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ELECTRON—ION INSTABILITIES IN A HIGH-CURRENT MODIFIED BETATRON

WALLACE M. MANHEIMER

Code 4790, Naval Research Laboratory Washington, D.C. 20375 U.S.A.

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The effect of ion-resonance and ion-streaming instabilities on the performance of a modified betatron is evaluated. It is shown that diffuse electron and ion profiles greatly reduce the growth rate of these instabilities. Ways of optimizing the performance of the accelerator in the presence of these instabilities are discussed.

I. INTRODUCTION

A new concept in high current particle accelerators is the modified betatron.[1–4] The idea here is to use a toroidal magnetic field to stabilize the particle orbits while the beam energy is relatively low and the self-fields are important. Once the particle has accelerated to high energy, the toroidal field is no longer necessary for orbit stability and can be removed. A high-current modified betatron is currently being planned at the Naval Research Laboratory. It is hoped to accelerate a 10 kA beam from $\gamma = 7$ to $\gamma = 100$ in a toroidal field of several kG in a betatron with a 1-m major radius.

One possible problem with this device is that a small ion contaminant could excite an ion-resonance[5,6] or two-stream instability.[7] Both of these instabilities have their origin in a slow mode (phase velocity less than azimuthal or axial streaming velocity) interacting with a lossy species (the ions). In the ion-resonance instability, there is no axial structure, but the mode rotates slower than the azimuthal velocity of the electrons. In the ion-streaming instability, there is no azimuthal structure, but the mode has an axial velocity slower than the electron streaming velocity. The ion contaminant could arise from either ionization of background gas, or else from ions pulled off the wall on beam injection. The ion-resonance instability is particularly worrisome, because it is now reasonably well established that this instability was responsible for the disruptions observed in both HIPAC,[8,9] a toroidal electron-cloud experiment and SPAC II, a slightly relativistic electron ring.[10] The two-stream instability was not observed in HIPAC, but one would not expect it because there is no axial streaming. In SPAC II, it was not observed despite the axial streaming. However, this might be due to the fact that the ion-resonance instability is seen, and it might be the stronger of the two. If the ion-resonance instability is stabilized some way, the ion two-stream instability could also pose a serious problem to the operation of the modified betatron.

Simple theories of both instabilities are usually done for cold homogeneous fluids. As such, these theories are extremely pessimistic as regards the operation of the high-current modified betatron. For instance, for the ion-resonance instability of a cylindrical nonrelativistic beam, an electrostatic perturbation having no axial structure, but having azimuthal structure $\exp(il\theta)$ and having beam radius r_b much less than the wall radius a, the dispersion relation, as derived in Refs. 5 and 6 is

$$1 + \frac{\omega_{pe}^2}{2\Omega_c(\omega - l\omega_e)} - \frac{\omega_{pi}^2}{2(\omega - l\omega_i)(\omega - (l-2)\omega_i)} = 0. \tag{1}$$

Here $\omega_{pe(i)}$ is the electron (ion) plasma frequency, Ω_c is the electron cyclotron frequency, ω_e is the diocotron frequency $\omega_{pe}^2/2\Omega_{ce}$ and ω_i is the ion rotation frequency, $\omega_i = (m/2M)^{1/2}\omega_{pe}$ for unmagnetized ions. Equation (1) shows that if $\omega \approx (l-1)\,\omega_e \approx l_{\omega i}$, the plasma is unstable with growth rate $\omega_{pi}(\omega_e/2\omega_i)^{1/2}$. Modes between $2 \leqslant l \leqslant \infty$ are unstable for densities or γ's in the range

$$\frac{1}{2} \leqslant \frac{\Omega_c}{\omega_{pe}}\left(\frac{m}{2M}\right)^{1/2} < 1. \tag{2}$$

and the growth time is of order 3 nanoseconds.

Now let us consider the ion-streaming instability. For an infinite homogeneous plasma, the dispersion relation for an electrostatic mode having wave number k parallel to the streaming velocity and $k_\perp$ perpendicular, as derived in Ref. 7 is

$$1 = \frac{(\omega_{pe}^2/\gamma)\dfrac{k_\perp{}^2}{k_\perp{}^2 + k^2}}{(\omega - kv_z)^2 - \Omega_c{}^2/\gamma^2} + \frac{(\omega_{pe}^2/\gamma^3)\dfrac{k^2}{k_\perp{}^2 + k^2}}{(\omega - kv_z)^2} + \frac{\omega_{pi}^2}{\omega^2} \tag{3}$$

Let us first consider an interaction with the parallel motion [the second term on the right of Eq. (3)]. Since the beam has radius r_b, $k_\perp r_b \gtrsim 1$. Then assuming $k \ll k_\perp$, Eq. (3) easily shows there is no parallel instability unless $r_b > \gamma^{3/2}c/\omega_{pe}$. Since the NRL modified betatron will be concerned with much smaller radius beams, there should be no two-stream instability driven by the purely parallel motion.

Now consider the perpendicular motion, the first term on the right of Eq. (3). It is not difficult to show that this predicts instability for

$$\frac{\Omega_c}{\gamma} < kv_z < \left(\frac{\Omega_c{}^2}{\gamma^2} + \frac{\omega_{pe}^2}{\gamma}\right)^{1/2}, \tag{4}$$

with growth rate $\sim (\omega_{pi}^2\omega_{pe}^2/\gamma\Omega_c)^{1/3}$. For parameters of the NRL modified betatron, this growth time is still of order 3 nanoseconds, and the beam is now unstable at all densities and energies.

These conclusions are clearly very pessimistic. The predicted growth times are much too fast to control in any way, and the ion-streaming instability is predicted to be unstable at some k no matter what the density or energy. However, the assumed cold-fluid, sharp-boundary, rigid-rotor model is tremendously destabilizing in that all ions are simultaneously resonant with the electron oscillation.

The purpose of this paper is to examine the effect of a diffuse profile on these instabilities. It seems extremely unlikely that the electron beam will have a sharp profile. When the ions are produced by ionization of the background gas, they will almost surely have nearly zero velocity, and therefore nearly zero angular momentum. Since the ion oscillation frequency is much larger than the ion cyclotron frequency, they are effectively unmagnetized. Therefore the ions simply oscillate through the center of the beam. They have zero axial velocity and zero angular momentum, but a large spread in energy, since they are produced at many different potential energies. Since for a diffuse profile, ions of different energies have different frequencies of oscillation, only a small fraction of ions can resonate with the slow electron wave. This can greatly reduce the growth rate.

The main purpose of this paper is calculate the effect of the diffuse ion and electron profiles on the growth rate of the modes. [In fact, the same technique as used here would give Eq. (1) and Eq. (3) in the appropriate limits.] We do find that the effects of a diffuse ion profile, as well as electromagnetic effects on the electrons can reduce the growth rates by several orders of magnitude. It turns out that it may be possible to completely stabilize the ion-resonance instability. It is not possible to stabilize the ion-streaming instability, but it is possible to slow it down so much as to make it feasible to accelerate the beam through the unstable region before the mode grows appreciably.

Section II describes the assumed equilibrium for both electrons and ions, Section III works out the theory of the ion-resonance instability. Section IV works out the theory of the ion-streaming instability. Section V outlines the conclusions. The Appendix gives some of the mathematical details not included in the text.

II. THE EQUILIBRIUM

We consider the equilibrium to consist of a cylinder of strongly magnetized $\mathbf{B} = Bi_z$ electrons with diffuse radial profile. The ion density will be assumed to be smaller than the electron density, so that the ions do not contribute to the equilibrium self-fields. Since the ion oscillation frequency in the self-fields is much larger than the ion cyclotron frequency for parameters of the NRL high-current modified betatron, the ions will be considered unmagnetized.

To calculate the electron equilibrium, use the electron density and momentum equations coupled to Poisson's and Ampere's equations

$$\frac{1}{r}\frac{\partial}{\partial r} r n_e v_r = 0 \tag{5a}$$

$$-\nabla\phi + \frac{\mathbf{v} \times \mathbf{B}}{c} = 0 \tag{5b}$$

$$\nabla^2\phi = 4\pi n_e e \tag{5c}$$

$$\nabla^2 A_z = 4\pi n_e e \frac{v_z}{c}, \tag{5d}$$

where we have neglected electron inertia, which means that the electrons rotate in the slow mode and are far below Brillouin flow,[6] and have also neglected the self B_z field generated by the rotation of the beam. Equations (5a–d) can be satisfied by specifying $n_e(r)$ and the z component of velocity V_z (assumed constant). Then Eq. (5c) can be integrated for ϕ. Since A_z obeys the same boundary conditions as ϕ for our assumed cylindrical model Eq. (5) gives the result $A_z = (v_z/c)\phi$. This then can be used in Eq. (5b) to solve for the rotation velocity of the electrons

$$v_\theta = \frac{c}{B\gamma^2}\frac{\partial\phi}{\partial r}, \tag{6}$$

where as usual, $\gamma = (1 - (v_z/c)^2)^{-1/2}$. We have further assumed that the relativistic γ is determined only v_z and not v_θ; that is $(v_\theta/c)^2 \ll 1 - (v_z/c)$.

For a monotonically decreasing density profile, the electron rotation frequency v_θ/r is a decreasing function of r. Assuming that near $r = 0$

$$n_e(r) = n_e(1 - r^2/r_\alpha^2), \tag{7}$$

where $r_\alpha^{-2} = -1/2\, n_e''(0)/n_e(0)$, we find that near the origin

$$E_r = -4\pi e n_e\left(\frac{r}{2} - \frac{r^3}{4r_\alpha^2}\right) \tag{8a}$$

$$\phi = 4\pi e n_e\left(\frac{r^2}{4} - \frac{r^4}{16r_\alpha^2}\right) \tag{8b}$$

$$\frac{v_\theta}{r} = \frac{4\pi e c n_e}{B\gamma^2}\left(\frac{1}{2} - \frac{r^2}{4r_\alpha^2}\right). \tag{8c}$$

Notice that one important effect of the diffuse radial profile is to generate a spread in the electron angular rotation frequency v_θ/r.

We now turn to a discussion of the ion equilibrium. Let us assume that once the beam is in place, ions are created by ionization of gas, which might be either background gas, or gas pulled off the wall upon beam injection. Since the ion oscillation frequency in the electric field is large compared with the ion cyclotron frequency, the ions will be considered unmagnetized. In addition the ion will be assumed to be created with no kinetic energy, so that when it is created, all its energy is potential energy, and also, it has zero angular momentum. Since a spread in parallel velocity and a spread in angular momentum will almost certainly have a stabilizing effect on any electron-ion streaming instability, the configuration we examine will give the most pessimistic results concerning stability. The equation of motion for an ion of mass M trapped near the origin and with zero angular momentum is

$$\frac{d^2r}{dt^2} = -\omega_I^2\left(1 - \frac{r^2}{2r_\alpha^2}\right) r \tag{9}$$

where $\omega_I^2 = 4\pi n_e e^2/2M$. The total energy of the ion, denoted by H, is given by

$$H = \frac{1}{2} M v_r^2 + \frac{M\omega_I^2 r^2}{2} - \frac{M\omega_I^2 r^4}{8r_\alpha^2}. \tag{10}$$

It is not difficult to show that for an ion trapped near the origin, the turning point and oscillation frequency as a function of H are given by

$$r^2(H) = \frac{2H}{M\omega_I^2} + \frac{H^2}{M^2\omega_I^4 r_\alpha^2} \tag{11a}$$

$$\omega(H) = \omega_I\left(1 - \frac{3H}{4M\omega_I^2 r_\alpha^2}\right). \tag{11b}$$

As the ions are produced, they oscillate at different frequencies through the center. We assume that after many such oscillations (tens of nanoseconds in the NRL modified-betatron experiment), the time-dependent phase mixes away and the ion distribution function becomes constant in time. That is, the distribution becomes a function of the constants of motion, H, L (angular momentum) any v_z. We assume that the v_z part simply decouples, so $f(H, L, v_z) \equiv f(H, L)\,\delta(v_z)$. Then the number of ions per unit length with energy between H and $H + dH$ and L and $L + dL$ is the number density of ions produced at radius $r(H)$ times the volume, so

$$N(H, L)\,dH\,dL = \frac{2\pi n_i(r(H))\delta(L)\,dH\,dL}{e\left|\dfrac{d\phi}{dr}\right|_{r=r(H)}}, \tag{12}$$

where $dH/e|d\phi/dr|_{r=r(H)}$ is the thickness of a cylindrical shell corresponding to ions produced with energies between H and $H + dH$. However, the quantity N is also given by an integral over the distribution function over the allowed range of L and H,

$$\begin{gathered} N(H, L) = \int d^2r \int dv_r'\,dv_\theta'\,f_i. \\ H < H' < H + dH \\ L < L' < L + dL. \end{gathered} \tag{13}$$

Using the fact that

$$dv_r' dv_\theta' = \frac{dL'\,dH'}{(2M^3)^{1/2} r\left(H' - \dfrac{L'^2}{2Mr^2} - e\phi\right)^{1/2}}, \tag{14}$$

it is a simple matter to show that

$$f_i(L, H) = \frac{n_i(r(H))r(H)\delta(L)(2M^3)^{1/2}}{e\left|\dfrac{\partial\phi}{\partial r}\right|_{r=r(H)}\left(\dfrac{2}{M}\right)^{1/2}\dfrac{\pi}{2\omega(H)}}, \tag{15}$$

where we have used the fact that

$$\int_0^{r(H)} \frac{dr}{(H - e\phi)^{1/2}} = \left(\frac{2}{M}\right)^{1/2} \frac{\pi}{2\omega(H)}.$$

Using the expressions for $\phi(r)$, $r(H)$ and $\omega(H)$ from Eqs. (8 and 11), it is not difficult to show that for ions trapped near $r = 0$ (that is, small H),

$$f(L, H) \equiv g(H)\delta(L) = \frac{2n_i(r(H))\left(1 + \dfrac{1}{4}\dfrac{H}{M\omega_i^2 r_\alpha^2}\right)M\delta(L)}{\pi\omega_I}. \tag{16}$$

Notice that the quantity multiplying $n_i(r(H))$ in Eq. (16) is an increasing function of H. This has to do with the fact that the restoring force is weaker than that of a harmonic oscillator. Hence the constant-H surfaces continuously move apart in radius as H increases. Whether f_i is an increasing or decreasing function of H will then depend upon the functional form of $n_i(r(H))$, and how rapidly it decreases as a function of H. Since the instabilities we consider arise from ion dissipation exciting slow (i.e., negative-energy) electron waves, a negative $\partial f/\partial H$ is destabilizing and a positive $\partial f/\partial H$ is stabilizing. Therefore the $[1 + H(4M\omega_I{}^2 r_\alpha{}^2)]$ term in Eq. (16) will be a stabilizing effect.

III. THE ION-RESONANCE INSTABILITY ($k = 0$)

This section discusses the instability of $k = 0$ modes due to the presence of a small, diffuse profile of ions with a distribution like that calculated in Section II. The section is divided into four subsections, (A) The Electron Mode, (B) Electron Landau Damping, (C) Calculation of the Ion-Resonance Instability for a Diffuse Profile, and (D) Review of Relevant Experimental Results on HIPAC and SPAC II.

A. Review of the Electron Oscillation

Here we assume an electron equilibrium like that calculated in the previous section, and perturb it in r and θ, assuming all perturbed quantities vary as $f(r) \exp i(l\theta - \omega t)$. As shown in the Appendix, the perturbed components of $\tilde{A}_\theta$ and $\tilde{A}_r$ are negligible, so that only perturbed components of $\tilde{\phi}$ and $\tilde{A}_z$ (but not the inductive field proportional to ωA_z) need be considered. Henceforth, a perturbed quantity will be denoted by a superscript $\sim$. It is also shown in the Appendix that for the perturbed current in the z direction, $\tilde{n}ev_z \gg ne\tilde{v}_z$ for $v_z \sim c$. Therefore $\tilde{A}_z = (v_z/c)\tilde{\phi}$. Making the usual assumption for the mode that electron inertia is negligible due to the strong magnetic field in the z direction, we find that the perturbed fluid equations for the electrons are

$$\left(-i\omega + \frac{ilv_\theta}{r}\right)\tilde{n} + \frac{il}{r} n\tilde{v}_\theta + \frac{1}{r}\frac{\partial}{\partial r} rn\tilde{v}_r = 0 \tag{17a}$$

$$\tilde{v}_r = \frac{ilc}{r\gamma^2 B}\tilde{\phi} \tag{17b}$$

$$\tilde{v}_\theta = \frac{c}{\gamma^2 B}\frac{\partial\tilde{\phi}}{\partial r} \tag{17c}$$

$$\nabla^2\tilde{\phi} = 4\pi e\tilde{n}. \tag{17d}$$

Combining Eqs. (17a–c) to determine n in terms of ϕ, we find the standard equation for the electron mode corrected for the relativistic motion of the beam

$$\nabla^2\tilde{\phi} = \frac{\partial^2\tilde{\phi}}{\partial r^2} + \frac{1}{r}\frac{\partial\tilde{\phi}}{\partial r} - \frac{l^2}{r^2}\tilde{\phi} = 4\pi e\frac{lc}{\gamma^2 eB}\frac{\dfrac{\partial n}{\partial r}}{\left(\dfrac{lv_\theta}{r} - \omega\right)}\tilde{\phi}. \tag{18a}$$

For some purposes, it is convenient to express the equilibrium quantities n and v_θ in terms of the equilibrium electric field E by using the equilibrium relations in the last section. Then Eq. (18a) becomes

$$\frac{\partial^2\tilde{\phi}}{\partial r^2} + \frac{1}{r}\frac{\partial\tilde{\phi}}{\partial r} - \frac{l^2}{r^2}\tilde{\phi} = -\frac{lc}{\gamma^2 rB}\frac{\tilde{\phi}}{\dfrac{lcE}{\gamma^2 rB} - \omega}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rE \tag{18b}$$

Equation (18a or b) are difficult to solve because of the singular nature of the right hand side. However there is one simple analytic solution[11,12] and one simple theorem worth writing here. For $l = 1$, it can be shown by direct substitution that

$$\tilde{\phi} = r\omega - \frac{cE}{\gamma^2 B} \tag{19}$$

is a solution to Eq. (18b). Since $\tilde{\phi}$ must be zero at the wall $r = a$, the frequency of the mode is given by

$$\omega = \frac{cE(a)}{\gamma^2 aB}. \tag{20}$$

For a wall radius a much larger than the beam radius r_b, as is expected in the NRL modified betatron, this frequency is very small, smaller than any rotation frequency in the beam, so that the resonant denominator vanishes at the wall where $n = 0$.

Secondly, it is a simple matter to show that for a monotonically decreasing electron density profile, there are no normal modes with ω_i (the imaginary part of ω) non-zero. To show this, multiply Eq. (18a) by $r\phi^*$, integrate over radius and take the imaginary part. The result is

$$\omega_i \int_0^a dr \frac{4\pi elc}{\gamma^2 B} \frac{\dfrac{\partial n}{\partial r}}{\left|\dfrac{lv_\theta}{r} - \omega\right|^2} |\phi|^2 = 0. \tag{21}$$

For $\partial n/\partial r < 0$ everywhere, the integral is negative, so the only way to solve Eq. (21) is to set $\omega_i = 0$. If however, n has a local maximum, at $r \neq 0$, then the density profile may be unstable to the standard diocotron instability. For a monotonic density profile, it is clearly not possible to have a nonsingular eigenfunction for real ω if $\partial n/\partial r \neq 0$ at the radial position $\omega = lv_\theta/r$. Thus any nonsingular initial perturbation must be decomposed into a superposition of many singular eigenfunctions with different frequencies. As these different frequency components phase-mix, the perturbation will damp away, as the Van Kampen modes in a plasma phase-mix to give rise to Landau damping. In the next subsection, we examine this process by looking at ω as a Laplace rather than a Fourier transform variable.

To conclude, we briefly review the electron mode for a uniform density profile out to $r = r_b$ and zero density outside. In this case, the beam rotation frequency v_θ/r is constant. It is a simple matter to integrate the equation across the delta function $\partial n/\partial r$ and arrive at the dispersion relation

$$\omega = (l-1)\frac{v_\theta}{r} + \left(\frac{v_\theta}{r}\right)\left(\frac{r_b}{a}\right)^{2l} \tag{22a}$$

and eigenfunction

$$\phi = \phi_0 \left(\frac{r}{r_b}\right)^l; \quad r < r_b$$

$$\phi = \phi_0 \left\{ \frac{1}{1 - \left(\frac{r_b}{a}\right)^{2l}} \left(\frac{r_b}{r}\right)^l - \frac{\left(\frac{r_b}{a}\right)^{2l}}{1 - \left(\frac{r_b}{a}\right)^{2l}} \left(\frac{r}{r_b}\right)^l \right\}; \quad r > r_b \tag{22b}$$

Notice that for $l = 1$, the wave frequency is very low, as we have discussed, while for $l > 1$, the frequency approaches the resonant frequency lv_θ/r, but always stays slightly below it. Thus the diocotron wave is a slow wave and can be destabilized by positive dissipation. This dissipation can arise from the ions or else other processes such as a resistive wall.[13–15]

B. *Landau Damping of the Electron Waves*

As was shown in the last subsection, as the mode number l increases, the frequency approaches the electron rotation frequency. For instance, while an $l = 1$ mode cannot resonate with any electron, an $l = 2$ mode resonates with an electron having roughly half the central rotation frequency, an $l = 3$ mode resonates with an electron at two thirds the central rotation frequency, etc. Clearly, as l increases, electron Landau damping then becomes more important. We will first consider the case $l = 2$ and show that the amount of Landau damping depends sensitively on the density profile. For instance, consider first a parabolic density profile

$$n = n_0 \left(1 - \frac{r^2}{{r_b}^2}\right); r < r_b \tag{23a}$$

$$E = 4\pi n_0 e \left(\frac{r}{2} - \frac{r^3}{4{r_b}^2}\right). \tag{23b}$$

Notice that at $r = r_b$, the rotation frequency, proportional to E/r, is just half its value at $r = 0$. Therefore, while there will be many resonant electrons for an $l \geq 3$ mode, there are no resonant electrons for $l = 2$ and the Landau damping should be very small. On this basis, we do not consider modes with $l \geq 3$.

On the other hand, for a Gaussian density profile

$$n = n_0 \exp\left(-\frac{r^2}{2{r_b}^2}\right) \tag{24a}$$

$$E = \frac{4\pi n_0 e r_b \left[1 - \exp\left(-\frac{r^2}{2{r_b}^2}\right)\right]}{r}, \tag{24b}$$

the electron rotation frequency is equal to half its value at $r = 0$ at approximately $r = 2r_b$, where there are many resonant electrons. Hence we expect electron Landau damping to be very strong for an $l = 2$ mode having a Gaussian density profile.

Let us now calculate the expression for the Landau damping rate for the diocotron wave. To do so, we assume, following Briggs, et al.[12] that the ω in Eq. (18) is a Laplace rather than a Fourier-transform variable. Hence, Eq. (18) as written is valid only for $Im\omega$ greater than the ω of all singular points of $\phi(r, \omega)$. The quantity $\phi(r, t)$ is obtained by performing an inverse Laplace transform, so that if the contour is deformed into the lower-half ω plane, one must use the analytic continuation of $\phi(r, \omega)$ into the lower-half ω plane. If the imaginary part of ω is small, this analytic continuation is obtained from the residue. Then performing the same integral as in Eq. (21), we find that

$$\omega_i = \frac{\pi \displaystyle\int \frac{lc}{\gamma^2 B} |\tilde{\phi}|^2 \, \delta\left(\omega - \frac{lv_\theta}{r}\right) \frac{\partial n}{\partial r}\, dr}{P \displaystyle\int \frac{lc}{\gamma^2 B} \frac{|\tilde{\phi}|^2}{\left(\dfrac{lv_\theta}{r} - \omega\right)^2} \frac{\partial n}{\partial r}\, dr}. \tag{25}$$

Using Eq. (22b) for $\tilde{\phi}$, Eq. (24a) for $\tilde{\phi}$ in the numerator, but the sharp boundary model for n in the denominator, assuming $l = 2$ and that the resonant position is at $r = 2r_b$, then we find

$$\omega_i \sim -\frac{\omega}{8}. \tag{26}$$

If near $r = r_b$, the eigenfunction is a smoother function of r than specified by Eq. (22b), as seems likely, the damping will be stronger still. Thus, for a Gaussian density profile, the Landau damping of an $l = 2$ diocotron mode is very strong, and it seems unlikely that a small ion population could drive the system unstable.

To summarize, we find that for $l \geqslant 3$ the Landau damping is so strong that these modes almost certainly will not be excited by a small ion population. For $l = 2$, the damping might or might not be strong; whether it is or not is sensitively dependent on the profile. For the case where the damping is weak, it is given approximately by Eq. (25). If Eq. (25) predicts strong damping, the actual expression for the damping will be inaccurate, but the damping will in fact be large. For $l = 1$, there is no Landau damping. However, as we will see shortly, the frequency of an $l = 1$ mode is so low that it is virtually impossible for an ion to resonate with this mode.

C. The Ion-Driven Instability

We now calculate the effect of a small ion population with an unperturbed distribution like that given in Section II. The possibility for instability exists whenever the ion-oscillation frequency $\omega(H)$, defined in Section II, is the same as the wave frequency ω. However, for $l \geq 3$, the electron Landau damping is almost certainly so strong that an instability will not be possible to excite. For $l = 1$ on the other hand, the frequency of the diocotron wave is very low. This frequency is roughly $\frac{1}{2}(\omega_{pe}^2/\gamma^2\Omega_{ce})(r_b/a)^2$, where ω_{pe} is the nonrelativistic electron plasma frequency and Ω_{ce} is the nonrelativistic electron cyclotron frequency. Since the ion-oscillation frequency is roughly equal to

$(m/2M)^{1/2}\omega_{pe}$, there can only be an interaction with an $l = 1$ mode if

$$\omega_{pe} \approx \gamma^2\Omega_{ce}\left(\frac{2m}{M}\right)^{1/2}\left(\frac{a}{r_b}\right)^2. \tag{27}$$

For large a/r_b (i.e., $\sim$ 10), the density predicted by Eq. (27) is much too high to be relevant to the NRL modified betatron accelerator, so the $l = 1$ mode should not be excited. This leaves only the $l = 2$ mode. As we will see, this mode is excited when $\omega = 2\omega(H)$, or at a density given roughly by

$$\omega_{pe} \sim 2\left(\frac{2m}{M}\right)^{1/2}\gamma^2\Omega_{ce}. \tag{28}$$

At low γ, approximately on or below the injection γ, this instability could be excited. However, once the beam is accelerated to γ's greater than about 5, there should be no interaction of the ions with an $l = 2$ diocotron mode. Thus an ion-resonance instability can only have an impact on the NRL modified-betatron program for (a) $l = 2$, (b) for electron density profiles with sufficiently small Landau damping, and (c) for electron density in the range given by Eq. (28), or $\gamma \lesssim 5$.

We now proceed to derive an approximate relation for the ion-generated growth rate where conditions for instability are met. The perturbed ion distribution function $\tilde{f}$ obeys the linearized Vlasov equation. By integrating along the unperturbed ion orbits, one finds the result

$$\tilde{f} = -\frac{e}{M}\int^{t} dt'\nabla\,\tilde{\phi}(r't')\cdot\nabla_v f, \tag{29}$$

where r' is an unperturbed ion orbit expressed as a function of t' having position r and velocity v at time $t' = t$. Notice that since the ions have $v/c \ll 1$, they respond only to the electrostatic part of the force. Making use of the fact that $d/dt = -i\omega + v\cdot\nabla$, and by using H, L instead of v_r, v_θ as independent velocity-space variables, we can rewrite $\tilde{f}$ as

$$\tilde{f} = -e\tilde{\phi}\frac{\partial f^0}{\partial H} - e\int^{t} dt'\,\tilde{\phi}(r', t')\left(i\omega\frac{\partial f^0}{\partial H} + il\frac{\partial f^0}{\partial L}\right), \tag{30}$$

where we assume the θ dependence of all quantities is $e^{il\theta}$. As discussed in Section II, the ions are assumed to have zero angular momentum. However, since $\partial f^0/\partial L$ appears in Eq. (30), it is necessary to consider also particle orbits for L just greater than zero. In the Appendix, we show that these make no contributions to $\tilde{f}_i$. Therefore we can drop the $\partial f^0/\partial L$ term in Eq. (30), and consider only ion orbits with $L = 0$. To make further analytic progress, we assume that first the ion motion $r'(t')$ is simple harmonic, but with an energy-dependent frequency $\omega(H)$ as discussed in Section II. Then

$$r(t') = r(H)\sin\left[\omega(H)(t' - t) + \beta\right], \tag{31}$$

where

$$\beta = \arctan\left(r\omega(H)/v\right), \tag{32}$$

and $r(H)$ is the maximum radius of an ion with total energy H. Secondly we assume the ion is trapped near the center of the column so that the eigenfunction $\tilde{\phi}(r) = \tilde{\phi}_0(r^2/r_b^2)$. Then the time integral in Eq. (30) is straightforward to do and the result is

$$\tilde{f} = -e\tilde{\phi}(r)\frac{\partial f^0}{\partial H} + e\tilde{\phi}_0 \frac{r^2(H)}{4r_b^2}\frac{\exp 2i\beta}{2\omega(H) - \omega}\,\omega\frac{\partial f^0}{\partial H}. \tag{33}$$

If there is a small perturbed ion density, then the expression for the potential is (assuming $l = 2$)

$$\frac{\partial^2\tilde{\phi}}{\partial r^2} + \frac{1}{\mathrm{r}}\frac{\partial\tilde{\phi}}{\partial r} - \frac{l^2\tilde{\phi}}{r^2} = 4\pi e\frac{2c}{rB\gamma^2}\frac{\partial n/\partial r}{\left(\dfrac{2v_\theta}{r} - \omega\right)}\tilde{\phi} - 4\pi\,\tilde{n}_i \tag{34}$$

where $\tilde{n}_i$ is the perturbed ion density

$$\tilde{n}_i = \sum_{\pm}\int \mathrm{dHdL}\frac{f_i}{(2M^3)^{1/2}\,r(H - e\phi)^{1/2}}, \tag{35}$$

and the summation over $\pm$ indicates a summation over particles with inward and outward radial velocity. To obtain an approximate expression for the growth rate, multiply Eq. (34) by $r\tilde{\phi}^*$ and integrate over radius. In the last term of Eq. (34), the $\tilde{n}_i$ term, it is easiest to change independent variables from r to β, $r = r(H)\sin\beta$, so $dr = r(H)\cos\beta\, d\beta$. Then do the β integral first at constant H. Using the definition $f^0(H, L) \equiv g(H)\,\delta(L)$, the imaginary part of the frequency is obtained from

$$\omega_i P\int_0^\infty \frac{4\pi e(2c)}{B\gamma^2}\frac{\dfrac{dn}{dr}|\phi(r)|^2}{\left(\dfrac{lv_\theta}{r} - \omega\right)^2} = -\frac{e|\tilde{\phi}_0|r^4(H)}{4M^2r_b^4\left|2\dfrac{\partial\omega(H)}{\partial H}\right|}\left(\frac{\pi}{2}\right)2\pi i\frac{\partial g_0}{\partial H}\bigg|_{H=H_0} \tag{36}$$

where H_0 is determined by $\omega = 2\omega(H_0)$. In calculating the right-hand side of Eq. (36), we have assumed $\omega_i \ll \omega$ so that a residue approximation is valid. If the left-hand side can be evaluated as in Eq. (25), (the sharp boundary model), the result is

$$\omega_i \approx -\omega^2\frac{\pi^2 B\gamma^2}{16c}\frac{e\,r^4(H_0)}{Mr_0^4\left|\dfrac{\partial\omega}{\partial H}\right|}\frac{1}{n_e}\frac{\partial g_0}{\partial H}\bigg|_{H=H_0} \tag{37}$$

Using the results of Section II, and further assuming $r(H) \sim r_b$ and $\partial g/\partial H \sim -\mathrm{n}_i/r_0^2\omega_I^3$, we find

$$\omega_i \sim \gamma^2\left(\frac{n_i}{n_e}\right)\Omega_{ci} \tag{38}$$

However, Eq. (38) is not valid for arbitrarily high γ, because as γ increases, the electron-ion interaction disappears according to Eq. (28). The maximum γ for interaction is given by Eq. (28). Thus the maximum growth rate is given by

$$\omega_i^{(\mathrm{MAX})} \sim \frac{1}{2\sqrt{2}} \frac{n_i}{n_e} \left(\frac{m}{M}\right)^{1/2} \omega_{pe}.$$

For a hydrogen-ion contaminant with a number density of 1% of the beam density, this growth rate is about 1 μsec. This is a fast enough growth rate to be dangerous to the operation of the device, but is still much slower than that which would be calculated by a sharp-profile model. As the growth rate is so large, this instability is best dealt with by avoiding it altogether. As we have seen, this can be done in several ways including the use of a sufficiently diffuse profile, or insuring that the γ stays sufficiently high.

D. Review of HIPAC and SPAC II Results

In this section we review briefly the results of the HIPAC[8,9] and SPAC II[10] experiments as they impact our studies of the ion-resonance instability. The former experiments attempted to produce an electron cloud with $\gamma = 1$ in a toroidal chamber of major radius 46 cm and minor radius 10 cm. The electron cloud is surrounded by a conducting liner with a toroidal slit through which the toroidal flux enters. Near this slit is an electron-emitting wire. As the toroidal flux enters, the wire is pulsed. The emitted electrons are swept into the toroidal chamber with the toroidal flux. This method of injection is called inductive charging. The maximum density produced was about 4×10^9 cm^{-3} and the magnetic field was several kG. Thus the ω_{pe}/Ω_{ce} ratio (about 1/10) is much lower than what the NRL modified betatron hopes to achieve. The densities were such that either the $l = 1$ or $l = 2$ diocotron mode could be excited. The electron-cloud radius was generally equal to the wall radius, except when compression was done. Then r_b/a could be reduced to about 1/2. These electron clouds were always well confined in that their lifetime could be just under 10 msec. They were finally disrupted by the ion-resonance instability. This was concluded by the comparison of disruption time with background-gas pressure. From both the frequency of the oscillation and the late-time azimuthal structure, it was ascertained that the mode was an $l = 1$ mode. At early times, the second harmonic was also observed. However, for most of the duration of the discharge, this second harmonic was not present. At no time was a third harmonic $l = 3$ structure ever observed. This lends credence to our conclusion that $l \geq 3$ should not be a problem. For $l = 1$, where $1 < a/r_b < 2$, and $\gamma = 1$, the density in HIPAC is such that an $l = 1$ or $l = 2$ mode could resonate with the ions.

SPAC II is a toroidal-containment device similar to the NRL modified betatron. A diode is placed just inside the toroidal shell at the meridian plane and a 450-keV, 16-kA, 25-nsec beam is fired in. Of the 16-kA injection current, a current of 300 A is trapped in the torus for about 20 μsec, or about 3000 beam transits. This beam current completely fills the toroidal shell. Once the ring is set up, its density begins to decay. Part way into this decay, there is a much more rapid disruption. This disruption is brought on by enhanced fluctuations. Both the frequency and wave number were measured, and it was concluded that the fluctuation is an $l = 1$, $k = 0$ electron mode, and that the frequency is nearly resonant with the (magnetized) ion-oscillation frequency. However, in the modified betatron, the parameters are such that an $l = 1$ mode should not be excited. Since this was the dominant mode in HIPAC and SPAC II, and apparently was always the one that triggered the final disruption when the ion density became sufficiently large, it is encouraging that this mode probably will not cause concern in the NRL modified betatron. The key to the difference is that in both HIPAC and SPAC II, $r_b/a \sim 1$, while the modified betatron is designed to have $r_b/a \ll 1$.

IV. THE ION-STREAMING INSTABILITY ($l = 0$)

In this section we discuss the other instability, the ion-streaming instability. As pointed out in the Introduction, the simple electrostatic instability in an infinite medium has two possible modes, the parallel two- stream interaction and the perpendicular (upper-hybrid) interaction with $k_\perp \gg k_\perp$. However, very simple geometric considerations show that the parallel instability cannot occur in the NRL high-current modified betatron. Therefore, in this section, we turn our attention to the upper-hybrid interaction, which the simple theory indicates is an area of great concern. We find that there are two very important stabilizing effects. First, there is the inclusion of the full electromagnetic effects on the electrons. These make it much more difficult for an upper-hybrid wave to have slow-enough phase velocity to resonate with the ions. Specifically, we find that there can be no ion interaction unless $k_r > 2\Omega_{ce}/r_b$. Secondly, the diffuse profile and the spread in ion energies is a very strong stabilizing effect, just as for the $k = 0$ ion-resonance instability. However, while we show that this instability is not nearly as pernicious as simple theory indicates, it is by no means benign either. In fact it appears to be of potentially great importance for the NRL modified-betatron program, and its presence dictates a minimum time for accelerating the beam. This section is divided up into three subsections, (A), the electron oscillation, (B) the ion driven instability, and (C), the impact on the NRL modified betatron.

A. The Electron Oscillation

To examine the electron oscillation in the absence of an ion contaminant, we will anticipate our principal results, namely, that the oscillation has a great deal of radial structure, so that we assume that all perturbed quantities have spatial and temporal dependence $f(r) \exp i(k_r r + kz - \omega t)$ where $k_r r_b \gg 1$, $k_r \gg k_z$ and $f(r)$ varies more slowly, on a space scale of order r_b. To get the properties of the electron oscillations, we can neglect the spatial dependence of $f(r)$. However, when considering the effect of the ion contaminant on this mode, it will be necessary to reintroduce it. To simplify the calculation, we will use the fact (proved in the Appendix) that $\tilde{A}_r$ can be neglected. The condition for the neglect of $\tilde{A}_r$ is marginally satisfied for the most unstable mode, but is much better satisfied for more slowly growing modes.

Using the fact that $\tilde{p}_\perp = \gamma m\tilde{\mathbf{v}}$, and $\tilde{p}_z = \gamma^3 m\tilde{v}_z$, the equations of motion for the perturbed velocities are

$$\gamma m(-i\omega + ikv_z)\tilde{v}_r + \frac{eB}{c}\tilde{v}_\theta = eik_r\left(\tilde{\phi} - \frac{v_z}{c}\tilde{A}_z\right) \tag{40a}$$

$$-\frac{eB}{c}\tilde{v}_r + m\gamma(-i\omega + ikv_z)\tilde{v}_\theta = -\frac{e}{c}i(kv_z - \omega)\tilde{A}_\theta \tag{40b}$$

$$\gamma^3 m(-i\omega + ikv_z)\tilde{v}_z = eik\tilde{\phi} - \frac{ie\omega}{c}\tilde{A}_z. \tag{40c}$$

These equations give for $\tilde{v}_r$, $\tilde{v}_\theta$ and $\tilde{v}_z$

$$\tilde{v}_r = \frac{m\gamma(-i\omega + ikv_z)eik\left(\tilde{\phi} - \frac{v_z}{c}\tilde{A}_z\right) + \frac{e^2B_z}{c^2}i(kv_z - \omega)\tilde{A}_\theta}{\left(\frac{eB_z}{c}\right) - \gamma^2m^2(\omega - kv_z)^2} \tag{41a}$$

$$\tilde{v}_\theta = \frac{-m\gamma(-i\omega + ikv_z)\frac{e}{c}i(kv_z - \omega)\tilde{A}_\theta - \frac{e^2B_z}{c}ik_r\left(\tilde{\phi} - \frac{v_z}{c}\tilde{A}_z\right)}{\left(\frac{eB_z}{c}\right)^2 - \gamma^2m^2(\omega - kv_z)^2} \tag{41b}$$

$$\tilde{v}_z = \frac{ei\left(k\tilde{\phi} - \frac{\omega}{c}\tilde{A}_z\right)}{\gamma^3m(-i\omega + ikv_z)}. \tag{40c}$$

The perturbed electron density, obtained from the mass-conservation equation, is

$$\tilde{n} = -\frac{kn}{(\omega - kv_z)^2}\frac{e\left(k\tilde{\phi} - \frac{\omega}{c}\tilde{A}_z\right)}{\gamma^3m} + \frac{ik_rn}{(\omega - kv_z)} \times \left[\frac{\frac{e^2B}{c^2}(kv - \omega)\tilde{A}_\theta - m\gamma(\omega - kv_z)eik_r\left(\tilde{\phi} - \frac{v_z}{c}\tilde{A}_z\right)}{\left(\frac{eB}{c}\right)^2 - \gamma^2m^2(\omega - kv_z)^2}\right]. \tag{42}$$

Note that because $k^2 \ll k_r^2$, the first term in $\tilde{n}$ arising from the parallel motion is negligible compared to that arising from the perpendicular motion. Similarly, in calculating the axial current, $\tilde{n}v_z \gg n\tilde{v}_z$. Then from Eqs. (41) and (42), it is a simple matter to calculate perturbed charge and current densities for use in Maxwell's equation. For Maxwell's equations, we assume the Lorentz gauge so that only currents in the θ and z direction need be considered. The quantity $\tilde{A}_r$ comes from the Lorentz condition

$$\nabla\cdot\tilde{A} + \frac{1}{c}\frac{\partial\tilde{\phi}}{\partial t} = 0. \tag{43}$$

However, as shown in the Appendix, the $\tilde{A}_r$ arising from Eq. (43) does not strongly couple back to the electron motion, so $\tilde{A}_r$ and Eq. (43) can be neglected. As in the case of the ion-resonance instability, we have shown that $\tilde{J}_z = -\tilde{n}ev_z$, so $\tilde{A}_z = (v_z/c)\tilde{\phi}$. Thus Maxwell's equations reduce to two equations for $\tilde{A}_\theta$ and $\tilde{\phi}$:

$$-k_r^2\tilde{\phi} = \frac{ik_r}{(\omega - kv_z)}\left\{\frac{\frac{(kv_z - \omega)\omega_{pe}^2}{\gamma c}\frac{\Omega c}{\gamma}\tilde{A}_\theta - (\omega - kv_z)\frac{\omega_{pe}^2}{\gamma^3}ik_r\tilde{\phi}}{(\Omega_c/\gamma)^2 - (\omega - kv_z)^2}\right\} \tag{44a}$$

$$-k_r^2\tilde{A}_\theta = \frac{\omega_{pe}^2(kv_z - \omega)^2\tilde{A}_\theta - ik_r\frac{\Omega_c}{\gamma^4c}\omega_{pe}^2\tilde{\phi}}{(\Omega_c/\gamma)^2 - (\omega - kv_z)^2} \tag{44b}$$

In Maxwell's equations, we have assumed that the mode has low frequency so that it can resonate with the ions. We have therefore neglected terms of order $(\omega/kc)^2$, but have retained terms of order (ω/kc). On the left hand sides of Eq. (44), we have also assumed also that $k_r^2 \gg k^2$ and have neglected terms of order $(k/k_r)^2$. The first thing is to find the wave numbers of modes which can resonate with the ions. To do this, set $\omega = 0$. This gives the relation between k_r and k for the electron oscillation to resonate with the ions. The result is

$$\left[\frac{\omega_{pe}^2}{\gamma} - \gamma^2 k^2 v_z^2 + \Omega_c^2\right]\left[\omega_{pe}^2\,\gamma\left(\frac{v_z}{c}\right)^2 \frac{k^2}{k_r^2} - \gamma^2 k^2 v_z^2 + \Omega_c^2\right] + \frac{\Omega_c^2 \omega_{pe}^4}{k_r^2 c^2 \gamma^2} = 0. \tag{45}$$

It is not difficult to show *a posteriori* that the first term in the second bracket is a small correction, so neglecting it we find

$$\Omega_c^2 - \gamma^2 k^2 v_z^2 = -\frac{\omega_{pe}^2}{2\gamma} \pm \left(\frac{\omega_{pe}^4}{4\gamma^2} - \frac{\omega_{pe}^4 \Omega_c^2}{\gamma^2 k_r^2 c^2}\right)^{1/2}. \tag{46}$$

There is therefore no root unless

$$k_r > \frac{2\Omega_c}{c}. \tag{47}$$

For parameters relevant to the NRL modified betatron, this means $kr_b > 1$. Thus electromagnetic effects (from the $\tilde{A}_\theta$) speed up the phase velocity of the wave and make it impossible to resonate with the ion unless there is a great deal of radial structure. For the most unstable wave, i.e., that with the least radial structure [but also where neglect of $\tilde{A}_r$ and the first term in the second bracket of Eq. (45) is least justified], we have that for a mode to resonate with the ions,

$$k_r = \frac{2\Omega_c}{c} \tag{48a}$$

$$k = \left[\left(\frac{\Omega_c}{\gamma}\right)^2 + \frac{\omega_{pe}^2}{\gamma^3}\right]^{1/2}\left(\frac{1}{v_z}\right). \tag{48b}$$

As k_r varies between its minimum value and infinity, k

$$\frac{\Omega^2}{\gamma^2} < k^2 v_z^2 < \frac{\Omega^2}{\gamma^2} + \frac{\omega_{pe}^2}{\gamma^3}. \tag{49}$$

B. *The Ion-Driven Instability*

To calculate the effect of a small population of ions on the electron mode, we assume that the ion density is so small that the fluctuating quantities are as given in the absence of the ions. Since the ions are non-relativistic, they respond only to the electrostatic part of the field $\tilde{\phi} = \tilde{\phi}_0(r)\exp(ik_r r + ikz - i\omega t)$ where k_r^{-1} is assumed small compared to radial scale size of the beam. The values of k_r and k are given by Eq. (48). Using H, L and v_z as independent velocity-space variables, (H being the energy in the transverse place) and assuming $f^0 = f^0(H, L, v_z) = g^0(H)\delta(L)\delta(v_z)$, we find the perturbed ion density is

$$\tilde{f} = -e\tilde{\phi}\frac{\partial f^0}{\partial H} - e\int^t dt'\,\tilde{\phi}(r')\exp i(kz' + ikr' - \omega t')$$
$$\times\left(i(\omega - kv_z)\frac{\partial f^0}{\partial H} + \frac{ik}{M}\frac{\partial f^0}{\partial v_z}\right). \tag{50}$$

We proceed by neglecting the r dependence of $\tilde{\phi}$, and for the unperturbed orbit, we use Eqs. (31 and 32). Then, making use of the fact that $\exp(i\alpha\sin\theta) = \sum_n J_n(\alpha)\exp(in\theta)$, the integral over t' in Eq. (50) can be done, giving the result (recall β is defined in Eq. (32)).

$$\tilde{f} = -e\tilde{\phi}\frac{\partial f^0}{\partial H} - e\sum_n \tilde{\phi}J_n(k_r\, r(H))\frac{e^{i(kz-\omega t+in\beta)}}{[kv_z + n\omega_0(H) - \omega]}$$
$$\times\left[(\omega - kv_z)\frac{\partial f^0}{\partial H} + \frac{k}{M}\frac{\partial f^0}{\partial v_z}\right] \tag{51}$$

Since the ion velocity is assumed to be very small, the ions will be assumed not to contribute to the perturbed current, but only to the perturbed density. The perturbed density is obtained by integrating over H and L as specified in Eq. (35) (and of course also integrating over v_z). In doing this integral, the perturbed density will have a complicated structure in r. The only Fourier component which can feed power into the electron oscillation is that part proportional to $\exp(ik_r r)$. Therefore, we will operate on Maxwell's equations with $\int_0^\infty r\,dr\exp(-ik_r r - ikz + i\omega t)$. The result is

$$\int_0^\infty r\,dr\exp(-ik_r r - ikz + i\omega t)\nabla^2\tilde{\phi} = 4\pi e\int r\,dr\,(\tilde{n}_e - \tilde{n}_i)\exp(-ik_r r - ikz + i\omega t). \tag{52a}$$

$$\int_0^\infty r\,dr\exp(-ik_r r - ikz + i\omega t)\nabla^2\tilde{A}_\theta = 4\pi e\int r\,dr\,n_e\tilde{v}_\theta\exp(-ik_r r - ikz + i\omega t). \tag{52b}$$

Within the beam, all quantities are assumed to vary as $\exp(ik_r r)$. Outside the beam, $n_e = \tilde{n}_e = \nabla^2\tilde{\phi} = \nabla^2\tilde{A}_\theta = 0$. Therefore Eqs. (52) reduce to

$$-\frac{r_b^2}{2}k_r^2\tilde{\phi} = \frac{4\pi e r_b^2}{2}\tilde{n}_e - 4\pi e\int_0^\infty r\,dr\,\tilde{n}_i\exp(-ik_r r - ikz + i\omega t) \tag{53a}$$

$$-\frac{r_b^2}{2}k_r^2\tilde{A}_\theta = 4\pi e\left(\frac{r_b^2}{2}\right)n_e\tilde{v}_\theta. \tag{53b}$$

Thus we need to calculate the last term on the right-hand side of Eq. (53a). To do so, change the independent variable from r to β as in Section IIIc. The result is

$$-4\pi e \int r\,dr\, \exp[-(ik_r r + ikz - i\omega t)]\, \tilde{n}_i =$$

$$-4\pi e^2 \int \frac{dH\tilde{\phi}}{M^2\omega_0(H)} \sum_n J_n^2\,(k_r r(H)) \left[\frac{\omega(\partial g^0/\partial H)}{n\omega(H) - \omega} + \frac{(k^2/M)g^0}{(n\omega(H) - \omega)^2}\right]. \quad (54)$$

Since we want to consider only the imaginary part of Eq. (54), namely that part from the resonant denominators, we did not write out the contribution from the first term in Eq. (50), which is pure real. It is not difficult to show that the second term in the square brackets in Eq. (54) is smaller by a factor $k^2 r_b^2$. Then, taking only the residue contribution, we find

$$-4\pi e \int r\,dr\, \exp[-i(k_r r + kz - \omega t)]\, \tilde{n}_i =$$

$$-4\pi e^2\,(\pi i) \int \frac{dH\tilde{\phi}}{M^2\omega_0(H)} \sum_n j_n^2\,(k_r r(H))\, \omega(\partial g^0/\partial H)\, \delta(n\omega(H) - \omega) \equiv \delta\tilde{\phi} \quad (54)$$

Using this extra contribution to the perturbed charge density in Eq. (53a), we can calculate the imaginary part of the frequency assuming k_r and k are given by Eqs. (48a & b). The imaginary part of the frequency comes from setting

$$D = \begin{vmatrix} \left(\frac{\Omega_c}{\gamma}\right)^2 + \frac{\omega_{pe}^2}{\gamma^3} - (\omega - kv_z)^2 & \frac{-i\omega_{pe}^2\Omega}{\gamma^2 c} \\ + \frac{\delta}{k_r^2(r_b^2/2)}\left(\frac{\Omega_c^2}{\gamma^2} - k^2 v_z^2\right) & k_r^2\left(\left(\frac{\Omega_c}{\gamma}\right)^2 - (\omega - kv_z)^2\right) \\ \frac{-i\Omega_c\omega_{pe}^2}{\gamma^4 c} & + \frac{\omega_{pe}^2(\omega - kv_z)^2}{\gamma c^2} \end{vmatrix} = 0. \quad (56)$$

A straightforward calculation then gives the result

$$\omega = \frac{-\delta\left[k_r^2\left(\frac{\Omega_c^2}{\gamma^2} - k^2 v_z^2\right) + \frac{\omega_{pe}^2 k^2 v_z^2}{\gamma c^2}\right]\left[\left(\frac{\Omega_c}{\gamma}\right)^2 - (kv_z)^2\right]}{k_r^2 \frac{\partial D}{\partial \omega}\left(\frac{r_b^2}{2}\right)} \quad (57)$$

assuming that ω is small. One can then calculate that

$$\frac{\partial D}{\partial \omega} = -2\frac{\omega_{pe}^2}{\gamma^3} k_r^2\, kv_z. \quad (58)$$

In the limit of small beam density $(\omega_{pe}^2/\gamma^3) \ll (\Omega_c^2/\gamma^2)$, we find

$$\omega_i \approx \delta \frac{\omega_{pe}^2}{k_r^2 r_b^2 \gamma^2 \Omega_c}. \quad (59)$$

Using the estimates in Section II for equilibrium quantities and assuming first that the resonant energy corresponds to a particle near $r(H) = r_b$, we find $\delta \sim n_i/(n_e k_r r_b)$, so

$$\omega_i \sim \frac{n_i}{n_e} \frac{\omega_{pe}^2}{(k_r r_b)^3 \gamma^2 \Omega_c}. \tag{60}$$

C. The Impact of the Ionic-Streaming Instability on the NRL High-Current Modified Betatron

Let us consider the effect of this instability on the performance of the NRL high-current modified betatron. Unlike the ion-resonance instability, the ion-streaming instability has a phase velocity very far from the electron axial velocity, so there is virtually no chance that this instability can be stabilized by electron Landau damping. The question then is whether the beam can be so rapidly accelerated that the instability never gets a chance to grow. To be specific, let us consider a 10-kA beam having a radius of 2 cm in a magnetic field of 3 kG. This gives the value $\omega_{pe} = 2.4 \times 10^{10}\ \text{sec}^{-1}$ and $\Omega = 5 \times 10^{10}\ \text{sec}^{-1}$. In this case, the unstable parallel wave numbers lie within the range

$$\frac{5}{3\gamma} < k(\text{cm}^{-1}) < \frac{5}{3\gamma}\left(1 + \frac{1}{8\gamma}\right) \tag{61}$$

and the growth rate according to Eq. (60) is given roughly by

$$\omega_i \sim 6 \times 10^7 \frac{n_i}{\gamma^2 n_e}\ \text{sec}^{-1}. \tag{62}$$

To continue, let us assume the density ratio of the contaminant is 10^{-2}. Then for $\gamma \approx 4$, corresponding to injection, the growth time is about 25 μsec. While this is not negligible, it is long enough to give at least hope of accelerating through the unstable region in k before the mode can grow appreciably. For instance, at $\gamma = 7$, the unstable k's are in a range having a width of about 2%. Thus if initially γ can be increased by 2% in 100 μsec, the instability should not have a chance to grow once $\gamma = 7$. Once, the beam is accelerated to still higher γ's, the conditions on the speed of acceleration are relaxed further.

V. CONCLUSIONS

In this paper, we have examined the ion-resonance ($k = 0$) and ion-streaming ($l = 0$) instability for a high-current modified betatron. Both instabilities are strongly stabilized by a diffuse profile. The ion-streaming instability is further stabilized by electromagnetic effects involving $\tilde{A}_\theta$. For a diffuse profile, the only mode of the ion resonance instability which can grow appears to be the $l = 2$, as long as (r_b/a) is sufficiently small. The ion-resonance instability can be stabilized by a sufficiently diffuse profile and can also be stabilized if the γ of the electron beam is above some critical value. The ion-streaming instability cannot be stabilized either way. However, it is unstable only in a fairly small γ-dependent range of wave numbers. If ω_{pe}/Ω_{ce} is sufficiently small, the ion density sufficiently low, and γ is sufficiently high, it appears to be possible to accelerate the beam through these unstable regions before the instability can grow. For either instability, the most dangerous time is at low γ. It seems reasonable to conclude that if these instabilities do not disrupt the beam during injection and self-field diffusion, they will not prevent acceleration either.

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APPENDIX

In the appendix, we begin by showing under what conditions electromagnetic terms can be neglected. We will start with the ion-resonance instability, where $\tilde{A}_r$ and $\tilde{A}_\theta$ were both neglected, but ϕ and $\tilde{A}_z$ were retained. Since variation is only in the r and θ direction, these quantities produce only a perturbed magnetic field in the Z direction and induced electric fields in θ and r directions. Since $\omega \ll lc/r$, Maxwell's equation for A_θ gives roughly

$$\tilde{A}_\theta \sim \frac{4\pi n_e e \tilde{V}_{e\theta} r^2}{cl^2} \tag{A1}$$

so

$$\tilde{B}_z \sim \frac{4\pi n_e e \tilde{V}_{e\theta} r}{lc}. \tag{A2}$$

[We have neglected the $\tilde{n}_e e V_{e\theta}$ contribution on the right hand side of Eq. (A1). Since $V_{e\theta}$ is small for the modified betatron, a similar calculation to this shows that this term is also negligible.] Since the $\tilde{A}_\theta$ and $\tilde{A}_r$ produce only an axial magnetic field, the perturbed force must involve the unperturbed $V_{e\theta}$. The radial force is then given roughly by

$$\tilde{F}_r \sim \frac{4\pi n_e e \tilde{V}_{e\theta} r}{cl} \frac{V_{e\theta}}{c}. \tag{A3}$$

Comparing this to the $e\tilde{V}_{e\theta}B/c$ radial force, we see that the former is negligible if

$$\frac{\omega_{pe}^2 r}{\Omega_c cl} \frac{V_{e\theta}}{c} \ll 1, \tag{A4}$$

a condition easily satisfied in the modified betatron, especially because $V_{e\theta}/c$ is so small. Now consider the effect of the induced electric field. Using the fact that, according to Eq. (17b and c), $\tilde{V}_r \sim \tilde{V}_\theta$, and the fact that $\omega \sim (l-1) V_{e\theta}/r$ for the electron mode, one can easily show that the condition for the neglect of the induced field is also given by Eq. (A4).

We continue by examing the neglect of the $-ne\tilde{v}_z$ in the perturbed axial current. Since there is no variation in the z direction, the axial current arises entirely from the induced electromagnetic field in the z direction, so

$$\tilde{v}_z = -\frac{e}{m}\frac{\omega}{c}\tilde{A}_z/\gamma^3\left(\frac{lv_\theta}{r} - \omega\right). \tag{A5}$$

The perturbed current from the other component, $-\tilde{n}ev_z$ is,

$$\frac{elcv_z}{\gamma^2 Br}\frac{\dfrac{\partial n}{\partial r}}{\left(\dfrac{lv_\theta}{r} - \omega\right)}\tilde{\phi} \tag{A6}$$

as derived in Section IIIA. Assuming $\tilde{\phi} = v_z/c\,\tilde{A}_z \approx A_z, \partial n/\partial r \sim n/r$, and making use of the dispersion relation for the $l = 2$ electron oscillation, we find that the $\tilde{v}_z$ contribution to the perturbed ion current is smaller by

$$\frac{\omega_{pe}^2 r^2}{4c^2\gamma^3} \sim \frac{\nu}{4\gamma^3} \ll 1. \tag{A7}$$

Thus the neglect of the $\hat{v}_z$ term in the perturbed current is justified.

Now let us turn to the electromagnetic effects on the ion-streaming instability. There, ϕ, $\tilde{A}_z$ and $\tilde{A}_\theta$ were retained, but $\tilde{A}_r$ was neglected. The perturbed $\tilde{A}_r$ can be calculated from the Lorentz condition, Eq. (43). Using the fact that $\tilde{\phi} \sim \tilde{A}_z$ and $\omega/c \ll k$, since $\omega \approx 0$ for resonance with the ions we find

$$\tilde{A}_r \sim (k/k_r)A_z. \tag{A8}$$

This perturbed $\tilde{A}_r$ gives rise to a perturbed $\tilde{B}_\theta \sim k\tilde{A}_r$, which in turn gives rise to a force in the radial direction of order $(ev_z k A_r/c) \sim (ev_z k\tilde{A}_z/ck_r)$. This is to be compared with the force in the radial direction which we included which is of order $ek_r\tilde{A}_z/\gamma^2$. Thus the effect of $\tilde{A}_r$ is negligible as long as

$$\gamma^2 k^2/k_r{}^2 \ll 1. \tag{A9}$$

For the minimum values of K_r, given by (48a), and with k given by Eq. (48b), Eq. (A9) reduces to $1/4 \ll 1$, and is therefore marginally satisfied. For the larger, more stable values of k_r however, Eq. (A9) is well satisfied.

We next turn to the question of the neglect of the $\partial f^0/\partial L$ term in Eq. (30). To examine this, it is necessary to examine the effects of particle orbits with L small. Such an orbit is shown schematically, along with the dependence of angle as a function of time in Fig. A1 for a particle with Hamiltonian H and small angular momentum L. Half way through each cycle of oscillation, the angle $\theta(t')$ changes by π over a time of order $\tau \sim L/H$. During this time, the particle is approximately a distance $L/(mH)^{1/2}$ from the origin. If the orbital frequency is given by $\omega(H)$, then the assumption of small L mean $\omega(H)\tau \ll 1$.

The actual orbit is then given as the sum of two components, the oscillating orbit considered in Sections II and IIIC, and the jumps in $\theta(t')$ by π in the time of order $\tau \sim L/H$. (Since l is even, $l = 2$), the jumps in $\theta(t')$ do not affect the orbit integral as computed in Section IIIc. Each jump in phase simply multiplies the integrand of

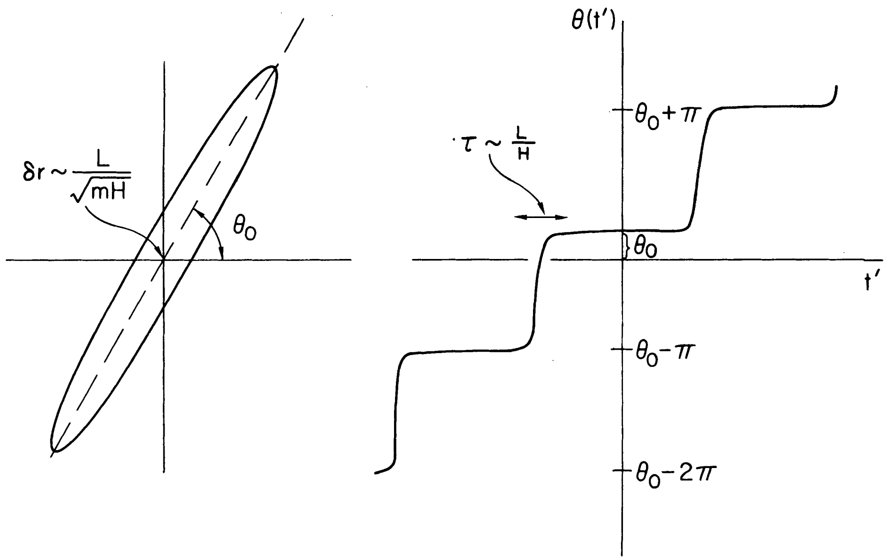

FIGURE A1 (A) The orbit of an ion with small, but non zero angular momentum, (B) the angle of the orbit as a function of time.

Eq. (30) by $\exp 2\pi i = 1$. To estimate the effect of the rapid phase variation, first assume that the jump in phase occurs while $r \sim L/(mH)^{1/2}$. Then doing the same calculation as in going from Eq. (30) to Eq. (33), we find that

$$\tilde{f}_i = -e\sum_n \int^t dt' \sim \frac{L^2}{mHr_b{}^2} \exp\big(i\theta + 2i\theta(t' - t_n) - i\omega t_n\big)\left(i\omega\frac{\partial f^0}{\partial H} + il\frac{\partial f^0}{\partial L}\right) \tag{A10}$$

where t_n is the time of the nth jump and the t^1 integral is carried out only over the jump time, of order L/H. Doing the integral over t^1 we find roughly

$$\tilde{f}_i \approx \sum_n^{t>t_n} - e\tilde{\phi}\frac{L^3}{mH^2r_b{}^2}\exp(2i\theta - i\omega t_n)\left(i\omega\frac{\partial f^0}{\partial H} + il\frac{\partial f_0}{\partial L}\right). \tag{A11}$$

Because of the L^3 dependence in front, this term integrates to zero if $f_0 \sim \delta(L)$. Note that that the origin of this is that $\tilde{\phi} \sim r^2$ for small r. Had the eigenfunction been uniform in r at smaller r, there would be a contribution from the $l(\partial f_0/\partial L)$ term. Since the other term calculated from the slow $\big(t^1 \sim \omega^{-1}(H)\big)$ oscillation has no explicit L dependence, [see Eq. (33)], the $\partial f_0/\partial L$ term does not contribute to $\tilde{f}_i$.

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