

LINEAR PROGRAMMING Vs LEAST SQUARES METHOD IN DETERMINING THE BEST TRIM COIL CURRENTS FOR THE NRL CYCLOTRON

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A description is given of a linear programming method which has been used in determining the best trim coil currents to produce the isochronous field needed to accelerate particular ions to a particular energy. The results using this sub-routine are compared with those using a least squares method. Although the least squares method had given good results, it contained a certain arbitrariness in choosing which trim coil to set at the limiting current value when two or more of the best trim coil currents fell outside the allowable limits. It was hoped that the linear programming method might give better results, since the requirement that all currents be less than certain values can be included among the constraints of the problem.

The two methods are compared in terms of the deviation from isochronism of the field produced. It was found that in cases where the general fit in either case was good, there was little difference between the methods, but when the general fit was poor, the least squares method gave over-all, less deviation from the calculated isochronous field.

INTRODUCTION

In order to accelerate ions in an isochronous cyclotron, it is necessary that the magnetic field in the accelerating region be precisely shaped, the shape depending on the type and final energy of the ions being accelerated. The basic field is obtained with the current around the main magnet pole pieces, and the difference between this and the needed isochronous field is supplied primarily by the trim coils. Computer programs to determine trim coil and other current settings have been written for different cyclotron facilities.¹⁻³ Some include optimization of the phase-energy curve and of the radial and vertical oscillations, as well as of the isochronous field averaged over the azimuthal angle.

We have considered only the last, and the method we have used has been described in a previous paper.⁴ It employs an iterative cycle which obtains an initial isochronous field and initial set of trim coil currents using the characteristics of the main field coil. These trim coil currents are used to obtain a more accurate flutter field, with which a better isochronous field is calculated. The flux of this new isochronous field is used to correct the gauss/ampere values of the trim coils for the flux they will be operating in, and finally the best values of the trim coils are obtained with these corrected

gauss/ampere values and the improved isochronous field.

In the least square method employed, the error in the average field at each radius was written as the difference between the desired isochronous field, $\bar{B}^{\text{iso}}(r_j)$ (the bar indicating average over angle at radius r_j), and the combined field produced by the main coil, $\bar{B}^0(r_j)$, and the trim coils,

$$\sum_{i=1}^{10} \frac{\partial B(r_j)}{\partial w_i} w_i.$$

The currents w_i in the ten trim coils are the independent variables which are being sought. Thus, at the radius r_j

$$e_j = \bar{B}^{\text{iso}}(r_j) - \bar{B}^0(r_j) - \sum_{i=1}^{10} \frac{\partial B(r_j)}{\partial w_i} w_i. \quad (1)$$

The trim coil currents were then chosen in the usual least square manner so as to make the sum of the squares of the errors, e_j , at radii one inch apart, a minimum.

However, additional constraints were necessary because of the current limitations of the trim coil power supplies. Therefore, if the first minimization of the square of the errors required any currents to be larger than those allowed, some adjustment had to be made. If only one current exceeded the current limit, this was set equal to the limiting current, i.e.,

made a constant instead of a variable, and the procedure repeated with one less current variable. However, if two or more of the best currents were greater than that allowed, a decision had to be made as to which to set at the maximum current and which to continue as variables. Our choice was to set that current equal to the maximum whose best value differed from the maximum by the largest amount.

For the most part, the results from our computer program, using the least squares subroutine, yielded current settings which were very successful in obtaining the desired beam. However, for some of the beams desired the program yielded three or more trim coils at the limiting values, and large root mean square errors. Moreover, at one time a breakdown in one of the coils made it necessary to operate with a lower limiting current. In some cases this changed the order in which trim coils were set at their limiting values. Since with linear programming the current constraints are included in the program from the beginning, it was thought that linear programming might yield better values.

We therefore wrote a very general linear programming subroutine,⁵ using the revised simplex method, and including a section to resolve degeneracies. (Our problem had a tenfold degeneracy at the very beginning.) The use of this subroutine is limited to problems of a size compatible with the size of the computer being used, since all the storage is in the memory of the computer. For our problem, the Naval Research Laboratory CDC 3800 had sufficient storage.

This subroutine was included in our main program for calculating fields and trim coil currents, described above, and was used to minimize the field errors, after setting up the problem in dual form, as will be described next.

SETTING UP THE LINEAR PROGRAMMING PROBLEM

The setting up of a problem somewhat similar to ours has been described by Garren.¹ Following Garren, we introduce a variable, λ , and specify that the absolute value of the field error at each radius must be less than λ . That is,

$$-\lambda \leq e_j \leq \lambda, \quad (2)$$

where e_j is given by Eq. (1). This condition can be written as the two conditions

$$\begin{aligned} e_j + \lambda &\geq 0 \\ e_j - \lambda &\leq 0, \end{aligned} \quad (3)$$

or, putting them both as greater-than conditions,

$$\begin{aligned} e_j + \lambda &\geq 0 \\ -e_j + \lambda &\geq 0. \end{aligned} \quad (4)$$

The object of the linear program is then to minimize the value of λ .

In addition to the limits on the errors it is necessary to put constraints on the size of the currents in the trim coils. If for the i th trim coil the magnitude of the maximum current allowed is W_i^{\max} , we must have

$$-W_i^{\max} \leq w_i \leq W_i^{\max}. \quad (5)$$

This again can be put as two conditions:

$$\begin{aligned} w_i &\geq -W_i^{\max} \\ w_i &\leq W_i^{\max}. \end{aligned} \quad (6)$$

Since the linear programming routine solves only for positive values of the variables w_i , we changed the problem to solve for w'_i where

$$w'_i = w_i + 800. \quad (7)$$

The number 800 was chosen because that was the maximum number of amperes the trim coil power supplies could carry. Rewriting the current conditions in terms of w'_i , and putting both as greater-than conditions, we have

$$\begin{aligned} w'_i &\geq 800 - W_i^{\max} \\ -w'_i &\geq -800 - W_i^{\max}. \end{aligned} \quad (8)$$

Replacing e_j by its value in Eq. (1), setting $w_i = w'_i - 800$,

$$\Delta B(r_j) = \bar{B}^{\text{iso}}(r_j) - \bar{B}^0(r_j), \quad (9)$$

and bringing all the constant values to the right-hand side of the expressions, the linear programming problem for obtaining the trim coil currents becomes:

Find positive values of w'_i which satisfy the following constraints:

$$\begin{aligned} -\sum_{i=1}^{10} \frac{\partial B(r_j)}{\partial w_i} w'_i + \lambda &\geq -\Delta B(r_j) - 800 \sum_{i=1}^{10} \frac{\partial B(r_j)}{\partial w_i} \\ & \quad j = 1, \dots, n, \\ \sum_{i=1}^{10} \frac{\partial B(r_j)}{\partial w_i} w'_i + \lambda &\geq \Delta B(r_j) + 800 \sum_{i=1}^{10} \frac{\partial B(r_j)}{\partial w_i} \\ & \quad j = 1, \dots, n, \end{aligned} \quad (10)$$

$$\begin{aligned} w'_i &\geq 800 - W_i^{\max} & i = 1, \dots, 10, \\ -w'_i &\geq -800 - W_i^{\max} & i = 1, \dots, 10, \end{aligned}$$

and minimize the objective function, λ . Here n is the number of radial positions in the region being fitted.

The radial region in which the fit was made was set by the data input cards, but was generally from 6 inches to 30 inches. Thus, each of the first two constraints consisted of 25 inequalities, while each of the last two consisted of 10 inequalities, since there are ten trim coils—a total of 70 inequalities.

In solving a linear programming problem, the inequalities are made into equalities by the addition of slack variables. If the inequality is $>$, a slack variable is subtracted, while for a $<$ inequality, a slack variable is added. In our case, 70 slack variables would be needed. In addition, some artificial variables would be needed in order to obtain an initial solution from which to proceed to others which would decrease the value of λ . With the problem thus set up, in order to include both slack and artificial variables, our main working matrix would have to have dimensions of at least 70 rows and 151 columns. To decrease this matrix we chose to put the problem in the dual form.⁶

The inequalities (10) are already in the proper form for changing to the dual problem, i.e., the inequalities are all $>$, and the objective function,

λ , is to be minimized. If we write the coefficients of the inequalities of (10) as a_{ij} , and b_i the inequalities become

$$\begin{array}{rccccccc}
 a_{11}w'_1 & + & a_{12}w'_2 & + \cdots + & a_{1,10}w'_{10} & + & \lambda & \geq & b_1 \\
 a_{21}w'_1 & + & a_{22}w'_2 & + \cdots + & a_{2,10}w'_{10} & + & \lambda & \geq & b_2 \\
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 a_{2n,1}w'_1 & + & a_{2n,2}w'_2 & + \cdots + & a_{2n,10}w'_{10} & + & \lambda & \geq & b_{2n} \\
 w'_1 & & & & & & & \geq & b_{2n+1} \\
 & & w'_2 & & & & & \geq & b_{2n+2} \\
 & & & \cdots & & & & & \vdots \\
 & & & & & & w'_{10} & \geq & b_{2n+10} \\
 -w'_1 & & & & & & & \geq & b_{2n+11} \\
 & & -w'_2 & & & & & \geq & b_{2n+12} \\
 & & & \cdots & & & & & \vdots \\
 & & & & & & -w'_{10} & \geq & b_{2n+20}
 \end{array} \tag{11}$$

The dual to a problem is formed by changing the column constants to the coefficients of new variables y_j , the inequalities thus formed being made \leq the coefficients of the objective function, which is to be maximized. Thus, our dual problem is to

maximize

$$b_1 y_1 + \cdots + b_{2n} y_{2n} + b_{2n+1} y_{2n+1} + \cdots + b_{2n+10} y_{2n+10} + b_{2n+11} y_{2n+11} + \cdots + b_{2n+20} y_{2n+20}$$

subject to the constraints

$$\begin{array}{rccccccc}
 a_{11}y_1 & + \cdots + & a_{2n,1}y_{2n} & + & y_{2n+1} & & & - & y_{2n+11} & \leq & 0 \\
 a_{12}y_1 & + \cdots + & a_{2n,2}y_{2n} & & & + & y_{2n+2} & & & - & y_{2n+12} & \leq & 0 \\
 \vdots & & \vdots & & & & \ddots & & & & \vdots & & \\
 a_{1,10}y_1 & + \cdots + & a_{2n,10}y_{2n} & & & + & y_{2n+10} & & & - & y_{2n+20} & \leq & 0 \\
 y_1 & + \cdots + & & & & & & & & & & & \leq & 1,
 \end{array}$$

or specifically:

maximize

$$\begin{aligned}
 & \left[-\Delta B(r_1) - 800 \sum_{i=1}^{10} \frac{\partial B(r_1)}{\partial w_i} \right] y_1 + \cdots + \left[-\Delta B(r_1) - 800 \sum_{i=1}^{10} \frac{\partial B(r_n)}{\partial w_i} \right] y_n \\
 & + \left[\Delta B(r_1) + 800 \sum_{i=1}^{10} \frac{\partial B(r_1)}{\partial w_i} \right] y_{n+1} + \cdots + \left[\Delta B(r_n) + 800 \sum_{i=1}^{10} \frac{\partial B(r_n)}{\partial w_i} \right] y_{2n} \\
 & + (800 - W_1^{\max}) y_{2n+1} + \cdots + (800 - W_{10}^{\max}) y_{2n+10} \\
 & + (-800 - W_1^{\max}) y_{2n+11} + \cdots + (-800 - W_{10}^{\max}) y_{2n+20}
 \end{aligned}$$

subject to the constraints

$$\begin{aligned}
 & -\frac{\partial B(r_1)}{\partial w_1} y_1 - \dots - \frac{\partial B(r_n)}{\partial w_1} y_n + \frac{\partial B(r_1)}{\partial w_1} y_{n+1} + \dots + \frac{\partial B(r_n)}{\partial w_1} y_{2n} + y_{2n+1} && - y_{2n+11} && \leq 0 \\
 & -\frac{\partial B(r_1)}{\partial w_2} y_1 - \dots - \frac{\partial B(r_n)}{\partial w_2} y_n + \frac{\partial B(r_1)}{\partial w_2} y_{n+1} + \dots + \frac{\partial B(r_n)}{\partial w_2} y_{2n} && + y_{2n+2} && - y_{2n+12} && \leq 0 \\
 & \vdots && \vdots && \vdots && \vdots \\
 & -\frac{\partial B(r_1)}{\partial w_{10}} y_1 - \dots - \frac{\partial B(r_n)}{\partial w_{10}} y_n + \frac{\partial B(r_1)}{\partial w_{10}} y_{n+1} + \dots + \frac{\partial B(r_n)}{\partial w_{10}} y_{2n} && + y_{2n+10} && - y_{2n+20} && \leq 0 \\
 & y_1 + && + y_n && + y_{n+1} && + y_{2n} && \leq 1.
 \end{aligned}$$

COMPARISON OF COMPUTER RESULTS

The computer printout of the results included the difference between the desired isochronous field, and the resultant field due to the main magnet and the trim coils. To compare the least squares and the linear programming methods, we have plotted this difference as a function of radius. A few of the results are shown in Figures 1 to 4. On each figure the value of the isochronous field, \bar{B}^{iso} ($r = 20$ inches) is given at the 20-inch radius. The arrow to the right of each figure gives the residual field error, ΔB , which would produce the fractional frequency change indicated, since $\Delta B/\bar{B}^{iso} \approx \Delta\omega/\omega^{iso}$.

Figures 1 and 2 give the results for 50.35-MeV protons and 43.23-MeV alphas, respectively. These are examples of good agreement between the two methods. Tables I and II give the trim coil values chosen by the two methods for these two examples,

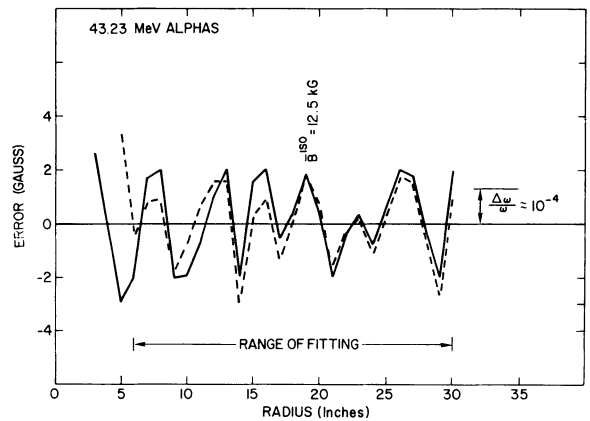


FIGURE 2 Comparison of residual errors for 43.23-MeV alphas with RISOC = 34 inches and the range of fitting from 6 inches to 30 inches. Full lines are linear programming, dashed lines are least squares.

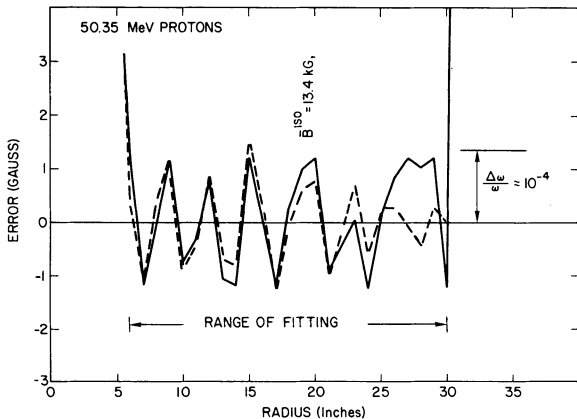


FIGURE 1 Comparison of residual errors for 50.35-MeV protons with RISOC = 32 inches and the range of fitting from 6 inches to 30 inches. Full lines are linear programming, dashed lines are least squares.

TABLE I

Trim coil currents obtained for 50.35-MeV protons with RISOC = 32 inches and the range of fitting from 6 inches to 30 inches. (See Figure 1)

Trim coil no.	Least squares (amperes)	Linear programming (amperes)
1	-411.40	-428.70
2	-186.14	-181.04
3	174.65	170.54
4	401.64	403.09
5	288.63	289.04
6	377.47	371.97
7	438.78	443.37
8	552.00	553.91
9	-289.81	-350.51
10	338.41	403.06
RMS error	0.7300 gauss	0.9413 gauss

TABLE II

Trim coil currents obtained for 43.23-MeV alphas with RISOC = 34 inches and the range of fitting from 6 inches to 30 inches. (See Figure 2.)

Trim coil no.	Least squares (amperes)	Linear programming (amperes)
1	-270.51	-223.30
2	46.16	26.53
3	437.06	449.62
4	652.91	650.46
5	557.93	554.93
6	586.19	589.32
7	721.00	720.27
8	800.00	800.00
9	345.59	350.41
10	-533.49	-539.42
RMS error	1.3410 gauss	1.5495 gauss

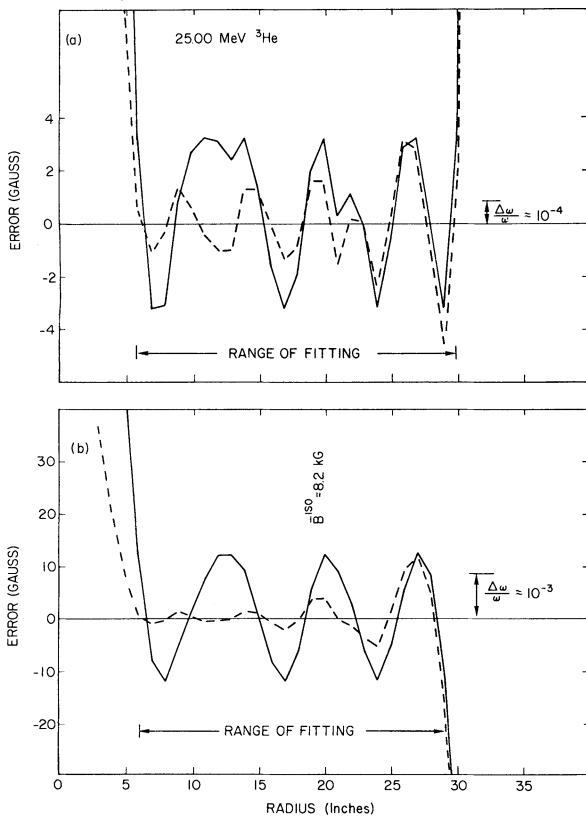


FIGURE 3 Comparison of residual errors for 25.00-MeV ^3He . (a) RISOC = 32 inches and range of fitting from 6 inches to 30 inches. (b) RISOC = 35 inches, range of fitting from 6 inches to 29 inches, and 9th trim coil (T_9) set equal to 0.0. Full lines are linear programming, dashed lines are least squares.

and also the root mean square error of the deviations for the results within the range of fitting.

Figure 3 and Table III give the results for 25.00-MeV ^3He ions under different conditions of calculation. Both the range of fitting and RISOC, the radius at which the calculated isochronous field was normalized to the main magnet field, i.e., the radius at which it was assumed no field correction was necessary, are different in the two cases. Also, in one, trim coil 9 (T_9) was set equal to zero. This shows that the best fit is dependent on the conditions used for the calculation. Our computer program, when asked to find the trim coil values for a certain energy, automatically does the calculations for two or three values of RISOC. Different ranges of fitting must be done in different runs.

TABLE III

Trim coil currents obtained for 25.00-MeV ^3He .

(a) RISOC = 32 inches and range of fitting 6 inches to 30 inches. [See Figure 3(a).]

Trim coil no.	Least squares (amperes)	Linear programming (amperes)
1	24.00	-61.10
2	231.77	263.50
3	486.09	481.82
4	417.61	402.20
5	308.81	325.31
6	437.36	424.84
7	619.53	625.75
8	800.00	800.00
9	771.60	767.64
10	-362.65	-360.99
RMS error	1.6463 gauss	2.5203 gauss

(b) RISOC = 35 inches, range of fitting 6 inches to 29 inches, and trim coil no. 9 set equal to 0.0 [See Figure 3(b)]

Trim coil no.	Least squares (amperes)	Linear programming (amperes)
1	25.31	-266.27
2	217.59	300.47
3	490.84	508.46
4	397.87	316.74
5	324.14	411.54
6	395.52	326.11
7	657.31	690.90
8	800.00	800.00
9	0.00	0.00
10	32.74	23.48
RMS error	5.0724 gauss	8.9511 gauss

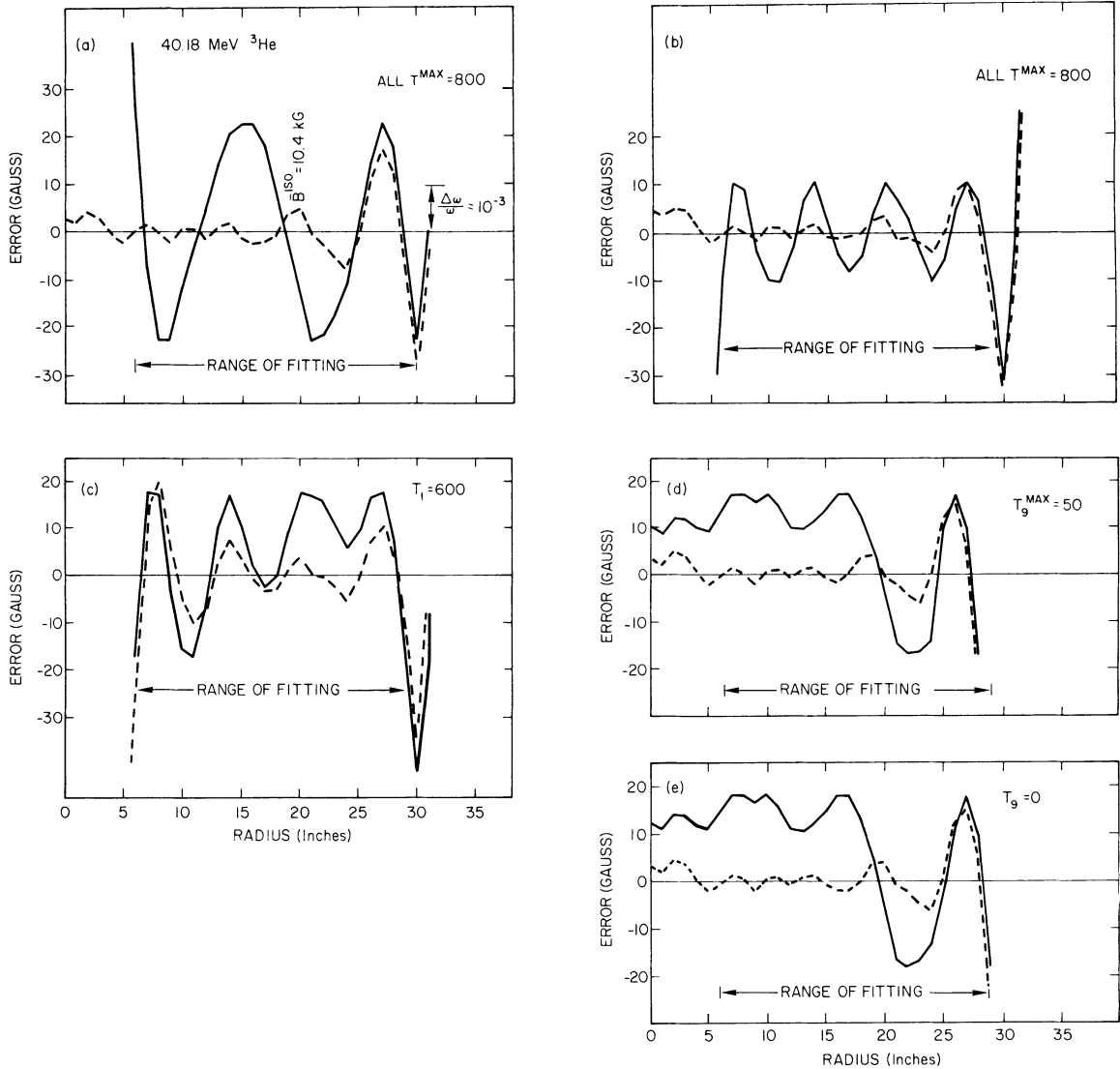


FIGURE 4 Comparison of residual errors for 40.18-MeV ${}^3\text{He}$. (a) Maximum of all trim coils 800 amperes, and range of fitting from 6 inches to 30 inches. (b) Maximum of all trim coils 800 amperes and range of fitting 6 inches to 29 inches. (c) First trim coil (T_1) set equal to 600 amperes, all other maxima 800 amperes, and range of fitting from 6 inches to 29 inches. (d) Maximum of 9th trim coil (T_9) set equal to 50 amperes, all other maxima 800 amperes, and range of fitting from 6 inches to 29 inches. (e) T_9 set equal to zero, all other maxima 800 amperes, and range of fitting from 6 inches to 29 inches. On all curves, RISOC = 35 inches, $B^{\text{iso}} = 10.4$ kG at 20 inches, and ΔB for $\Delta\omega/\omega \approx 10^{-3}$ is the same as on (a). Full lines are linear programming and dashed lines are least squares.

Figure 4 gives results for 40.18-MeV ${}^3\text{He}$ ions under different conditions. In all cases, RISOC was 35 inches. Figures 4a and 4b compare different ranges of fitting when all trim coils are used, while Figures 4c, 4d, and 4e compare the results for a fitting range of 6 inches to 29 inches, when certain trim coils are set to fixed values or have a lower allowed maximum. None of the results were in

this case very good, as evidenced by the large rms errors given in Table IV, which also gives the trim coil values determined by the two methods.

In looking at the representative results given in Figures 1 to 4 and Tables I to IV, it is evident that when the fit to the isochronous field is good, as shown by small rms errors, there is little difference between the results of least squares and linear pro-

TABLE IV
Trim coil currents obtained for 40.18-MeV ^3He .
RISOC = 35 inches.

(a) Maximum of all trim coils 800 amperes and range of fitting from 6 inches to 30 inches [See Figure 4(a).]

Trim coil no.	Least squares (amperes)	Linear programming (amperes)
1	-37.99	-501.79
2	146.78	232.63
3	597.34	627.90
4	662.79	697.41
5	548.64	415.68
6	511.54	586.82
7	800.00	800.00
8	800.00	800.00
9	434.13	420.80
10	-800.00	-800.00
RMS error	7.907 gauss	16.782 gauss

(b) Maximum of all trim coils 800 amperes and range of fitting from 6 inches to 29 inches. [See Figure 4(b).]

Trim coil no.	Least squares (amperes)	Linear programming (amperes)
1	-43.13	341.41
2	150.53	-34.17
3	592.01	730.84
4	671.73	569.94
5	532.31	620.14
6	535.91	470.00
7	768.05	800.00
8	800.00	800.00
9	456.16	451.49
10	-800.00	-800.00
RMS error	4.338 gauss	7.866 gauss

(c) 1st trim coil (T_1) set equal to 600 amperes, maximum of all others 800 amperes, and range of fitting from 6 inches to 29 inches. [See Figure 4(c).]

Trim coil no.	Least squares (amperes)	Linear programming (amperes)
1	600.00	600.00
2	-83.66	-147.13
3	714.54	800.00
4	617.17	556.38
5	561.02	616.44
6	523.62	487.66
7	772.67	800.00
8	800.00	736.19
9	458.33	495.97
10	-800.00	-800.00
RMS error	9.069 gauss	13.019 gauss

(d) Maximum of 9th trim coil (T_9^{max}) set equal to 50 amperes, maximum of all others 800 amperes, and range of fitting from 6 inches to 29 inches. [See Figure 4(d).]

Trim coil no.	Least squares (amperes)	Linear programming (amperes)
1	-41.95	-27.41
2	145.14	153.02
3	590.63	549.46
4	659.51	721.93
5	539.11	438.48
6	511.12	541.76
7	785.99	800.00
8	800.00	800.00
9	50.00	50.00
10	-342.49	-352.45
RMS error	6.317 gauss	13.596 gauss

(e) 9th trim coil (T_9) set equal to zero, maximum of all others 800 amperes, and range of fitting 6 inches to 29 inches. [See Figure 4(e).]

Trim coil no.	Least squares (amperes)	Linear programming (amperes)
1	-42.15	-30.37
2	144.74	154.30
3	590.25	544.68
4	658.47	728.08
5	539.13	427.18
6	508.87	544.42
7	786.99	800.00
8	800.00	800.00
9	0.00	0.00
10	-285.02	-294.80
RMS error	6.472 gauss	14.712 gauss

gramming. But, when these rms errors are large, the least squares method gives over-all, less deviation from the calculated isochronous field.

OBTAINING CYCLOTRON BEAMS USING COMPUTER RESULTS

When operation of the cyclotron was first begun in 1967, a systematic test of the computer results using least squares was made with numerous main coil current settings ranging from the lowest to the highest. When the computed trim coil currents were dialed in, some adjustment was needed for the settings of trim coils No. 1 and No. 10, but in all cases a beam was obtained with only very minor

twiddling of the other trim coil dials. No such systematic test was made using the trim coil currents obtained by the linear programming method, since by that time settings were known for most of the beams desired, and obtaining them had become routine. However, when values for new ions or energies were needed, the linear programming routine was frequently used, and no particular difficulty was found in obtaining the beams.

The success of our computer program is evidenced by the fact that never has any time had to be set aside for beam development. New beams are usually obtained from the computed dial settings within less than 15 minutes.

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