmatrix} 1 \\ \dfrac{ib\nu_s}{\nu_s^{\,2} - n} \end{pmatrix} e^{i\nu_s\theta} + c.c., \tag{10}$$

where A_f and A_s are complex numbers depending on particle initial conditions and the frequencies are given by

$$\nu_{f_s} = \left[\frac{b^2 + 1 \pm [(b^2+1)^2 - 4n(1-n)]^{1/2}}{2}\right]^{1/2}. \tag{11}$$

The subscripts f and s are used here and below to label the amplitudes and frequencies of the fast and slow oscillation modes. We will assume that the linear motion is stable, that is $n(1 - n) > 0$.

We may calculate the correction to Eq. (10) due to the nonlinear terms by inserting Eq. (10) in Eqs. (8) and (9) and resolving. The resulting equations will be inhomogeneous with various "driving" terms at the frequencies $2\nu_f, 2\nu_s, 0$, and $\nu_f \pm \nu_s$. Consequently, the nonlinear correction to Eq. (10) will remain small unless it happens that

$$\nu_f - \nu_s = \nu_s \tag{12}$$

the condition for which, from Eq. (11), is

$$b^2 = \frac{5}{2}[n(1-n)]^{1/2} - 1. \tag{13}$$

In the absence of a toroidal field, Eq. (13) is satisfied for $n = 0.2$ or 0.8, which we identify as the Walkinshaw resonance,[6] the consequences of which were first observed in cyclotrons.[12] We proceed to examine particle behavior on this resonance in the modified betatron.

On resonance, conventional perturbation theory fails and one must resort to some other method. A multiple "time" scale analysis of the problem gives a solution of the form (10) in which the complex amplitudes A_f and A_s are no longer strictly constant but vary slowly with θ; they are found to obey the equations

$$\frac{dA_s}{d\theta} = i\Gamma_1(n, n_2)A_f A_s^* \tag{14}$$

$$\frac{dA_f}{d\theta} = i\Gamma_2(n, n_2)A_s^2, \tag{15}$$

where $\Gamma_{1,2}$ are two real-valued functions of the field indices n and n_2,

$$\Gamma_1 = \frac{\nu_s^2 - n}{6\nu_s^3}\left[2n + 6\nu_s^2 + n_2\left(\frac{2\nu_s^2 + n}{\nu_s^2 - n}\right)\right]$$
$$= -4\left(\frac{\nu_s^2 - n}{\nu_f^2 - n}\right)\Gamma_2, \tag{16}$$

and where, on resonance, we have

$$\nu_f^2 = 4\nu_s^2 = 2(n(1 - n))^{1/2}. \tag{17}$$

[An asterisk denotes complex conjugate in Eq. (14).]

The question of orbital stability on resonance is thus reduced to the question of the behavior of the mode amplitudes in Eqs. (14) and (15).

The equations (14) and (15) may be completely solved in a straightforward manner; the solution is obtained and discussed in the Appendix. To settle the stability issue, however, it is sufficient to note that there is a simple integral of motion

$$\frac{1}{2}\Gamma_1|A_f|^2 + \frac{1}{2}\Gamma_2|A_s|^2 \equiv D/\Gamma_1, \tag{18}$$

and consequently the motion is necessarily bounded if $\Gamma \equiv \Gamma_1\Gamma_2 \geqslant 0$ which, is in fact true for all n and n_2, as follows from Eq. (16). On this resonance, energy is simply exchanged back and forth between the fast and slow modes of motion.

Although we have argued that particle motion is bounded on this resonance, we have not in fact specified a bound or showed that the bound is acceptable, in terms of some machine aperture. One might conjecture, from Eq. (18), that if one of $\Gamma_{1,2}$ were significantly larger than the other, then transfer of energy from the more "stiff" (larger Γ coefficient) mode to the less "stiff" mode would result in increasing particle oscillation amplitude. Hence one would be concerned if, from Eq. (16), either $\nu_s^2 = n$ or $\nu_f^2 = n$. It follows from Eqs. (17) and (18), however, that this can occur only for $b = 0$, $n = 0.2$ or 0.8. If n is chosen so that b is O(1) when the resonance is crossed, then Γ_1 and Γ_2 are of the same order of magnitude and one expects this resonance to be quite harmless. For specific initial conditions, it is possible to find a bound by calculating the turning point of a certain particle-in-a-well problem, as shown in the Appendix.

Curiously, one can render this resonance completely inoperative for any particular n by choosing n_2 so that Γ_1 and Γ_2 both vanish. From Eqs. (16) and (17) this value is found to be

$$n_2 = \hat{n}_2 \equiv \frac{1}{2}\left[\frac{7n - 3 + 4[(1 - n)n]^{1/2}}{1 + [(1 - n)/n]^{1/2}}\right]. \tag{19}$$

Choosing this value ensures that the mode amplitudes remain constant when passing through the "exchange" resonance.

We proceed to illustrate some of these results using a simple single-particle numerical orbit integration. The algorithm includes the fields of Eqs. (5) to (7) but does not use an expansion of the force or acceleration. Figures 2 and 3 show the solutions to Eqs. (14–15) and (1–2) respectively for the case $n = 0.5, n_2 = \hat{n}_2 = 0.625, b = 0.5$. The mode amplitudes are strictly constant, no exchange occurs, and the particle-orbit projection retraces itself in a stable manner over and over again. We contrast this case

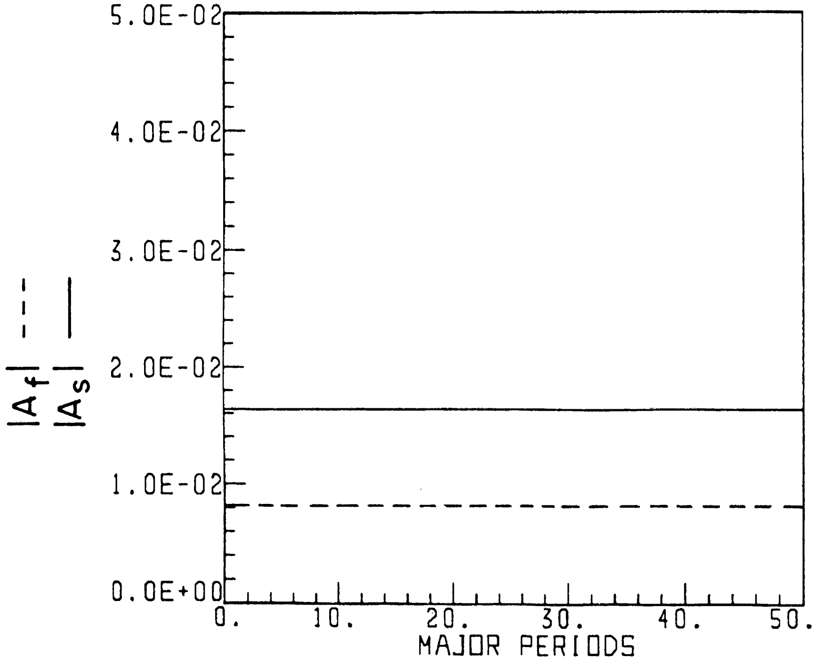

FIGURE 2 $|A_f|$, $|A_s|$ vs. major periods (A major period is a change of θ by 2π) $n = \frac{1}{2}$; $n_2 = \frac{5}{8}$. Initial ($\theta = 0$) values of $A_f = 5.7735 \times 10^{-3}(1 - i)$ and $A_s = 2A_f^*$ correspond to those of a particle initially at $x = y = 0.02$ with zero transverse velocity. The same initial values are used in all subsequent figures.

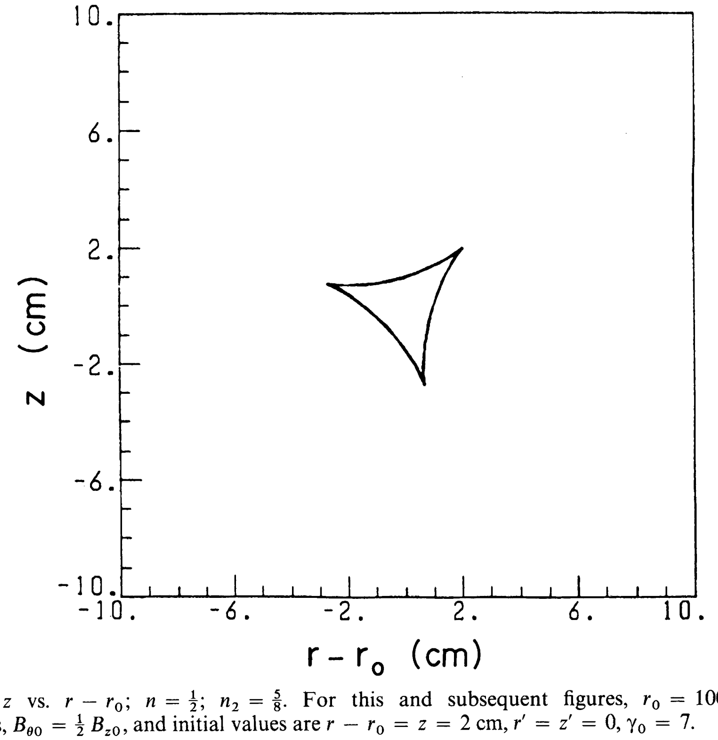

FIGURE 3 z vs. $r - r_0$; $n = \frac{1}{2}$; $n_2 = \frac{5}{8}$. For this and subsequent figures, $r_0 = 100$ cm, $B_{z0} = 118.092$ gauss, $B_{\theta 0} = \frac{1}{2} B_{z0}$, and initial values are $r - r_0 = z = 2$ cm, $r' = z' = 0, \gamma_0 = 7$.

with that illustrated in Figs. 4 and 5 for which the parameters are again $n = 0.5$ and $b=0.5$ but now with $n_2=0$. Now the mode amplitudes oscillate; one rises while the other falls in order to conserve D [Eq. (18)]. The oscillation period, from Eq. (A6) in the Appendix, is 46.6 major periods for the particular initial conditions chosen. The particle-orbit projection now simply (and harmlessly) rotates slowly counter-clockwise.

Our conclusion is that in the case of azimuthally symmetric fields the generalized Walkinshaw resonance is quite harmless in the modified betatron. The exchange of energy between fast and slow modes is expected to cause no major changes in the beam dimensions. When azimuthal field variations are present, however, the situation changes dramatically, due to a coincidence described in the next section.

IV. A TRIPLE COINCIDENCE RESONANCE

When $n = \frac{1}{2}$ (the case illustrated in Figs. 2 through 5) the values of the tunes ν_f and ν_s at the exchange resonance of Eq. (13) are $\nu_f = 1$, $\nu_s = \frac{1}{2}$; therefore in the presence of field and focusing errors the generalized Walkinshaw resonance coincides with an integer and a half-integer orbital resonance. This triple coincidence allows us to study in detail in this special case the effect of mode coupling (and the amplitude dependence of the betatron frequencies) on the orbit at an integer and half-integer resonance. Though the restriction to $n = \frac{1}{2}$ is necessary for there to be a true coincidence, if n is near but not exactly $\frac{1}{2}$, the three resonances will be "nearby" and will occur nearly simultaneously and the analysis below should still hold in an approximate way.

At the triple coincidence resonance, the mode-evolution equations become

$$\frac{dA_s}{d\theta} = i\Gamma_1 A_f A_s^* + \epsilon_1 A_s^* \tag{20}$$

$$\frac{dA_f}{d\theta} = i\Gamma_2 A_s^2 + \epsilon_2 + \epsilon_3 A_f^*. \tag{21}$$

The ϵ's in Eqs. (20) and (21) are complex constants proportional to certain Fourier coefficients in the expansions of the fields and their gradients; specifically, ϵ_1 is due to an $l = 1$ term in the field gradient, leading to a half-integer resonance, ϵ_2 is due to an $l=1$ term in the field, leading to an integer resonance, and ϵ_3 is due to an $l=2$ term in the field gradient, leading to a "2-halves" integer resonance. Were they present alone (i.e. with no nonlinearity) in Eqs. (20) and (21), the field-imperfection terms would lead to the usual linear or exponential growth characteristic of integer or half-integer resonances. In the presence of mode coupling, the situation is much less clear. Since, as they stand, Eqs. (20) and (21) cannot be solved analytically, we must in general resort to numerical integration. First let us comment on what we might expect to see in the solution.

The field imperfection terms in Eqs. (20) and (21) act, roughly speaking, as source terms, pumping energy from longitudinal to transverse motion. If this energy flow continues, and if there is no mechanism to return this energy to longitudinal motion, the result is disastrous—a linear orbital resonance. Non-linearities, however, can shift the betatron frequency of the resonant particle off resonance for some finite amplitude of betatron oscillation. (The quantity $(n_2 - \hat{n}_2)$ is presumably a measure of the frequency shift induced by a given amplitude oscillation.) In practice, though, one

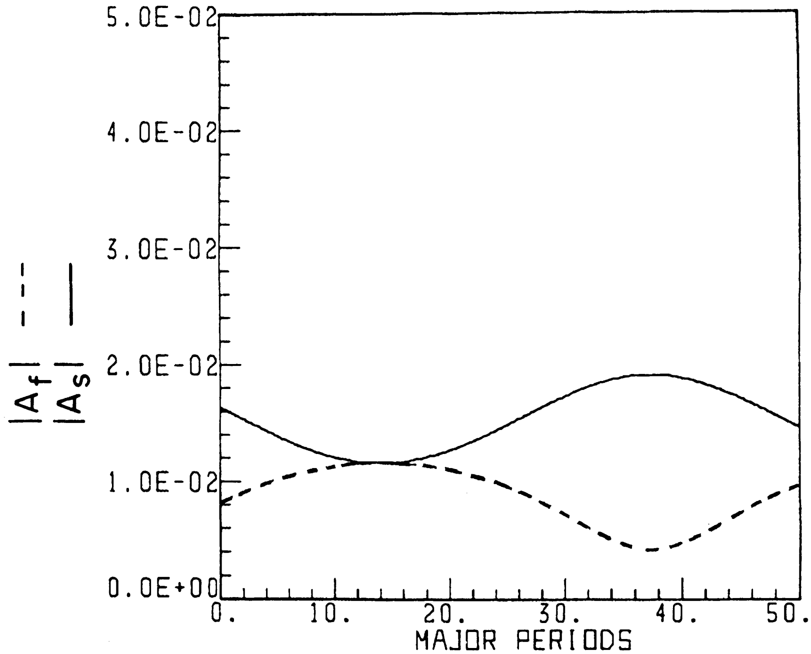

FIGURE 4 $|A_f|$, $|A_s|$ vs. major periods; $n = \frac{1}{2}$; $n_2 = 0$.

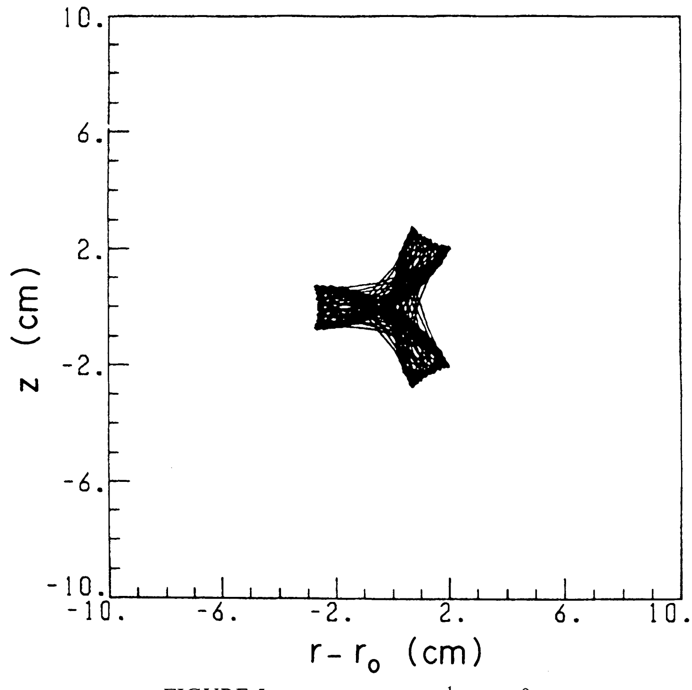

FIGURE 5 z vs. $r - r_0$; $n = \frac{1}{2}$; $n_2 = 0$.

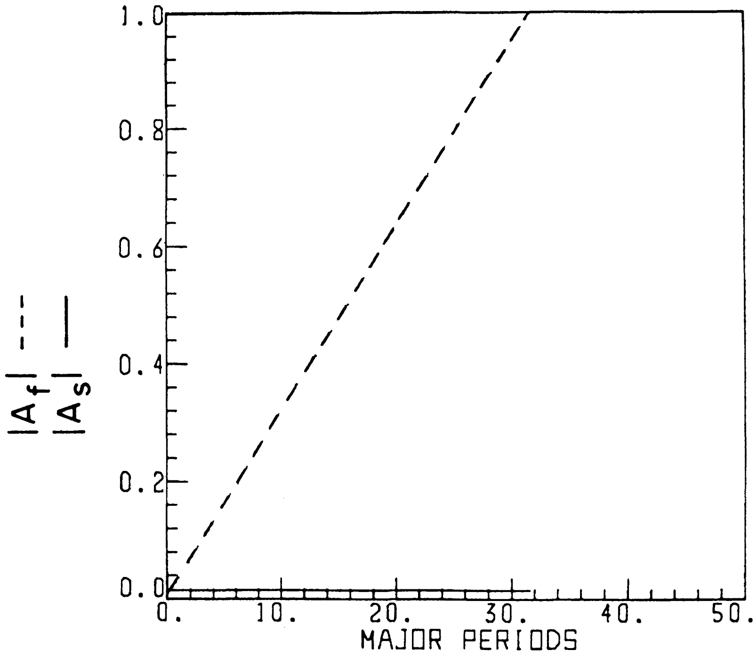

FIGURE 6 $|A_f|$, $|A_s|$ vs. major periods; $n = \frac{1}{2}$; $n_2 = \frac{5}{8}$; $\epsilon_1 = \epsilon_3 = 0$; $\epsilon_2 = 5 \times 10^{-3}$.

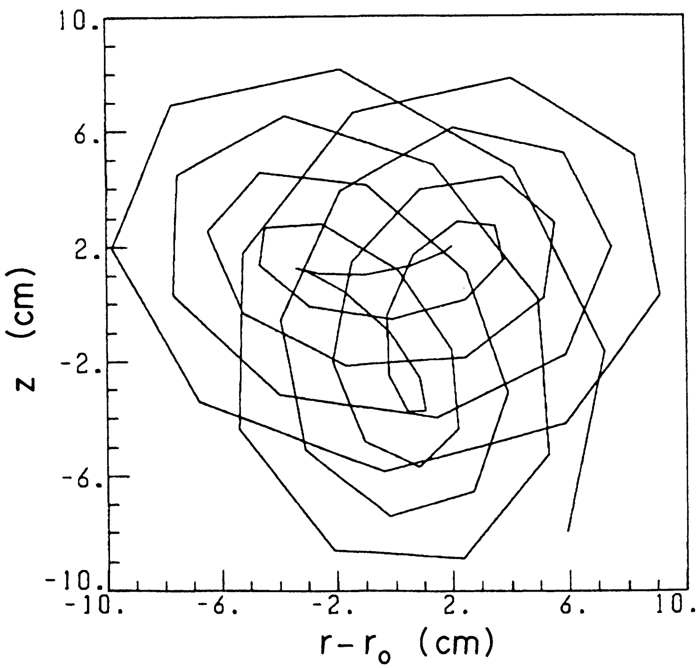

FIGURE 7 z vs. $r - r_0$; $n = \frac{1}{2}$; $n_2 = \frac{5}{8}$; $\delta B = 0.5$ gauss in $l = 1$ Fourier mode. Particle leaves plot area after ~ 9 major periods.

cannot say a priori how much frequency shift will be sufficient to terminate the growth of the resonant mode. A numerical study seems to be essential.

For a numerical example, we will examine the effect of the nonlinear terms in Eqs. (20) and (21) on an integer resonance; that is, in the following we shall take $\epsilon_1 = \epsilon_3 = 0$. Cases have been examined numerically for various other combinations of values for the ϵ's with no major differences appearing in the results.

In Figs. 6 and 7, we illustrate the mode amplitudes and orbit projection for a pure integer resonance with no mode coupling ($n_2 = \hat{n}_2 = 0.625$, $\epsilon_2 = 0.005$). The (resonant) fast mode amplitude grows linearly without limit; the (decoupled) slow mode stays at a fixed, small value. The particle-orbit size (Fig. 7) consequently grows continuously.

Turning on the mode coupling changes the behavior of the mode amplitudes dramatically but has little apparent effect on the particle orbit, which still appears to grow to intolerable size. This is illustrated in Figs. 8 and 9 where we see that the mode amplitudes grow to a certain size and then turn over—presumably a reflection of the detuning of the resonance due to the frequency shift. The "turnover", however, is at extremely large amplitudes (recall that the mode amplitudes are normalized to the radius of the device, r_0). Therefore the particle motion appears to be relatively unaffected, practically speaking, by changing n_2 from $\frac{5}{8}$ to 0. This result does suggest, though, that by increasing $|n_2 - \hat{n}_2|$ we might reduce the resonant response to a tolerable value.

Figures 10–11, 12–13, and 14–15 show our results for $n_2 = -1$, -4, and -10 respectively. We see that as $|n_2 - \hat{n}_2|$ is increased, the mode-turnover amplitude is reduced and the particle orbit becomes somewhat more compact, staying within the plot boundaries for significantly longer times. (Even so, the transverse orbit size is rather large, even for the largest values of $|n_2 - \hat{n}_2|$ we have tried.)

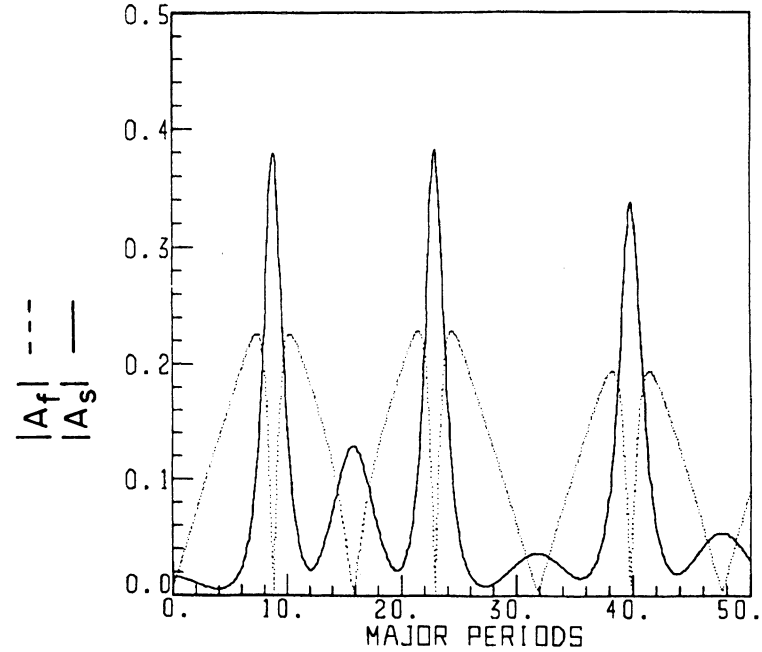

FIGURE 8 $|A_f|$, $|A_s|$ vs. major periods; $n = \frac{1}{2}$, $n_2 = 0$; $\epsilon_1 = \epsilon_3 = 0$; $\epsilon_2 = 5 \times 10^{-3}$.

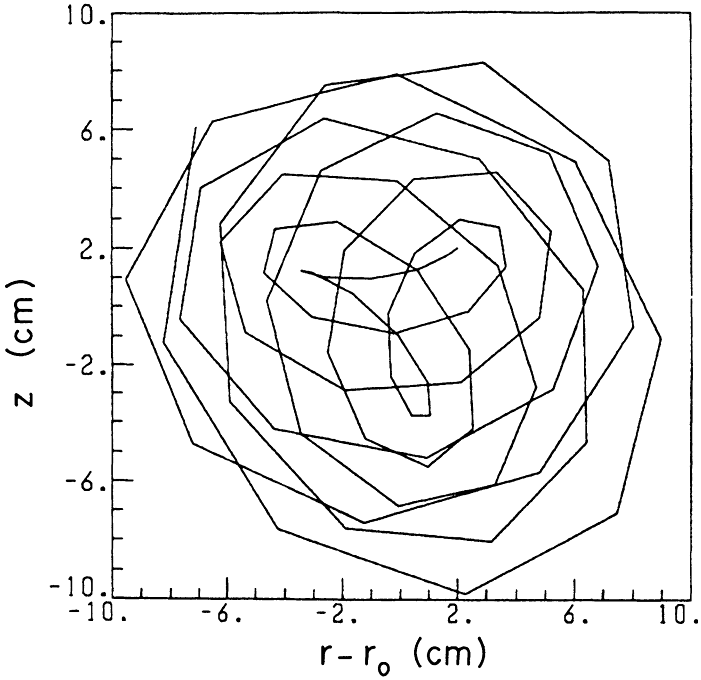

FIGURE 9 z vs. $r - r_0$; $n = \frac{1}{2}$; $n_2 = 0$; $\delta B = 0.5$ gauss. Particle leaves plot area after ~ 10 major periods.

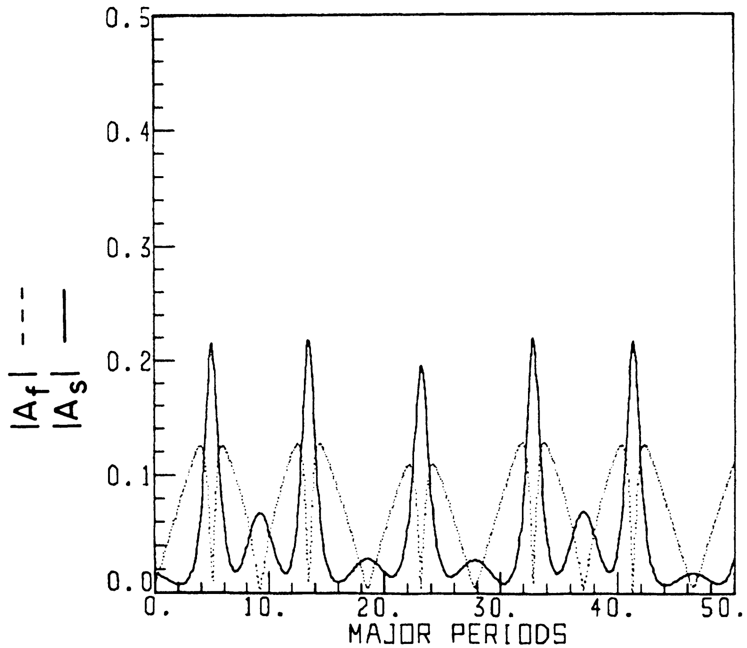

FIGURE 10 $|A_f|$, $|A_s|$ vs. major periods; $n = \frac{1}{2}$; $n_2 = -1$; $\epsilon_1 = \epsilon_3 = 0$; $\epsilon_2 = 5 \times 10^{-3}$.

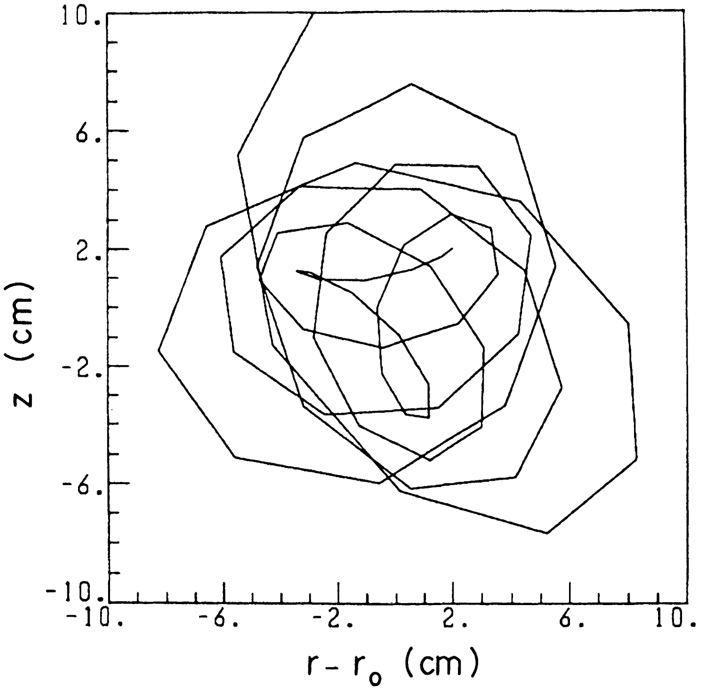

FIGURE 11 z vs. $r - r_0$; $n = \frac{1}{2}$; $n_2 = -1$; $\delta B = 0.5$ gauss. Particle leaves plot area after ~ 8 major periods.

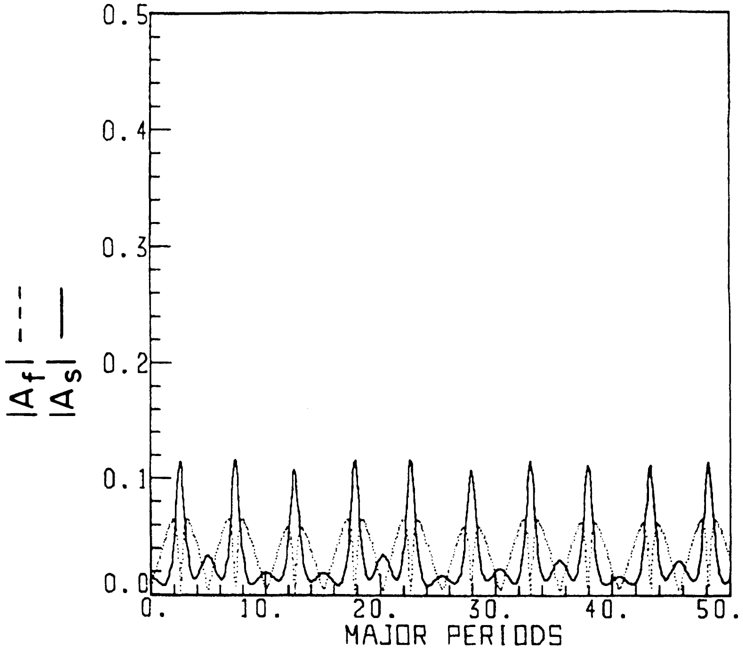

FIGURE 12 $|A_f|$, $|A_s|$ vs. major periods; $n = \frac{1}{2}$; $n_2 = -4$; $\epsilon_1 = \epsilon_3 = 0$; $\epsilon_2 = 5 \times 10^{-3}$.

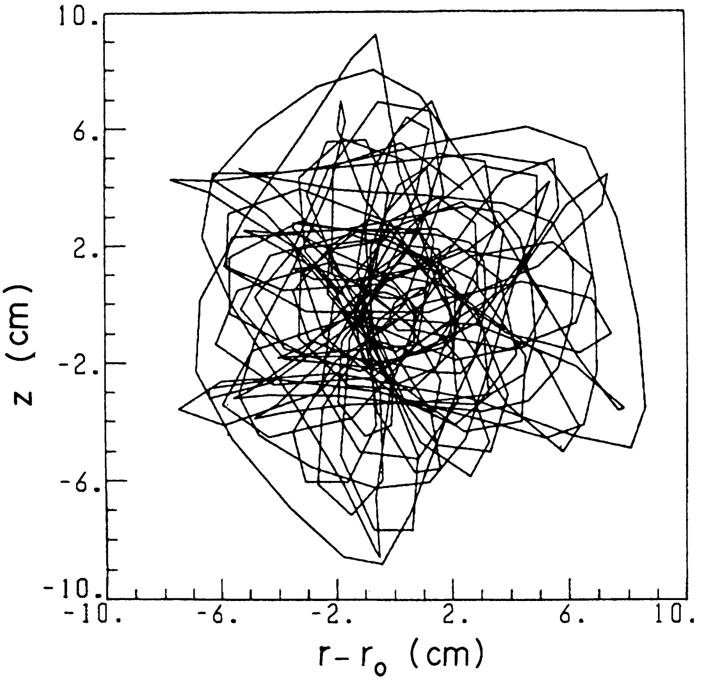

FIGURE 13 z vs. $r - r_0$; $n = \frac{1}{2}$; $n_2 = -4$; $\delta B = 0.5$ gauss. Particle remains in plot area for 50 major periods.

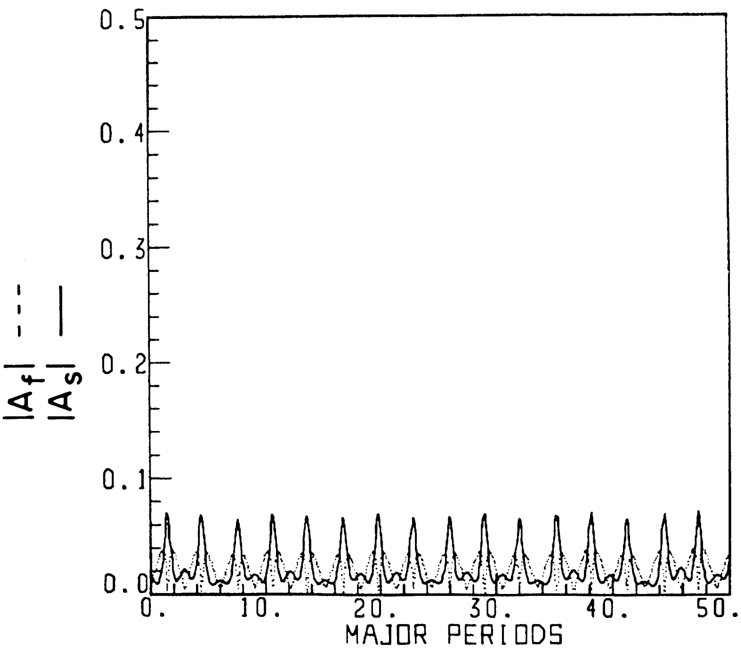

FIGURE 14 $|A_f|$, $|A_s|$ vs. major periods; $n = \frac{1}{2}$; $n_2 = -10$; $\epsilon_1 = \epsilon_3 = 0$; $\epsilon_2 = 5 \times 10^{-3}$.

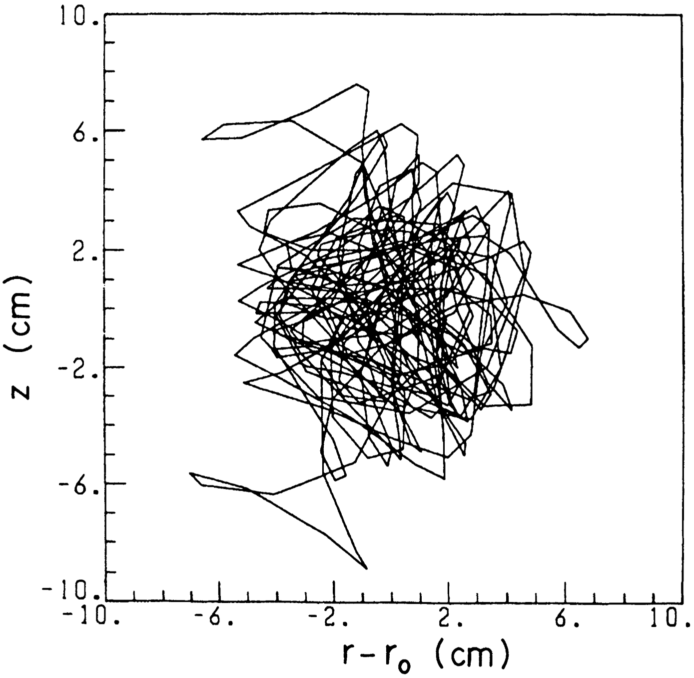

FIGURE 15 z vs. $r - r_0$; $n = \frac{1}{2}$; $n_2 = -10$; $\delta B = 0.5$ gauss. Particle remains in plot area for 50 major periods.

These results suggest that to stabilize the $l = 1$ resonance, a significant nonlinearity (large value of n_2) could be intentionally introduced in the betatron field. One must be careful in drawing this conclusion, however, because such a nonlinearity has well-known adverse effects, among them a sensitivity of the behavior of the orbit to initial conditions; that is, only some special class of particles may be confined, while others are lost. In addition, if n_2, which is effectively the radial derivative of n, is very large, it then becomes difficult to keep n itself within the stable range $0 \leqslant n \leqslant 1$ everywhere within the aperture. Consequently we conclude that, as a practical matter, it is best *not* to rely on nonlinearities to stabilize the $l = 1$ resonance in the modified betatron and to design the machine with a flat radial-index profile. Avoidance of this resonance as well as other low-order resonances, which then becomes the only reasonable experimental alternative, is possible in principle by accelerating with constant b (i.e., $B_{\theta 0} \propto B_{z0}$), thereby keeping the tunes fixed[5]—except for the tune shift due to space charge, which affects the fast-mode tune only very slightly for large B_θ; the slow-mode tune can generally be chosen to be very small ($\sim$0.2 to 0.3) for all time.

V. SUMMARY AND CONCLUSIONS

We have examined the effects of mode coupling on single-particle orbits in the modified betatron. We find that a generalization of the Walkinshaw (exchange) resonance can occur for any value of field index in the range $0.2 \leqslant n \leqslant 0.8$, but that its effect on particle orbits in general is quite modest (Figs. 2–5) and may be rendered completely ineffective by a special choice of the field-gradient index, Eq. (19).

When n is near $\frac{1}{2}$, the exchange resonance coincides with both integer and half-integer resonances. An examination of orbit behavior at this triple coincidence shows that, as a practical matter, the amplitude-dependent frequency shift in the betatron oscillation due to mode coupling is *not* sufficient to stabilize the $l = 1$ integer resonance (though presumably, as in the case of accelerators not employing toroidal fields, higher-order resonances will be subject to nonlinear stabilization[8]). This fact makes it advisable to allow, in the design of an experiment, for acceleration with constant or nearly constant ratio $B_{\theta 0}/B_{z0}$, thereby holding the tunes approximately fixed in time.

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APPENDIX

In this Appendix, we discuss the solution to Eqs. (14) and (15) in the text. Writing

$$A_f = a_f \exp(i\phi_f) \qquad A_s = a_s \exp(i\phi_s), \tag{A1}$$

where $a_{f,s}$ and $\phi_{f,s}$ are real we find in a straightforward way an equation for $\rho \equiv \frac{1}{2}a_s^2$

$$\frac{1}{2}\rho'^2 + V(\rho) = 0, \tag{A2}$$

where

$$V(\rho) = 4\Gamma\rho^3 - 4D\rho^2 + \frac{1}{2}C^2,$$

$$\Gamma = \Gamma_1\Gamma_2,$$

$$C = \Gamma_1 a_f a_s^2 \cos(\phi_f - 2\phi_s) = \text{constant},$$

and the constant D is defined in the text in Eq. (18). The other quantities are given in terms of ρ by

$$a_s = (2\rho)^{1/2} \tag{A3}$$

$$\phi_s' = C/(2\rho) \tag{A4}$$

$$A_f = \frac{1}{2\rho\Gamma_1}(C - i\rho')\exp(2i\phi_s). \tag{A5}$$

These expressions hold for $\rho > 0$. If $\rho = 0$ at $\theta = 0$, say, then, from Eqs. (14) and (15) $A_s \equiv 0$ and A_f remains fixed for all $\theta > 0$. The modal frequency shifts are given directly by Eq. (A4) for the slow mode and may be obtained from Eq. (A5) for the fast mode.

It may be shown that $V(\rho)$ has one negative and two positive roots. Denoting these by $\rho_{1,2,3}$, with $\rho_1 > \rho_2 > 0 > \rho_3$, we find that the exchange period [period of $\rho(\theta)$] is given by

$$\left(\frac{2}{\Gamma(\rho_1 - \rho_3)}\right)^{1/2} K(m), \tag{A6}$$

where $m = (\rho_1 - \rho_2)/(\rho_1 - \rho_3)$ and where K is the usual complete elliptic integral.[13]

The special cases $C = 0$ and $\Gamma = 0$ lead to motion of infinite period and $\rho =$ constant, respectively.