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STABILITY PROPERTIES OF INTENSE NONNEUTRAL ION BEAMS FOR HEAVY-ION FUSION

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The transverse stability properties of an intense ion beam in a quadrupole magnetic field are investigated within the framework of the Vlasov-Maxwell equations, including the important influence of beam rotation and transverse beam temperature on stability behavior. It is shown that the strongest instability occurs for zero rotational frequency ($\omega_b = 0$). The system can, however, be easily stabilized by slightly detuning the rotational frequency from $\omega_b = 0$.

High-energy, low-current, heavy-ion beams have been considered as the source of ignition energy for inertially confined fusion reactions.[1] In this regard, it appears that heavy-ion beams with suitable energies and densities can be used as viable drivers. One of the major limitations on beam transport, however, may be due to the transverse instability[2,3] that originates from beam self-field effects.[4] In this context, it is important to examine the transverse instability for an ion beam in a quadrupole magnetic field. Several theoretical and experimental studies of this instability are being carried out.[5–8] In this paper, we investigate the important influence of beam rotational motion and transverse beam temperature on stability behavior, making use of the linearized Vlasov-Maxwell equations for a nonneutral ion-beam system.[4] It is found that the value of beam rotational frequency ω_b plays a decisive role in determining ion-beam stability behavior in a quadrupole magnetic field, and that the strongest instability occurs for zero rotational frequency ($\omega_b = 0$). The system can be easily stabilized, however, by a slight detuning of the rotational frequency from $\omega_b = 0$.

In order to simplify the analysis, we consider ion motion in the average external magnetic field[3] produced by periodic quadrupole magnets. In this regard, the focusing associated with the applied quadrupole field can be determined from the axial component of the effective vector potential $A_z^{ext}(r) = -(\gamma_b m/2e\beta_b)\omega_f^2 r^2$, where the oscillation frequency ω_f is related to the quadrupole field gradient, $\gamma_b = 1/(1 - \beta_b^2)^{1/2} = \text{const.}$ is the relativistic mass factor, and e and m are the ion charge and rest mass, respectively. Moreover, it is also assumed that $\nu/\gamma_b \ll 1$, where $\nu = N_b e^2/mc^2$ is Budker's parameter, $N_b = 2\pi\int_0^{R_c} dr\, r\, n_b^0(r)$ is the number of ions per unit axial length, and c is the speed of light in *vacuo*. Cylindrical polar coordinates (r,θ,z) are used, with the z-axis (the beam propagation direction) directed along the axis of symmetry; r is the radial distance from the z-axis, and θ is the polar angle in a plane perpendicular to the z-axis. The equilibrium and stability analysis presented here parallels the general formalism developed in substantially more detail for intense relativistic electron beams in Ref. 8.

EQUILIBRIUM

Equilibrium and stability properties are calculated for the specific choice of equilibrium distribution function

$$f_b^0(H,P_\theta,P_z) = \frac{\hat{n}_b}{2\pi\gamma_b m}\,\delta(H - \omega_b P_\theta - \hat{\gamma} mc^2) \times \delta(P_z - \gamma_b m\beta_b c), \quad (1)$$

where $H = (m^2c^4 + c^2\mathbf{p}^2)^{1/2} + e\phi_0(r)$ is the total energy, $P_\theta = rp_\theta$ is the canonical angular momentum, $P_z = p_z - (\gamma_b m/2\beta_b c)\omega_f^2 r^2 + (e/$

$c)A_z^s(r)$ is the axial canonical momentum, $\hat{n}_b$ = const. is the ion density, ω_b = const. is the beam rotational frequency, $\mathbf{p} = (p_r, p_\theta, p_z)$ is the mechanical momentum, $\hat{\gamma}(>\gamma_b)$ is a constant, $\phi_0(r)$ is the equilibrium electrostatic potential, and $A_z^s(r)$ is the axial component of vector potential for the equilibrium azimuthal self-magnetic field. Making use of Eq. (1) and the fact that the $r - \theta$ kinetic energy is small in comparison with the characteristic directed beam energy $\gamma_b mc^2$, we obtain the equilibrium ion density profile[8] $n_b^0(r) = \int d^3p\, f_b^0 = \hat{n}_b U(R_b - r)$, where the beam radius R_b is defined by $R_b^2 = 2c^2(\hat{\gamma} - \gamma_b)/\gamma_b\Omega_b^2$, Ω_b^2 and $\hat{\omega}_b^2$ are defined by $\Omega_b^2 = \hat{\omega}_b^2 - \omega_b^2$ and $\hat{\omega}_b^2 = \omega_f^2 - \omega_{pb}^2/2\gamma_b^2$, $\omega_{pb}^2 = 4\pi\hat{n}_b e^2/\gamma_b m$ is the ion plasma frequency squared, and $U(x)$ is the Heaviside step function defined by $U(x > 0) = 1$ and $U(x < 0) = 0$. Evidently, radially confined equilibrium solutions exist only for rotational frequency ω_b satisfying $-\hat{\omega}_b < \omega_b < \hat{\omega}_b$. After some straightforward algebra, the transverse temperature profile can be approximated by $T_\perp^0(r) = (2\gamma_b m)^{-1}\int d^3p\, f_b^0[p_r^2 + (p_\theta - \gamma_b m\omega_b r)^2] = \hat{T}_\perp(1 - r^2/R_b^2)$ for $0 < r < R_b$, where the maximum temperature $\hat{T}_\perp$ (at $r = 0$) is defined by $\hat{T}_\perp = \gamma_b m\Omega_b^2 R_b^2/2$. Combining the above results, the beam rotational frequency ω_b is determined from

$$\omega_b^2 = \omega_f^2 - (\omega_{pb}^2/2\gamma_b^2) - 2\hat{T}_\perp/\gamma_b m R_b^2, \quad (2)$$

which is a statement of radial force balance (of centrifugal, magnetic, electric, and pressure gradient forces) on an ion-beam fluid element. In order for the equilibrium to exist, the temperature $\hat{T}_\perp$ in Eq. (2) is restricted to the range $0 \leq 2\hat{T}_\perp/\gamma_b m R_b^2 \leq \omega_f^2 - \omega_{pb}^2/2\gamma_b^2$. Finally, from Eq. (1), the mean equilibrium azimuthal motion of the ion beam is rigid rotor with $V_\theta^0(r) = (\int d^3p v_\theta f_b^0)/(\int d^3p f_b^0) = \omega_b r$, and the mean axial velocity can be approximated by $V_z^0(r) = (\int d^3p v_z f_b^0)/(\int d^3p f_b^0) = \beta_b c$ = const. for $\nu/\gamma_b \ll 1$.

STABILITY

In the present stability analysis, flute perturbations with $\partial/\partial z = 0$ are considered. For perturbations with azimuthal harmonic number l, perturbed quantities are expressed as $\delta\Phi(\mathbf{x},t) = \hat{\Phi}(r) \exp\{i(l\theta - \omega t)\}$, where ω is the complex eigenfrequency. In order to make the stability analysis tractable, we assume low-frequency perturbations characterized by $|\omega R_b|^2 \ll (l^2 + 1)c^2$. In this limit, together with $\nu/\gamma_b \ll 1$, the axial component of the perturbed current density $\hat{J}_z(r)$ is related to the perturbed charge density $\hat{\rho}(r)$ by the approximate relation[8] $\hat{J}_z(r) = \beta_b c\hat{\rho}(r)$. Moreover, the linearized Maxwell equations for the perturbed fields can be combined to give the approximate eigenvalue equation[8]

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{l^2}{r^2}\right)\psi(r) = -\frac{4\pi}{\gamma_b^2}\hat{\rho}(r), \quad (3)$$

where $\psi(r) = (ir/l)[\hat{E}_\theta(r) + \beta_b\hat{B}_r(r)]$. Here $\hat{E}_\theta(r)$ is the azimuthal component of the perturbed electric field, $\hat{B}_r(r)$ is the radial component of the perturbed magnetic field, and $\hat{\rho}(r) = e\int d^3p\hat{f}_b(\mathbf{x},\mathbf{p})$ is the perturbed charge density. Neglecting initial perturbations, the perturbed distribution function is given by

$$\hat{f}_b(\mathbf{x},\mathbf{p}) = -e\int_{-\infty}^{0} d\tau \exp(-i\omega\tau) \times \left\{\hat{\mathbf{E}}(\mathbf{x}') + \frac{\mathbf{v}' \times \hat{\mathbf{B}}(\mathbf{x}')}{c}\right\} \cdot \frac{\partial}{\partial \mathbf{p}'} f_b^0, \quad (4)$$

where Im $\omega > 0$, $\tau = t' - t$, $\delta f_b(\mathbf{x},\mathbf{p},t) = \hat{f}_b(\mathbf{x},\mathbf{p}) \exp(-i\omega t)$, and the particle trajectories[4,8,9] in the equilibrium fields satisfy the "initial" conditions $\mathbf{x}'(\tau = 0) = \mathbf{x}$ and $\mathbf{v}'(\tau = 0) = \mathbf{v}$. Evaluating $\hat{\rho} = e\int d^3p\hat{f}_b$, after some straightforward algebra Eq. (3) reduces to the eigenvalue equation[8]

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r} - \frac{l^2}{r^2}\right)\psi(r) = \frac{\omega_{pb}^2\delta(r - R_b)}{\gamma_b^2\Omega_b^2 R_b}[\psi(R_b) + (\omega - l\omega_b)\hat{I}(R_b)] + 2m^2\omega_{pb}^2(\omega - l\omega_b)U(R_b - r)(d\hat{I}/dp_\perp^2)_{p_\perp^2 = p_0^2}. \quad (5)$$

In Eq. (5), the orbit integral $\hat{I}$ is defined by[9]

$$\hat{I}(r) = \int_0^{2\pi}\frac{d\phi}{2\pi}\int_{-\infty}^{0} d\tau i\psi(r') \times \exp\{i[l(\theta' - \theta) - \omega\tau]\}. \quad (6)$$

Moreover, $p_0^2 = \gamma_b^2 m^2\Omega_b^2(R_b^2 - r^2)$, and the polar momentum variables $(p_\perp, \phi)$ in the rotating frame are defined by $p_x + \gamma_b m\omega_b y = p_\perp \cos\phi$, and $p_y - \gamma_b m\omega_b x = p_\perp \sin\phi$.

In the remainder of this paper, the stability analysis is restricted to azimuthally symmetric perturbations ($l = 0$) characterized by the eigenfunction

$$\psi(r) = \begin{cases} \sum_{j=0}^{n} a_j (r/R_b)^{2j}, & 0 < r < R_b , \\ \ln(r/R_c) \sum_{j=0}^{n} a_j / \ln(R_b/R_c), & R_b < r < R_c , \end{cases} \tag{7}$$

where R_c is the radius of the outer conductor, and the coefficients $a_j(\omega)$ will be determined self-consistently. In general, solutions to Eq. (5) exist for arbitrary radial mode number n. For present purposes, we consider the $n = 3$ eigenfunction in the subsequent analysis. Substituting Eq. (7) into Eq. (6), the orbit integral $\hat{I}$ can be evaluated explicitly in terms of the coefficients a_j and the eigenfrequency ω.[8] Substituting Eq. (7) into Eq. (5) we then solve Eq. (5) inside the beam ($0 \leq r < R_b$). Moreover, the appropriate boundary condition at $r = R_b$ is determined by multiplying Eq. (5) by r and integrating from $R_b(1 - \epsilon)$ to $R_b(1 + \epsilon)$ with $\epsilon \to 0_+$. After some tedious but straightforward algebraic manipulation, we obtain the matrix relation

$$\begin{pmatrix} x_{00} & x_{01} & x_{02} & x_{03} \\ 0 & x_{11} & x_{12} & x_{13} \\ 0 & 0 & x_{22} & x_{23} \\ 0 & 0 & 0 & x_{33} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 , \tag{8}$$

where the matrix elements x_{ij} are defined by $x_{00} = [\ln(R_b/R_c)]^{-1}$, $x_{01} = x_{00} + 2(\xi_1 - 1)$, $x_{02} = x_{00} + 4\xi_1(6\xi_2 + 1) - 4$, $x_{03} = x_{00} + 6\xi_1(120\xi_2\xi_3 + 12\xi_2 + 1) - 6$, $x_{11} = \xi_1 - 1$, $x_{12} = -24\xi_1\xi_2$, $x_{13} = 1080\xi_1\xi_2\xi_3$, $x_{22} = \xi_1(18\xi_2 + 1) - 1$, $x_{23} = 9(x_{22} - x_{33})/5$, and $x_{33} = \xi_1(1200\xi_2\xi_3 + 48\xi_2 + 1) - 1$. Here, $\xi_1 = (\omega_{pb}^2/\gamma_b^2)/(\omega^2 - 4\omega_b^2)$, $\xi_2 = \Omega_b^2/(\omega^2 - 16\hat{\omega}_b^2)$, and $\xi_3 = \Omega_b^2/(\omega^2 - 36\hat{\omega}_b^2)$. Evidently, from Eq. (8), the dispersion relation can be expressed as $x_{11}x_{22}x_{33} = 0$, where $x_{jj}(\omega) = 0$ corresponds to the dispersion relation for perturbations with radial mode number $n = j$.

(a) $n = 1$ *Perturbations*. In this case, $x_{11}(\omega) = 0$ gives the dispersion relation $\omega^2 = 4\omega_f^2 - \omega_{pb}^2/\gamma_b^2$, which does not exhibit instability ($Im\omega > 0$) for *any* allowed value of equilibrium beam temperature $\hat{T}_\perp$ or rotational frequency ω_b consistent with Eq. (2). For $n = 1$, the coefficient a_1 in Eq. (7) is related to a_0 by $a_1 = -a_0$.

(b) $n = 2$ *Perturbations*. The dispersion relation $x_{22}(\omega) = 0$ can be expressed as

$$(\omega^2 - 16\hat{\omega}_b^2)(\omega^2 - 4\hat{\omega}_b^2) = (\omega_{pb}^2/\gamma_b^2) \times (\omega^2 + 2\hat{\omega}_b^2 - 18\omega_b^2) . \tag{9}$$

Since the rotational frequency ω_b is related to $\hat{T}_\perp$ [Eq. (2)], it is evident from Eq. (9) that the beam temperature plays a significant role in determining stability behavior. As a particular case, it is instructive to examine Eq. (9) for the specific value of $\hat{T}_\perp$ corresponding to the maximum allowable beam density, i.e., $(\omega_{pb}/\gamma_b)^2 = \omega_f^2 - 2\hat{T}_\perp/\gamma_b m R_b^2$. In this case, the beam rotation frequency is given by $\omega_b = 0$ [Eq. (2)], and Eq. (9) reduces to the result obtained by Gluckstern[3] for perturbations about the Kapchinsky-Vladimirsky[10] equilibrium distribution function. For $\omega_b = 0$, instability ($Im\omega > 0$) readily follows from Eq. (9).

In general, Eq. (9) is a simple quadratic equation for ω^2, and the necessary and sufficient condition for instability can be expressed as

$$(\gamma_b\omega_b/\omega_{pb})^2 < h(\omega_f^2\gamma_b^2/\omega_{pb}^2) . \tag{10}$$

where $h(x)$ is defined by $h(x) = (x - 0.5)(17 - 32x)/9$. Note that when Eq. (10) is satisfied, the perturbations are purely growing, i.e., $\omega_r = Re\omega = 0$. The function $h(x)$ assumes its maximum value, $h_m = 1/1152$, at $x = 33/64$. It is evident from Eq. (10) that beam rotation (ω_b) plays a critical role in determining stability behavior. Moreover, the $n = 2$ perturbations can be completely stabilized by a modest increase in rotational frequency to values satisfying $(\gamma_b\omega_b/\omega_{pb})^2 > h_m = 1/1152$. Making use of Eqs. (8) and (9) it follows that $a_1 = -4a_0$ and $a_2 = 3a_0$.

(c) $n = 3$ *Perturbations*. After some straightforward algebra, the dispersion relation $x_{33}(\omega) = 0$ can be expressed as

$$(\omega^2 - 4\hat{\omega}_b^2)(\omega^2 - 16\hat{\omega}_b^2)(\omega^2 - 36\hat{\omega}_b^2) = (\omega_{pb}^2/\gamma_b^2)[(\omega^2 - 16\hat{\omega}_b^2)(\omega^2 - 36\hat{\omega}_b^2) + 48\Omega_b^2(\omega^2 - 36\hat{\omega}_b^2) + 1200\Omega_b^4] . \tag{11}$$

where $\Omega_b^2 = \hat{\omega}_b^2 - \omega_b^2$, and the coefficients are related by $a_1 = -9a_0$, $a_2 = 18a_0$ and $a_3 = -10a_0$. The growth rate $\omega_i = Im\omega$ and real oscillation frequency $\omega_r = Re\omega$ have been obtained numerically from Eq. (11) for a broad range of beam parameters $(\omega_b/\omega_f)^2$ and $\omega_{pb}^2/2\gamma_b^2\omega_f^2$. In order to illustrate the influence of beam rotation on stability behavior, the stability boundaries in the parameter space $(\omega_b^2/\omega_f^2, \omega_{pb}^2/2\gamma_b^2\omega_f^2)$ are shown in Fig. 1. Since the beam temperature $\hat{T}_\perp$ in Eq. (2) is positive, equilibrium solutions are not allowed in the cross-hatched region in Fig. 1 where $\omega_b^2 > \omega_f^2 - \omega_{pb}^2/2\gamma_b^2$. The horizontal axis in Fig. 1 represent the Kapchinsky-Vladimirsky (K-V) equilibrium distribution ($\omega_b = 0$). The solid curves in Fig. 1 correspond to the stability boundaries $Im\omega = 0$ obtained from Eq. (11). We note from Fig. 1 that there exist two unstable regions in the parameter space $(\omega_b^2/\omega_f^2, \omega_{pb}^2/2\gamma_b^2\omega_f^2)$. Perturbations in Region 1 have $Im\omega > 0$ and a non-zero real oscillation frequency $Re\omega \neq 0$ (see Fig. 2). On the other hand, perturbations in Region 2 are characterized by $Im\omega > 0$ and $Re\omega = 0$. For the K-V distribution ($\omega_b = 0$), we note from Fig. 1 that the onset of instability occurs for beam density satisfying $\omega_{pb}^2 = 1.7\gamma_b^2\omega_f^2$. The instability corresponding to Region 1 in Fig. 1 can, however, be circumvented by increasing the rotational frequency ω_b to values satisfying $\omega_b^2/\omega_f^2 \geq 0.003$. Therefore, for $\omega_b^2 \geq 0.003\ \omega_f^2$, a high-density ion beam with $\omega_{pb}^2 = 1.96\gamma_b^2\omega_f^2$ is not subject to this instability. Finally, for a specified value of rotational frequency ω_b, we also note from Eq. (2) that the transverse beam temperature $\hat{T}_\perp$ can be significantly reduced by increasing the beam density to values approaching $\omega_{pb}^2 = 2\gamma_b^2(\omega_f^2 - \omega_b^2)$, thereby substantially decreasing the beam emittance for stable propagation.

In conclusion, we summarize the following main points. First, an intense ion beam in a quadrupole magnetic field can be subject to a strong transverse instability for radial mode numbers $n = 2$ and $n = 3$. Second, the value of beam rotational frequency ω_b plays a decisive role in determining ion beam stability behavior. Third, generally speaking, the strongest instability occurs for the K-V distribution function ($\omega_b = 0$). However, the system can be easily stabilized by

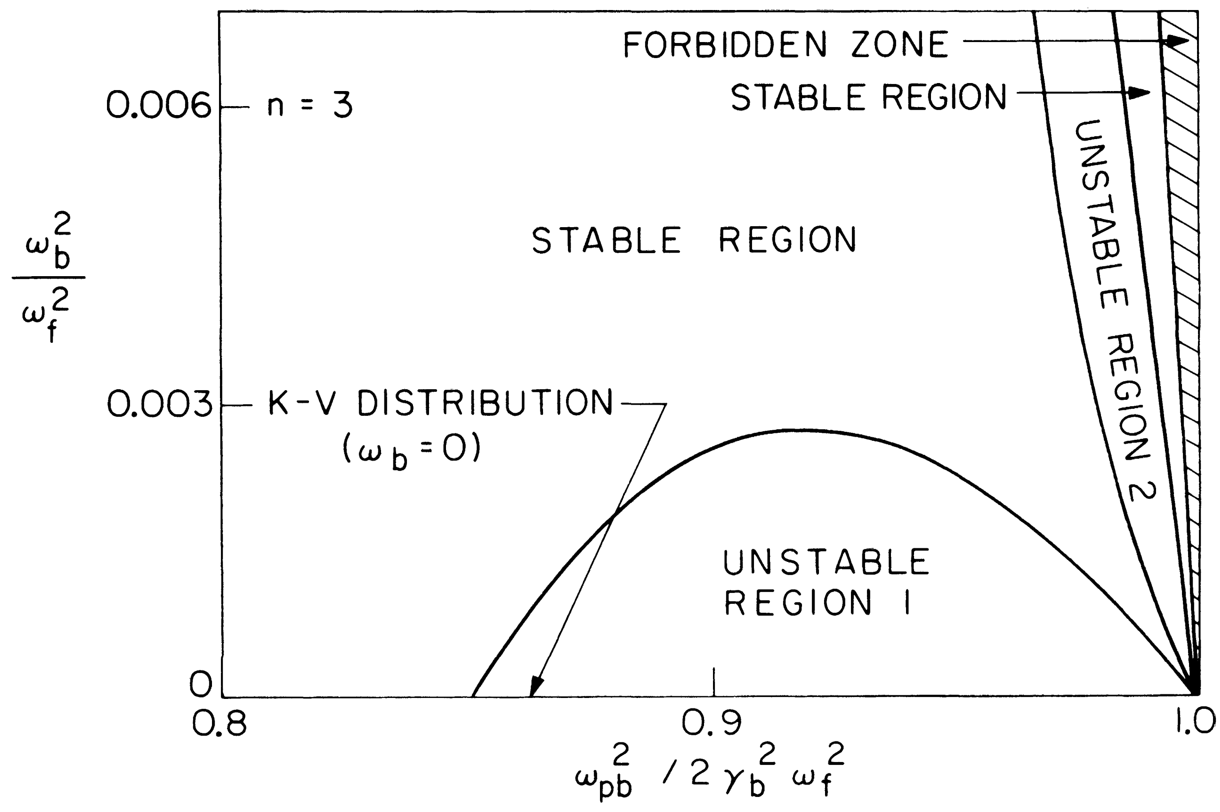

FIGURE 1 Stability boundaries [Eq. (11)] in the parameter space $(\omega_b^2/\omega_f^2, \omega_{pb}^2/2\gamma_b^2\omega_f^2)$ for azimuthally symmetric $n = 3$ perturbations.

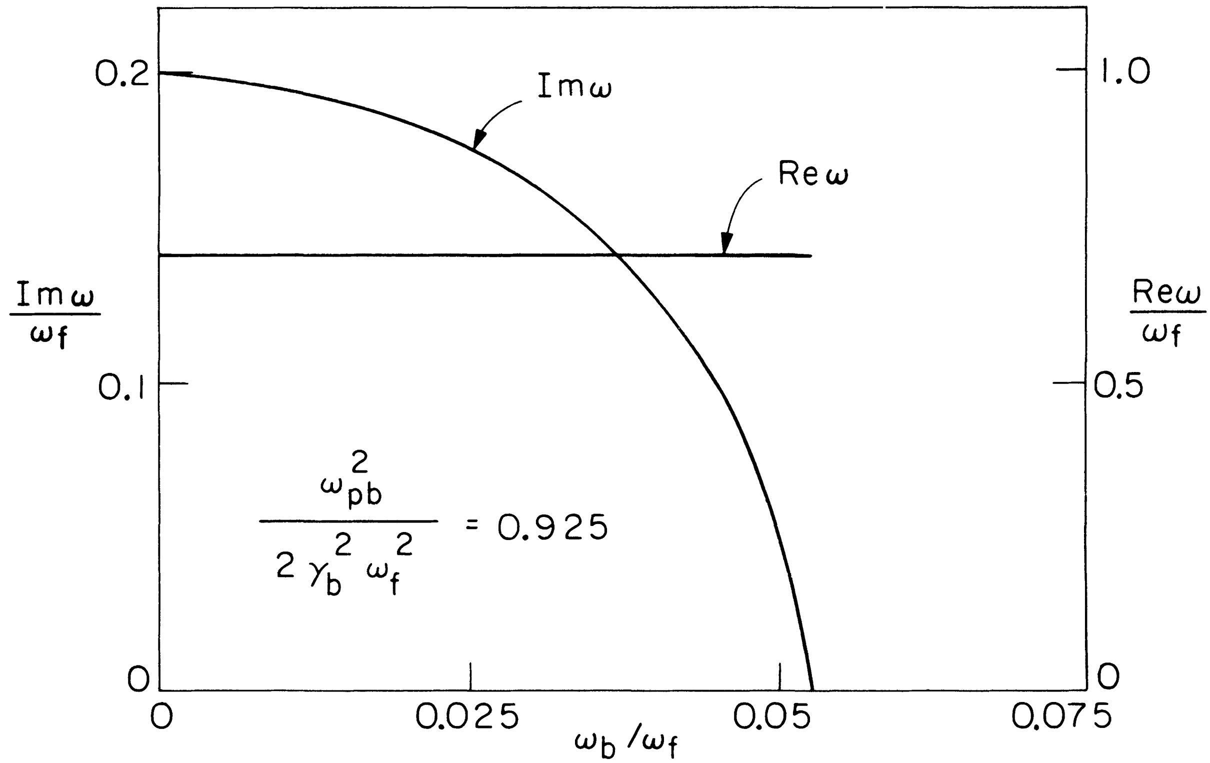

FIGURE 2 Plot of growth rate ω_i = Im ω and real frequency ω_r = Re ω versus ω_b^2/ω_f^2 for $\omega_{pb}^2/2\gamma_b^2\omega_f^2 = 0.925$ [Eq. (11)].

slightly detuning the rotational frequency from $\omega_b = 0$. In this regard, both the beam emittance and stability properties can be substantially improved.

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