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RANDOM PERTURBATIONS OF THE TRANSVERSE MOTION OF PROTONS IN A LINEAR ACCELERATOR AND THEIR CORRECTION

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The influence of random perturbations of focusing channel parameters on the transverse dimensions of a beam in a linear proton accelerator is calculated by a new method. The authors did not investigate the isolated particle behaviour but investigated the four-dimensional phase volume. This allowed estimation of the increase of the beam radius and of the emittance without any essential limitations of the size of the random perturbations. Simulation by means of computers is used in parallel with analytical methods. The possibility of correcting the centre of gravity of the transverse position of the beam and its phase volume transformation is briefly discussed.

INTRODUCTION

Quadrupole focusing in linear proton accelerators is particularly sensitive to random errors in strong-focusing channels. Even if the stability of the focusing fields is high and the manufacture and installation of the focusing elements are held within close tolerances, the random errors increase the radius of the beam and the emittance as the beam travels through the accelerator.

Analytical investigation of the influence of random errors on proton beam parameters appears in Refs. 1–3. The major errors which contribute to the increase of the radius of the beam, were analysed. They are: the displacement of the magnetic axes of the quadrupole lenses relative to the accelerator axis; the rotation of the median axes of the lenses around the longitudinal axis, and the deviation of the magnetic field gradients in the lenses from their nominal values. The first leads to coherent beam oscillations; the rest lead to a distortion of the form of its phase volume without displacing the beam.

In Refs. 1–3, the beam parameters were calculated assuming that the increase in the radius of the beam was small. If the tolerances and the number of focusing periods are large or if the focusing is done with quadrupole lenses which are short in comparison with the period length, the formulae of Refs. 1–3 can be seriously inaccurate and also cease to describe the character of the fluctuation of the beam parameters. As long as the output energy and the average particle current are small and the loss of particles does not lead to important radioactive contamination of the accelerator, the present theory satisfies practical demands. But when the energy and the average intensity of the beam increase, a smaller loss of particles is required. This leads to the necessity to investigate in detail random perturbations of the transverse motion of the beam.

In the present article, this problem is solved without any essential limitations of random perturbation values. The coherent oscillations of the beam and the distortion of the form of its phase volume have been investigated. Along with analytical methods, statistical models using computers (Monte Carlo method) were applied. The transverse motion was considered in the linear approximation. In contrast to the enumerated references in which only probable values and dispersion of parameters were calculated, we find probable limits of the beam parameters. We also discuss beam correction problems.

As in Refs. 1–3 we, also, did not take the Coulomb interaction into account. This does not seriously restrict the field of application of our results, since the effects under consideration are predominant in the high energy section of the 'meson factory' type accelerators, in which the influence of Coulomb forces is small.

1. TRANSVERSE DISPLACEMENTS OF QUADRUPOLE LENSES

In order to determine the displacement of the centre of the beam, let us consider a quadrupole lens, whose action is described by a matrix M in the phase plane (x, x'). If we denote the displacements along the x-axis of this lens at the beginning and the end by Δx_B and Δx_E respectively, the relation between the initial and final coordinates of the particle which passes through this lens will be given by:

$$\begin{pmatrix} x_E \\ x'_E \end{pmatrix} = \begin{pmatrix} \Delta x_E \\ \Delta x'_E \end{pmatrix} + M \begin{pmatrix} x_B \\ x'_B \end{pmatrix} - M \begin{pmatrix} \Delta x_B \\ \Delta x'_B \end{pmatrix}. \quad (1.1)$$

Here $\Delta x'_E = \Delta x'_B = (\Delta x_E - \Delta x_B)/\xi$; ξ is the lens length, expressed in units of focusing period length. The action of the displaced lens on the beam is reduced to a nondisplaced lens and some equivalent transformation of the coordinates at the chosen point of the focusing period. The displacement errors of all the focusing period lenses can be reduced to equivalent coordinates at a chosen point of the period.

Let us consider the errors in a channel with a focusing period consisting of two quadrupole lenses as a special case. The system $(x, x'/v)$ is chosen as a coordinate system, and we take as primary lens errors the linear combinations of the displacements of the lenses, $\Delta = (\Delta x_B + \Delta x_E)/2$ and $\Delta_1 = (\Delta x_E - \Delta x_B)/2$. As a reference point inside the focusing lens, we choose the point at which the beam envelope reaches an extremum.

Two such neighbouring points are considered as the bounds of the focusing period. Then

$$\begin{pmatrix} \Delta x \\ \dfrac{\Delta x'}{v} \end{pmatrix} = M_{p/2D} \left[M_{DE}^{-1} \begin{pmatrix} \Delta x_E \\ \dfrac{\Delta x'_E}{v} \end{pmatrix}_D - M_{DB} \begin{pmatrix} \Delta x_B \\ \dfrac{\Delta x'_B}{v} \end{pmatrix}_D \right]$$
$$- M_{FB} \begin{pmatrix} \Delta x_B \\ \dfrac{\Delta x'_B}{v} \end{pmatrix} + M_{FE}^{-1} \begin{pmatrix} \Delta x_E \\ \dfrac{\Delta x'_E}{v} \end{pmatrix}. \quad (1.2)$$

Here $M_{p/2D}$ is the matrix of the half-period, which starts at a point inside the defocusing lens and lies between the points in which the crossovers of the beam are located:

$$M_{p/2D} = \begin{pmatrix} \gamma \cos \mu_1 & \dfrac{1}{\gamma} \sin \mu_1 \\ -\gamma \sin \mu_1 & \dfrac{1}{\gamma} \cos \mu_1 \end{pmatrix} \quad (1.3)$$

v is the product of the minimum frequency of the transverse oscillations and the time of the beam passing through a focusing period; γ is the coefficient of the modulation of the envelope[2]; μ_1 is the angular advance of the transverse oscillations per half-period; $M_{DB,K}$ and $M_{FB,E}$ are the matrices of the defocusing and focusing lenses of the initial and final parts, respectively,

$$M_{DB,E} = \begin{pmatrix} \cosh \Lambda \xi_{1,2} & \dfrac{v}{\Lambda} \sinh \Lambda \xi_{1,2} \\ \dfrac{\Lambda}{v} \sinh \Lambda \xi_{1,2} & \cosh \Lambda \xi_{1,2} \end{pmatrix},$$

$$M_{FB,E} = \begin{pmatrix} \cos \Lambda \xi_{1,2} & \dfrac{v}{\Lambda} \sin \Lambda \xi_{1,2} \\ -\dfrac{\Lambda}{v} \sin \Lambda \xi_{1,2} & \cos \Lambda \xi_{1,2} \end{pmatrix}.$$

ξ_1 and ξ_2 are the relative lengths of the initial and final parts of the lenses and the indices F and D refer to the focusing and defocusing lenses, respectively. The dimensionless parameter Λ is equal to

$$\Lambda = \left[\frac{eGL^2(1-\beta^2)^{1/2}}{m_0 c^2 \beta} \right]^{1/2}.$$

Here G is the gradient of the magnetic field in the lenses, e and m_0 are the proton charge and mass, at rest, respectively, β the ratio of the particle velocity to the velocity of light c, and L is the length of the focusing period.

If the equivalent displacements in all the periods are known, it is possible to obtain the beam displacement in a channel with n periods:

$$\begin{pmatrix} \Delta x \\ \dfrac{\Delta x'}{v} \end{pmatrix}_n = \sum_{j=1}^{n} \begin{pmatrix} \cos \mu(n-j) & \sin \mu(n-j) \\ -\sin \mu(n-j) & \cos \mu(n-j) \end{pmatrix} \begin{pmatrix} \Delta x \\ \dfrac{\Delta x'}{v} \end{pmatrix}_j. \quad (1.4)$$

The amplitude of the oscillations of the centre of the beam along the x-axis, after traversing a channel with n periods, can be calculated according to the formula

$$x_n^2 = (\Delta x, \Delta x'/v) \begin{pmatrix} \Delta x \\ \dfrac{\Delta x'}{v} \end{pmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{n} (\Delta x, \Delta x'/v)_j$$

$$\times\begin{pmatrix} \cos\mu(j-i) & \sin\mu(j-i) \\ -\sin\mu(j-i) & \cos\mu(j-i) \end{pmatrix}\begin{pmatrix} \Delta x \\ \dfrac{\Delta x'}{v} \end{pmatrix}_j, \tag{1.5}$$

where $(\Delta x, \Delta x'/v)$ is a row-vector. Assuming that the primary errors related to different focusing periods are independent and have the same dispersion, we obtain from (1.2) and (1.5) for an average value of x_n^2 the expression:

$$\overline{x_n^2} = n\left[\overline{(\Delta x)^2} + \overline{\left(\frac{\Delta x'}{v}\right)^2}\right] = n\overline{\Delta^2}\sum_{i=1}^{8} m_i^2.$$

Here the coefficients

$$m_1 = -(\cos\Lambda\xi_1 - \cos\Lambda\xi_2)$$

$$m_2 = (\cos\Lambda\xi_1 + \cos\Lambda\xi_2) - \frac{2}{\Lambda\xi}(\sin\Lambda\xi_1 + \sin\Lambda\xi_2)$$

$$m_3 = \frac{\Lambda}{v}(\sin\Lambda\xi_1 + \sin\Lambda\xi_2)$$

$$m_4 = \frac{\Lambda}{v}\left[(\sin\Lambda\xi_1 - \sin\Lambda\xi_2) + \frac{2}{\Lambda\xi}(\cos\Lambda\xi_1 - \cos\Lambda\xi_2)\right]$$

determine the contribution of the focusing lens, the coefficients

$$m_5 = \gamma(\cosh\Lambda\xi_1 - \cosh\Lambda\xi_2)$$

$$m_6 = \gamma(\cosh\Lambda\xi_1 + \cosh\Lambda\xi_2) - \frac{2}{\Lambda\xi}(\sinh\Lambda\xi_1 + \sinh\Lambda\xi_2)$$

$$m_7 = \frac{\Lambda}{v\gamma}(\sinh\Lambda\xi_1 + \sinh\Lambda\xi_2)$$

$$m_8 = -\frac{\Lambda}{v\gamma}\left[(\sinh\Lambda\xi_1 - \sinh\Lambda\xi_2) - \frac{2}{\Lambda\xi}(\cosh\Lambda\xi_1 - \cosh\Lambda\xi_2)\right]$$

determine those of the defocusing lens, $\overline{\Delta^2} = \overline{\Delta_1^2}$, where Δ and Δ_1 were defined above.

If the lens asymmetry with respect to the cross-over points is very small or zero and if the angles of deflection in the half-lenses are small enough so that it is possible to use $\sin(\Lambda\xi/2) = \sinh(\Lambda\xi/2) = \Lambda\xi/2$ and $\cos(\Lambda\xi/2) = \cosh(\Lambda\xi/2) = 1$ (thin lenses), the expression for $\overline{x_n^2}$ can be reduced to

$$\overline{x_n^2} = \frac{\Lambda^4\xi^2}{v^2}\left(1+\frac{1}{\gamma^2}\right)n\overline{\Delta^2},$$

or by using the gap coefficient $\varepsilon = 1-2\xi$

$$\overline{x_n^2} = \frac{\Lambda^4(1-\varepsilon)^2}{4v^2}\left(1+\frac{1}{\gamma^2}\right)n\overline{\Delta^2}. \tag{1.6}$$

For symmetrical focusing structures $\overline{x_n^2} = \overline{y_n^2}$.

It is impossible to calculate the probability that the quantities x_n and y_n will not exceed the given confidence values x_n^* and y_n^* if only the quantities $\overline{x_n^2}$ and $\overline{y_n^2}$ are known (without the knowledge of a distribution law). It is necessary to use the relation

$$x_n^2 = \Delta x_n^2 + \frac{1}{v^2}(\Delta x_n')^2$$

in order to determine a distribution law of the quantity x_n. The quantities Δx_n and $\Delta x_n'$ are linear combinations of a great number of initial errors with zero mean values and that is why they distributed according to the normal law with zero mean values. If the relation between the dispersions of the Δx_n and $\Delta x_n'$ are known the law of distribution can be determined. It is practically very complicated to calculate this relation and therefore we shall consider two extreme cases:

1. $\overline{\Delta x_n^2} = \overline{(\Delta x_n'/v)^2}$
2. $\overline{(\Delta x_n'/v)^2} = 0$ (or $\overline{\Delta x_n^2} = 0$).

In the first case, we have the following expressions for the probability integral of x_n and its mean-square value,

$$F(x_n) = 1 - \exp(-x_n^2/\overline{x_n^2}).$$

In the second case, x_n is distributed similarly to the module of the quantity distributed according to the normal law. Then the equation of the probability integral takes the form

$$F(x_n) = \mathrm{erf}\,[x_n/(2\overline{x_n^2})^{1/2}].$$

The values of x_n^*, corresponding to different values of the probability integrals for both cases are given in Table I. It is easy to see that the difference of estimates for both cases are small.

TABLE I

Confidence Estimates x_n^*

$F(x_n)$	$K = x_n^*/(\overline{x_n^2})^{1/2}$ First Case	Second Case
0.90	1.51	1.64
0.95	1.73	1.96
0.98	2.14	2.35

If the upper estimate of the amplitude of oscillations along every coordinate is known, it is possible to obtain an upper estimate of the radius of the beam as well.

For the cases considered above, R_n^* will have the value

$$R_n^* = x_n^*[1+(1/\gamma)^2]^{1/2}.$$

In the thin lens approximation

$$R_n^* = Kn^{1/2}\Lambda^2(1-\varepsilon)[1+(1/\gamma)^2](\overline{\Delta^2})/(2v). \quad (1.7)$$

Here the coefficient K is chosen from the table according to the accepted value of the confidence estimate.

2. INFLUENCE OF INSTABILITIES IN MAGNETIC FIELD GRADIENTS IN QUADRUPOLE LENSES

Instabilities of magnetic fields in quadrupole lenses do not lead to a coupling between the particle oscillations along the x- and y-axes. Therefore it is possible to solve problems concerning the increase of the dimensions of the beam independently along any axis. Let us consider an increase of the dimension of the beam along the x-axis.

We shall determine the increase in the dimension of a beam whose dimension in an ideal channel changes periodically and the period of this change is equal to the focusing period. Such a beam is said to be matched to the channel. The beam crossover points, in which its dimension is maximal, are chosen as focusing period boundaries. We choose $(x, x'/v)$ as a coordinate system. In this system the matrix of the focusing period is a rotation matrix and the boundary of the consistent phase volume is a circle. We shall consider the ratio x_m/x_{m0} as a coefficient θ of the increase in the dimension of the beam due to errors that arise in the section of a strong-focusing channel containing n focusing periods. Here x_m is the radius of the circle which circumscribes the beam phase volume at the exit of the section of the channel under consideration; x_{m0} is the beam radius at the boundaries of the focusing period in an ideal channel.

Let us relate the increase of the beam dimension with the coefficients of the reciprocal matrix of the ellipse which limits the beam phase volume at the exit of the error-containing channel. This matrix has the form

$$M_E^{-1} = x_{m0}^2\begin{pmatrix} e_{11} & e_{12} \\ e_{12} & e_{22}\end{pmatrix} = x_{m0}^2 N_E^{-1},$$

where N_E^{-1} is the reciprocal matrix of the ellipse of the beam. It is implied that the dimension of the beam in an ideal channel is equal to one (we shall call this matrix the 'normalized inverse matrix').

When the beam moves through an ideal channel, the ellipse representing it rotates in each focusing period through an angle μ and becomes a canonical ellipse at a certain point of the channel. The inverse matrix of the ellipse at this point has the form

$$M_{EK}^{-1} = x_{m0}^2\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2\end{pmatrix} = x_{m0}^2 N_E^{-1}.$$

Assume $\lambda_1 \geq \lambda_2$. In this case, $x_{m0}^2\lambda_1 = x_m^2$ and, consequently, λ_1 is the square of the coefficient of the increase in the dimension of the beam.

The ellipse N_{EK}^{-1} is obtained from the ellipse N_E^{-1} by rotation. Therefore, the trace of the matrix and its determinant are conserved in the matrix N_{EK}^{-1} of the ellipse. If we take into account that the determinant of the inverse matrix of the ellipse does not change either, when the beam moves through an error-containing channel, and that Det $N_{E0}^{-1} = 1$, we can write

$$e_{11}e_{22}-e_{12}^2 = \lambda_1\lambda_2, \quad e_{11}+e_{22} = \lambda_1+\lambda_2.$$

From here we obtain

$$\theta = \sqrt{\lambda_1} = \sqrt{\frac{\mathrm{Sp}\, N_E^{-1}}{2} + \sqrt{\left(\frac{\mathrm{Sp}\, N_E^{-1}}{2}\right)^2 - 1}}. \quad (2.1)$$

θ is a random positive quantity; its average value

$\bar{\theta}$ can be used as an estimate of the increase in the beam dimension. It is very complicated to calculate the average value; therefore we shall determine the value of $\bar{\bar{\theta}}$ which can be obtained from (2.1) if the mean value of $\mathrm{Sp}\, N_{\mathrm{E}}^{-1}$ is inserted. Further, we shall show that this replacement does not increase the value of $\bar{\theta}$ appreciably.

Let us calculate $\mathrm{Sp}\, N_{\mathrm{E}}^{-1}$ (the average value of the trace of the normalized inverse matrix of the ellipse). The matrix of the ellipse which represents the beam which has passed through n focusing periods is related to the ellipse representing the beam which has passed through $(n-1)$ periods. This relation is given by

$$N_{\mathrm{E},n}^{-1} = \tilde{M}_n N_{\mathrm{E},n-1}^{-1} \tilde{M}_n{}^T.$$

Here $\tilde{M}_n$ is the matrix of the nth error-containing period, and the index 'T' designates the transposition of the matrix. Therefore,

$$\mathrm{Sp}\, N_{\mathrm{E},n}^{-1} = \mathrm{Sp}\,[N_{\mathrm{E},n-1}^{-1}(\tilde{M}_n{}^T \tilde{M}_n)].$$

Let us write

$$N_{\mathrm{E},n}^{-1} = \begin{pmatrix} a_n & b_n \\ b_n & d_n \end{pmatrix}, \qquad \tilde{M}_n{}^T \tilde{M}_n \equiv K_n = \begin{pmatrix} \alpha_n & \beta_n \\ \beta_n & \delta_n \end{pmatrix}.$$

With these notations

$$\tfrac{1}{2}\mathrm{Sp}\, N_{\mathrm{E},n}^{-1} = \tfrac{1}{2}\mathrm{Sp}\, N_{\mathrm{E},n-1}^{-1} \cdot \tfrac{1}{2}\mathrm{Sp}\, K_n + \tfrac{1}{4}(a_{n-1}-d_{n-1})(\alpha_n - \delta_n) + b_{n-1}\beta_n. \tag{2.2}$$

Let us rotate the ellipses described by the matrices $N_{\mathrm{E},n-1}^{-1}$ and K_n through angles ψ and φ, respectively. Then, the matrices describing them will have the form

$$(N_{\mathrm{E}}^{-1})_n{}^{\mathrm{K}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}_n, \qquad K_n{}^{\mathrm{K}} = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}_n$$

with

$$a_n - d_n = (\lambda_1 - \lambda_2)\cos 2\psi \quad \alpha_n - \delta_n = (\omega_1 - \omega_2)\cos 2\varphi$$
$$b_n = \tfrac{1}{2}(\lambda_1 - \lambda_2)\sin 2\psi \quad \beta_n = \tfrac{1}{2}(\omega_1 - \omega_2)\sin 2\varphi.$$

Equation (2.2) can be rewritten

$$\tfrac{1}{2}\mathrm{Sp}\, N_{\mathrm{E},n}^{-1} = \tfrac{1}{2}\mathrm{Sp}\, N_{\mathrm{E},n-1}^{-1} \cdot \tfrac{1}{2}\mathrm{Sp}\, K_n + \tfrac{1}{2}(\lambda_1 - \lambda_2)_{n-1}(\omega_1 - \omega_2)_n \cos 2(\psi_{n-1} - \varphi_n). \tag{2.3}$$

The angle $\psi_{n-1} - \varphi_n$ increases by $\mu + \Delta\varphi$ after every focusing period as the beam moves through the channel. Here $\Delta\varphi$ is an error-dependent random increment. It is essentially smaller than μ. If we neglect $\Delta\varphi$, we may average Eq. (2.3) over the primary errors, and obtain

$$\tfrac{1}{2}\overline{\mathrm{Sp}\, N_{\mathrm{E},n}^{-1}} = \tfrac{1}{2}\mathrm{Sp}\, N_{\mathrm{E},n-1}^{-1} \cdot \overline{\tfrac{1}{2}\mathrm{Sp}\, K \cdot X},$$

where

$$X = 1 + \overline{\left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\right)_{n-1}} \cdot \overline{\left(\frac{\omega_1 - \omega_2}{\omega_k + \omega_2}\right)_n} \cos 2n\mu.$$

This means that the coefficient of the increment of the average value of the trace of the matrix of the ellipse after one focusing period is equal to the product of the trace of the matrix of the ellipse and X. It is implied here that the ellipse has passed through one focusing period. $X - 1$ is the product of the slightly changing quantity $\overline{(\lambda_1 - \lambda_2)/(\lambda_1 + \lambda_2)}$, the constant quantity $(\omega_1 - \omega_2)/(\omega_1 + \omega_2)$ and the cosine of the angle, which changes from period to period through certain slowly changing values. The average value of $X - 1$ over a few focusing periods is nearly zero, and so we can consider approximately

$$\tfrac{1}{2}\overline{\mathrm{Sp}\, N_{\mathrm{E},n}^{-1}} = \tfrac{1}{2}\mathrm{Sp}\, N_{\mathrm{E},n-1}^{-1} \cdot \overline{\tfrac{1}{2}\mathrm{Sp}\, K},$$
$$\tfrac{1}{2}\overline{\mathrm{Sp}\, N_{\mathrm{E},n}^{-1}} = (\tfrac{1}{2}\overline{\mathrm{Sp}\, K})^n,$$

and, according to (2.1)

$$\bar{\bar{\theta}} = \sqrt{(\tfrac{1}{2}\overline{\mathrm{Sp}\, K})^n + \sqrt{(\tfrac{1}{2}\overline{\mathrm{Sp}\, K})^{2n} - 1}}. \tag{2.4}$$

Thus the calculation of $\bar{\bar{\theta}}$ is reduced to calculating the trace of the matrix of one focusing period. The final expression $\overline{\mathrm{Sp}\, K}$, under the condition that the beam crossover point lies within the lens (but not necessarily in the centre), and the focusing period contains two quadrupole lenses, is the following:

$$\begin{aligned}\overline{\mathrm{Sp}\, K} = 2 + \tfrac{1}{4}\{ & 2\sin\Lambda\xi[1 + \sin^2\Lambda(\xi_1 - \xi_2)] \\ & - 2\sinh\Lambda\xi[1 - \sinh^2\Lambda(\xi_1 - \xi_2)] \\ & + \left(\frac{\nu}{\Lambda}\right)^2 [\Lambda\xi - \cos\Lambda(\xi_1 - \xi_2)\sin\Lambda\xi]^2 \\ & + \left(\frac{\Lambda}{\nu}\right)^2 [\Lambda\xi + \cos\Lambda(\xi_1 - \xi_2)\sin\Lambda\xi]^2 \\ & + \left(\frac{\nu\gamma^2}{\Lambda}\right)^2 [\Lambda\xi - \cosh\Lambda(\xi_1 - \xi_2)\sinh\Lambda\xi]^2\end{aligned}$$

$$+\left(\frac{\Lambda}{\nu\gamma^2}\right)^2[\Lambda\xi+\cosh\Lambda(\xi_1-\xi_2)\sinh\Lambda\xi]^2\}$$

$$\cdot\overline{\left(\frac{\Delta H}{H}\right)^2}. \quad (2.5)$$

Here $\overline{(\Delta H/H)^2}$ is the standard deviation in the magnitude of the magnetic field in the lenses.

When the focusing period is symmetrical ($\xi_1=\xi_2$), and the increase of the phase of the transverse vibrations over the length of the lenses is small (so that we can consider the lens as a thin one), Eq. (2.5) becomes much simpler:

$$\overline{\mathrm{Sp}\,K}=2+\frac{\Lambda^4(1-\varepsilon)^2}{4\nu^2}(1+1/\gamma^4)\overline{\left(\frac{\Delta H}{H}\right)^2}. \quad (2.6)$$

In order to find the difference between $\bar{\bar{\theta}}$, which is calculated according to (2.4) and (2.5) and the value $\bar{\theta}$, we have determined $\bar{\theta}$ by the method of probability simulation using a computer. For this purpose, we calculated the random simulation of the matrix N_{E}^{-1} by multiplying the matrices of the different focusing periods. Their elements were taken as functions of a primary error amplitude and a random number. The value of $\bar{\theta}$ was obtained by averaging over 500 random simulations. The results of the simulation are given in Figure 1 for the cases of (a) small and (b) large errors. The curves for $\bar{\bar{\theta}}$ are drawn on the same figure for comparison purposes. From this figure it follows that when the values of θ are small, the curves coincide almost completely. When they are large, the curve which is obtained by an analytical method is higher than the computed one. Large values of θ are not used in practice, and we have to calculate them only in order to estimate an upper boundary of the tolerable values of the primary errors. It is seen from the graphs that $\bar{\theta}$ increases much more rapidly than do the primary errors. In this example, after 200 focusing periods, if the primary errors increase five-fold, then the radius of the beam increases 45-fold.

Although $\bar{\theta}$ measures the increase in radius of the beam in a channel, this value is not of practical importance in determining the diameter of the aperture. For this purpose, it is important to know the probability that the quantity θ will not exceed the confidence estimate θ^*. The distribution law of θ has to be known in order to obtain this estimate. It

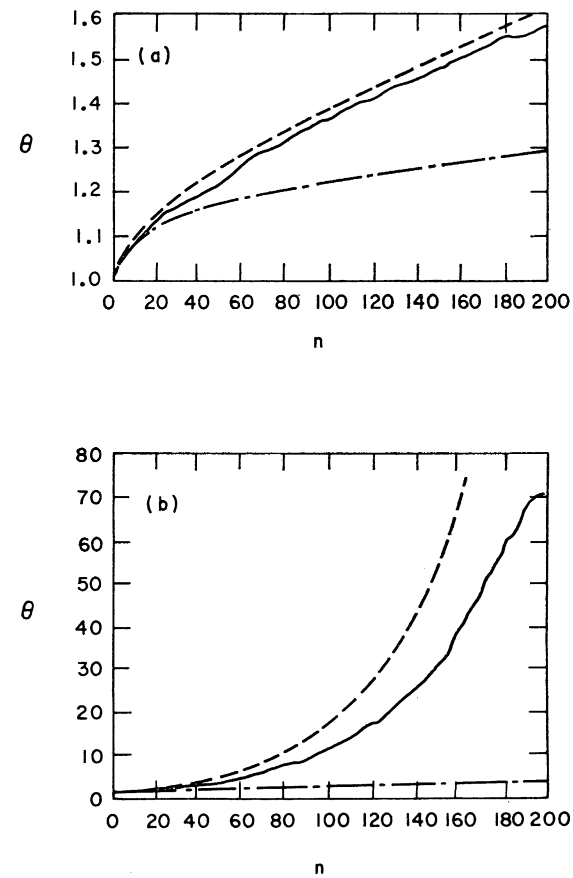


FIGURE 1 Dependence of the increase in the dimension of a beam on the number of focusing periods: (a) $[\overline{(\Delta H/H)^2}]^{1/2}=1.2$ per cent, (b) $[\overline{(\Delta H/H)^2}]^{1/2}=6$ per cent. The dashed curve is the theoretical curve; the solid curve is the simulated curve. The curve with alternating dots and dashes is the theoretical curve given by Refs. 1–3.

is difficult to calculate it analytically; therefore we use the probability simulation. We checked that the probability integrals of θ are exactly determined by two parameters: the value of $\bar{\theta}$ and the number of focusing periods n by which θ is calculated. The probability integral curves are given in Figure 2. The probability that θ will not exceed $\bar{\theta}$ is not greater than 0.6 to 0.7, as is seen from this figure. Let the probability that the confidence estimate θ^* will not be exceeded be equal to 0.98. Then the values of $\theta-1$ lie within the limits (2.75–5.9)$(\bar{\theta}-1)$ for given probability integrals. It is possible, by using the graphs of Figure 2, to determine $\bar{\theta}$ from θ^* for any value of the confidence probability.

It follows that we are able to determine the

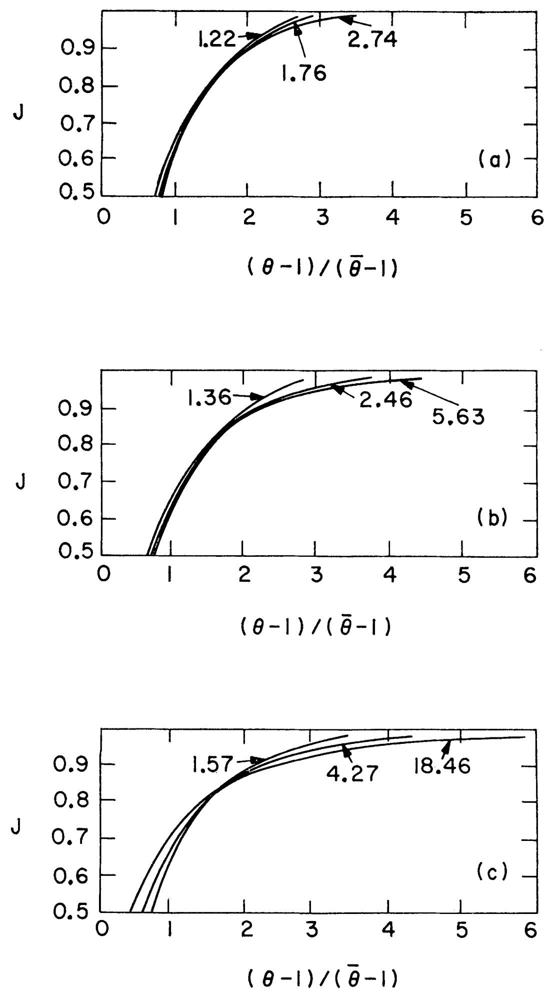


FIGURE 2 Probability integrals J of $(\theta-1)/(\bar{\theta}-1)$ for gradient errors when $n = 50$ (a), $n = 100$ (b), and $n = 200$ (c); $\bar{\theta}$ has different values, which are indicated on the curves.

increase of the beam dimension along one of the coordinate axes. Let us now find a confidence estimate of the increase of the beam radius. In a four-dimensional phase space, the matrix of the ellipsoid at the exit of the channel is related to the matrix of the ellipsoid at the entrance by

$$M_{\mathrm{EK}}^{-1} = \tilde{M}_{\mathrm{K}} M_{\mathrm{E0}}^{-1} \tilde{M}_{\mathrm{K}}{}^{T}.$$

Here $\tilde{M}_{\mathrm{K}}$ is the matrix of the error-containing channel. It is equal to the product of the matrix of the focusing period and the error matrix. Since the matrix M_{E0}^{-1} and the matrices of channel elements have the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where A and B are second-order matrices and 0 is a zero second-order matrix, the matrix M_{EK}^{-1} has the same form. The projection of the ellipsoid on the (x, y) plane has the form

$$a_{33} x^2 + a_{11} y^2 + 2a_{13} xy = a_{11} a_{33} - a_{13}{}^2,$$

where a_{ij} are the coefficients of the matrix M_{EK}^{-1}. Since $a_{13} = 0$, the projection of the ellipsoid on the (x, y) plane is always a canonical ellipse. Therefore we must consider the confidence estimate of the dimension of the beam along that coordinate axis, which is larger than the others, as the confidence estimate of the beam radius.

For symmetrical structures of the focusing periods (FODO and FOD belong to those structures) $\theta_{\mathrm{R}}{}^* = \theta_x{}^* = \theta_y{}^*$.

3. ROTATIONS OF QUADRUPOLE LENSES AROUND THE LONGITUDINAL AXES

The rotations of quadrupole lenses around their longitudinal axes lead us to the coupling between the oscillations of the particles along the x- and y-axes. Therefore we must consider the problem in the four-dimensional system, in which the vectors of all the particles have the form $\mathbf{V}(x, x', y, y')$.

We shall consider focusing periods consisting of two quadrupole lenses and especially, those whose stability diagrams coincide with the coordinate axis. These demands are satisfied for both FODO and FOD structures, which are used most frequently in modern linear accelerators. We shall assume as well that the locations of the crossover points of the beam along both coordinates are approximately the same.

Initially, let us consider an increase in the beam dimension along one coordinate axis, e.g., the x-axis. We shall start the focusing period from the crossover point, which is located within the lens which focuses along the x-axis. It is convenient to consider the problem in the coordinate system, for which the focusing period matrix is a rotation matrix. In order to pass to such a system, it is necessary to carry out the transformation $\mathbf{V}_1 = B\mathbf{V}$, where

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/v\gamma^2 \end{pmatrix}.$$

We keep the previous notations in the new coordinate system. The matrix of an ideal focusing period is described in the new coordinate system by

$$M = \begin{pmatrix} U(\mu) & 0 \\ 0 & U(\mu) \end{pmatrix},$$

where $U(\mu)$ is a two-dimensional rotation matrix which rotates through an angle μ.

We shall consider the ratio x_m/x_{m0} as the coefficient of the increase of the beam dimension along, for example, the x-axis, due to errors in a strong focusing channel section with n focusing periods. Here x_m is the dimension which the beam would reach in an ideal channel located after the error-containing section. x_{m0} is the beam dimension in an ideal channel.

We assume that the initial beam is a beam whose phase volume boundary is a four-dimensional ellipsoid. The parameters of this ellipsoid are chosen so that in an ideal channel, the dimensions of the beam along every coordinate will not exceed x_{m0} and that the ellipsoid has a maximum phase volume.

It can be shown that the ellipsoid

$$M_{\mathrm{E}0} = \frac{1}{x_{m0}^2}\begin{pmatrix} E & 0 \\ 0 & \gamma^2 E \end{pmatrix} \tag{3.1}$$

(E is a second-order unit matrix) satisfies these conditions.

It is possible to relate the increase in the beam dimension along each coordinate with the coefficient of the ellipsoid inverse matrix. We refer to the ellipsoid which limits the phase volume of the beam passing through an error-containing channel. We consider the x-axis. Let the matrix be of the form

$$M_{\mathrm{E}}^{-1} = x_{m0}^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12}^T & E_{22} \end{pmatrix} = x_{m0}^2 N_{\mathrm{E}}^{-1}, \tag{3.2}$$

where N_{E}^{-1} is the inverse matrix of the ellipsoid which represents the beam whose dimension in an ideal channel does not exceed unity (we shall call it the 'normal reciprocal matrix'). The matrices E_{ij} are of second order.

The projection of the ellipsoid, characterized by N_{E}^{-1}, on the (x, x') plane, can be described by the elements of the matrix:

$$a_{22}x^2 + a_{11}x'^2 - 2a_{12}xx' = \mathrm{Det}\, E_{11},$$

where a_{ij} are the coefficients of the matrix N_{E}^{-1}. The projection of the ellipsoid is an ellipse with the following inverse normalized matrix:

$$N_{\mathrm{E},xx'}^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = E_{11}.$$

Therefore the normalized inverse matrix of the ellipse which is a projection of an ellipsoid on the (x, x') plane is the element E_{11} of the ellipsoid normalized inverse matrix.

When a beam passes through an ideal channel, the ellipsoid rotates in the (x, x') and (y, y') planes. The projection on the (x, x') plane rotates as well. Therefore, we can obtain with our method, analogous to the method of the previous section:

$$\theta = \sqrt{\frac{\mathrm{Sp}\, E_{11}}{2} + \sqrt{\left(\frac{\mathrm{Sp}\, E_{11}}{2}\right)^2 - \mathrm{Det}\, E_{11}}}. \tag{3.3}$$

For this case $\mathrm{Det}\, E_{11}$ changes while the beam moves through an error-containing channel.

We shall compute $\bar{\bar{\theta}}$, instead of $\bar{\theta}$, as we did in the case of the gradient errors. $\bar{\bar{\theta}}$ is the value of θ which is computed from the average values of Sp E_{11} and $[(\mathrm{Sp}\, E_{11}/2)^2 - \mathrm{Det}\, E_{11}]^{1/2}$.

Let us first find the average value of the trace. The inverse matrix of the ellipsoid of the beam which has passed n focusing periods, is related to the matrix of the beam ellipsoid, which has passed $n-1$ focusing periods by

$$N_{\mathrm{E},n}^{-1} = \tilde{M}_n N_{\mathrm{E},n-1}^{-1} \tilde{M}_n^{\,T},$$

where $\tilde{M}_n$ is the matrix of the nth error-containing period. If we write the matrix of the period as

$$\tilde{M}_n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$$

then

$$\begin{aligned} E_{11,n} &= A_n E_{11,n-1} A_n^{\,T} + B_n E_{12,n-1} A_n^{\,T} \\ &\quad + A_n E_{12,n-1} B_n^{\,T} + B_n E_{22,n-1} B_n^{\,T} \\ E_{22,n} &= C_n E_{11,n-1} C_n^{\,T} + D_n E_{12,n-1} C_n^{\,T} \\ &\quad + C_n E_{12,n-1} D_n^{\,T} + D_n E_{22,n-1} D_n^{\,T}. \end{aligned}$$

The average values of the traces of these matrices are

$$\begin{aligned} \overline{\mathrm{Sp}\, E_{11,n}} &= \overline{\mathrm{Sp}(A_n E_{11,n-1} A_n^{\,T})} + \overline{\mathrm{Sp}(B_n E_{22,n-1} B_n^{\,T})} \\ \overline{\mathrm{Sp}\, E_{22,n}} &= \overline{\mathrm{Sp}(C_n E_{11,n-1} C_n^{\,T})} + \overline{\mathrm{Sp}(D_n E_{22,n-1} D_n^{\,T})}. \end{aligned} \tag{3.4}$$

The average values of the other traces are zero, since the coefficients of the matrices B_n and C_n contain only the first powers of the primary errors and cancel out when averaged.

Similarly to the previous section, we can consider that approximately,

$$\overline{\tfrac{1}{2}\mathrm{Sp}(A_n E_{11,n-1} A_n^T)} = \overline{\tfrac{1}{2}\mathrm{Sp}\, E_{11,n-1}} \cdot \overline{\tfrac{1}{2}\mathrm{Sp}(A_n A_n^T)}.$$

Then we can write the system of difference equations (3.4) in the form

$$\begin{pmatrix} \overline{\frac{1}{2}\mathrm{Sp}\, E_{11,n}} \\ \overline{\frac{1}{2}\mathrm{Sp}\, E_{22,n}} \end{pmatrix} = \overline{K_n} \begin{pmatrix} \overline{\frac{1}{2}\mathrm{Sp}\, E_{11,n-1}} \\ \overline{\frac{1}{2}\mathrm{Sp}\, E_{22,n-1}} \end{pmatrix}$$

where

$$\overline{K_n} = \begin{pmatrix} \overline{\frac{1}{2}\mathrm{Sp}\, A_n A_n^T} & \overline{\frac{1}{2}\mathrm{Sp}\, B_n B_n^T} \\ \overline{\frac{1}{2}\mathrm{Sp}\, C_n C_n^T} & \overline{\frac{1}{2}\mathrm{Sp}\, D_n D_n^T} \end{pmatrix}.$$

From these relations, it follows that

$$\begin{pmatrix} \overline{\frac{1}{2}\mathrm{Sp}\, E_{11,n}} \\ \overline{\frac{1}{2}\mathrm{Sp}\, E_{22,n}} \end{pmatrix} = \prod_{i=n}^{1} \overline{K_i} \begin{pmatrix} \overline{\frac{1}{2}\mathrm{Sp}\, E_{11,0}} \\ \overline{\frac{1}{2}\mathrm{Sp}\, E_{22,0}} \end{pmatrix}, \tag{3.5}$$

where, according to (3.1)

$$\tfrac{1}{2}\mathrm{Sp}\, E_{11,0} = 1, \quad \tfrac{1}{2}\mathrm{Sp}\, E_{22,0} = 1/\gamma^2.$$

The calculation of $\bar{\bar{\theta}}$ is reduced to computing the product of the matrices $\overline{K_i}$. Each one characterizes one focusing period. We shall omit cumbersome calculations and write for the period which consists of two lenses

$$\overline{\tfrac{1}{2}\mathrm{Sp}\, E_{11,n}} = \left[\overline{\tfrac{1}{2}\mathrm{Sp}(AA^T)} + \frac{1}{2\gamma^2}\overline{\mathrm{Sp}(BB^T)}\right]^n, \tag{3.6}$$

where

$$\overline{\tfrac{1}{2}\mathrm{Sp}(AA^T)} = 1 + 2\overline{\chi}^2(\cos\Lambda\xi \cosh\Lambda\xi - 1)$$

$$\overline{\tfrac{1}{2}\mathrm{Sp}(BB^T)} = \overline{\chi^2}\{(1+\gamma^4)[(\cos\Lambda\xi_1 \cosh\Lambda\xi_1$$
$$-\cos\Lambda\xi_2 \cosh\Lambda\xi_2)^2 + (\sin\Lambda\xi_k \sinh\Lambda\xi_1$$
$$-\sin\Lambda\xi_2 \sinh\Lambda\xi_2)^2]$$
$$+\frac{\nu\gamma^2}{\Lambda^2}(\cos\Lambda\xi_1 \sinh\Lambda\xi_1 - \sin\Lambda\xi_1 \cosh\Lambda\xi_1$$
$$+\cos\Lambda\xi_2 \sinh\Lambda\xi_2 + \sin\Lambda\xi_2 \cosh\Lambda\xi_2)^2$$
$$+\frac{\Lambda^2}{\nu^2}(\cos\Lambda\xi_1 \sinh\Lambda\xi_1 + \sin\Lambda\xi_1 \cosh\Lambda\xi_1$$
$$+\cos\Lambda\xi_2 \sinh\Lambda\xi_2 + \sin\Lambda\xi_2 \cosh\Lambda\xi_2)^2\}.$$

Here χ is a random quantity which is equal to the rotation angle of the median plane of the lens. If the lenses are thin and $\xi_1 = \xi_2$, the formula becomes

$$\overline{\tfrac{1}{2}\mathrm{Sp}\, E_{11,n}} = \left(1 + \frac{4\Lambda^4\xi^2}{\nu^2\gamma^2}\overline{\chi^2}\right)^2. \tag{3.7}$$

The analytical calculation of the average value of $[(\mathrm{Sp}\, E_{11}/2)^2 - \mathrm{Det}\, E_{11}]^{1/2}$ is valid only if $\mathrm{Sp}\, E_{11}$ does not differ much from two. In this case we may approximate the matrix E_{11} by

$$E_{11} = \begin{pmatrix} 1+\Delta_1 & \Delta_2 \\ \Delta_3 & 1+\Delta_4 \end{pmatrix},$$

where the Δ_i's are quantities, mostly less than unity and dependent on primary errors. For this matrix

$$\mathrm{Sp}\, E_{11} = 2 + \Delta_1 + \Delta_4,$$
$$\mathrm{Det}\, E_{11} = 1 + \Delta_1 + \Delta_4 + \Delta_1\Delta_4 - \Delta_2\Delta_3,$$

i.e., we can approximately consider

$$\mathrm{Sp}\, E_{11} = \mathrm{Det}\, E_{11} + 1. \tag{3.8}$$

Then

$$\sqrt{\left(\frac{\mathrm{Sp}\, E_{11}}{2}\right)^2 - \mathrm{Det}\, E_{11}} = \tfrac{1}{2}\mathrm{Sp}\, E_{11} - 1. \tag{3.9}$$

In order to estimate the accuracy of the relation (3.9), we used the random simulation. It showed that for the average value of the left and right sides, the equality holds quite well when the values of the traces are very large.

On the basis of (3.3) and (3.9), we may write:

$$\bar{\bar{\theta}} = \sqrt{\overline{\mathrm{Sp}\, E_{11,n}} - 1}. \tag{3.10}$$

$\bar{\bar{\theta}}$ and $\bar{\theta}$, calculated by the random simulation, are very close to $\bar{\bar{\theta}} = 20$, as can be seen from their comparison. Comparison curves appear in Figure 3. As in the case of the gradient errors, the beam radius increases faster, in general, than do the primary errors.

In order to determine confidence estimates θ^* of the increase of the beam dimension, we have obtained by the simulation method a few curves of the probability integrals of θ for different $\bar{\theta}$ and different numbers of focusing periods (see Figure 4).

Let us now calculate the confidence estimate of

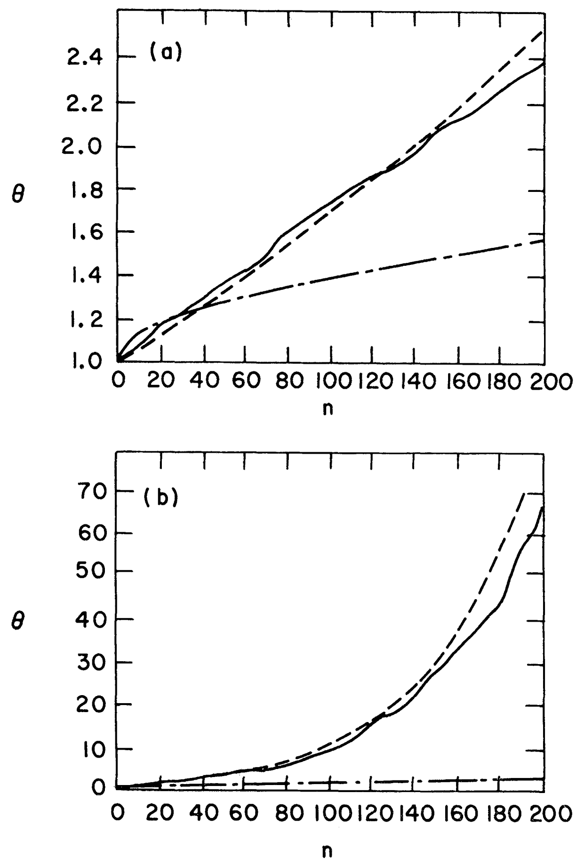


FIGURE 3 Dependence of the increase in the beam dimension on the number of focusing periods: (a) $\overline{(\chi^2)}^{1/2} = 1°$, (b) $\overline{(\chi^2)}^{1/2} = 2.5°$. The dashed curve is the theoretical curve; the solid curve is the simulated curve. The curve with alternating dots and dashes is the theoretical curve given by Refs. 1–3.

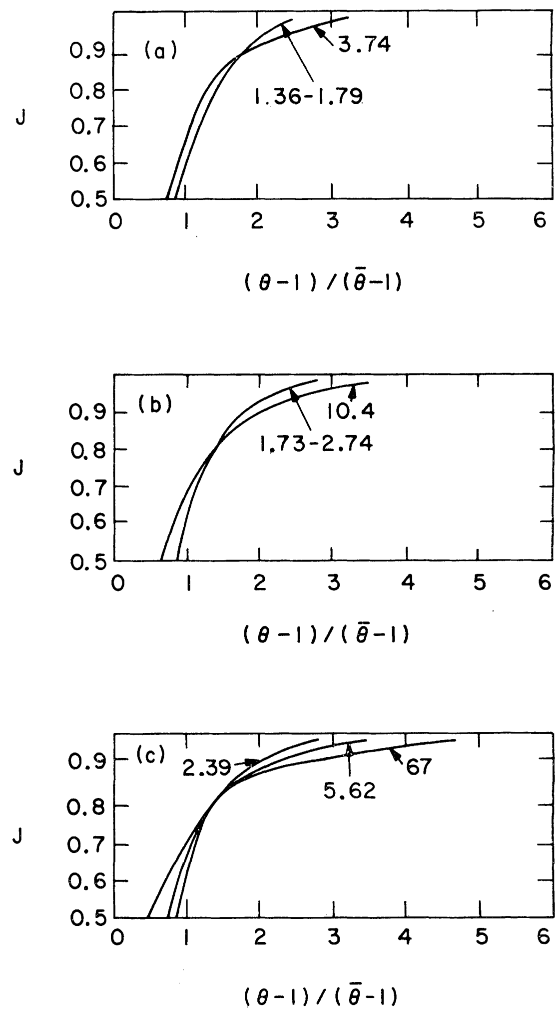


FIGURE 4 Probability integrals J of $(\theta-1)/(\bar{\theta}-1)$ for rotations of lenses around the longitudinal axis, when $n = 50$ (a), $n = 100$ (b), $n = 200$ (c); θ has different values, as indicated on the curves.

the increase of the beam radius. In contrast to the gradient errors case, the projection of the four-dimensional ellipsoid on the (x, y) plane is not a canonical ellipse. We shall consider only symmetrical structures of focusing period $\theta_x{}^* = \theta_y{}^*$. We must then assume as a confidence estimate of the beam radius, the semi-major axis of the ellipse, which is inscribed in the rectangle whose side dimensions equal $R_0\theta_x{}^*$ and $R_0\theta_y{}^*/\gamma$.

In the worst case this value is equal to

$$R^* = R_0\,\theta_x{}^*\sqrt{1+1/\gamma^2},$$

in the best case

$$R^* = R_0\,\theta_x{}^*.$$

If the increase in the beam radius is small, the projection of the ellipsoid on the (x, y) plane is nearly a canonical ellipse, and the second estimate is better.

The problem of the increase in the beam emittance was investigated separately, that is, the area of the projection of the four-dimensional ellipsoid on the (x, x') plane. This quantity can be measured with existing accelerators. The area of the projection of the ellipsoid on the (x, x') plane is the square root of the determinant of the element E_{11} of the matrix of the ellipsoid M_E^{-1}. We investigated the behaviour of this quantity with increasing numbers of focusing

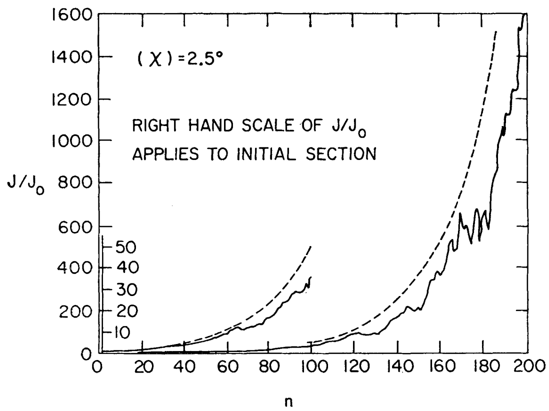


FIGURE 5 The change of the beam emittance along an accelerator when the lens rotates around the longitudinal axis. The dashed curve is the theoretical curve; the solid curve is the simulated curve.

periods, using a computer, and obtained the curve given in Figure 5.

This curve is quite well described by

$$\overline{(\mathrm{Det}\, E_{11,n})^{1/2}} = [\overline{(\mathrm{Det}\, E_{11,1})^{1/2}}]^n \qquad (3.11)$$

Its graph is denoted in this figure by the dotted line. The quantity $\overline{(\mathrm{Det}\, E_{11,1})^{1/2}}$ (it is the average value of the emittance of the beam having passed through one focusing period), can be obtained by

$$\overline{(\mathrm{Det}\, E_{11,1})^{1/2}} = \overline{(\mathrm{Sp}\, E_{11,1} - 1)^{1/2}}. \qquad (3.12)$$

The trace can be obtained by (3.6) or (3.7).

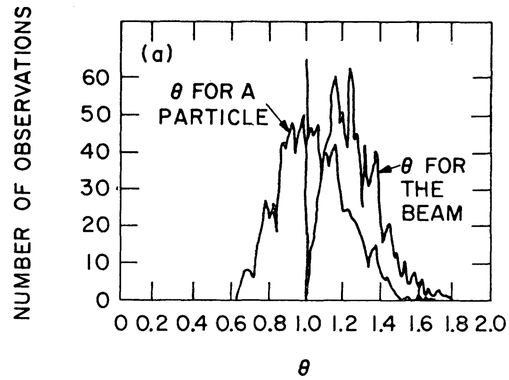
We see from the graph, that when the errors and the number of focusing periods is large, the beam emittance can increase manyfold.

4. DISCUSSION OF RESULTS

Let us now compare our results to those of Refs. 1–3. We note that the beam radius is estimated using different average parameters here and in Refs. 1–3. Our formulae enable us to calculate the average value of the beam radius, that is, the radius of the envelope of the trajectories of the particles, which are located on the surface of the phase volume of the beam, consistent with the channel.

The results of Refs. 1–3 concern the mean-square value of the oscillation amplitudes of the various particles. A comparison of the radius confidence estimates would be more graphic. However, in Refs. 1–3, confidence estimates were not computed. Therefore we can only compare the results concerning the average characteristics.

In order to clarify the relationship between the average value of the beam radius and the mean-square value of the oscillation amplitude of an isolated particle, we used random simulation. We obtained, also, distribution histograms of both the beam radius and the oscillation amplitude. These histograms for the errors of the magnetic field gradients appear in Figure 6. It follows from the



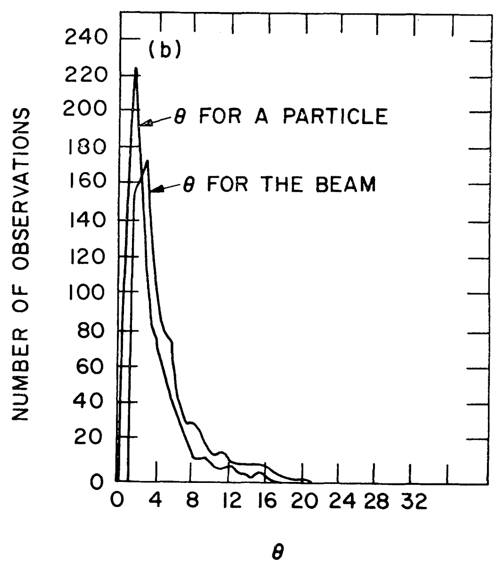


FIGURE 6 Histograms of the distribution of the various oscillation amplitudes of the particle and the beam dimensions when the beam perturbations are (a) small, and (b) large.

figure that the assumption that the average value of the change of the oscillation amplitude of each particle is zero (the same assumption is made in Refs. 1–3), is correct as long as the increase in the oscillation amplitude is small compared to the initial amplitude. The probability that the beam radius exceeds a certain value is larger than the probability that the oscillation amplitude exceeds this value.

From the above, it follows that if confidence estimates of the beam radius were deduced by investigating the behaviour of isolated particles, the results would be underestimated.

It is seen from Figures 1 and 3 that if the increase in the beam radius is large, the formulae, which were obtained previously, yield mainly underestimated results and therefore cannot be used in choosing tolerances.

The increase of the beam radius and the emittance renders difficult the construction of proton linear accelerators of energies of a few hundred MeV and higher. In such cases, the number of periods is large, and these periods are independent sources of random errors. Moreover, the influence of random errors becomes stronger in these accelerators, since for energies of 100 MeV or so, strong-focusing channels are used. In such channels the part of the focusing period, which is occupied by the quadrupole lens, is small.

The old theory predicted that with the tolerances commonly accepted, high energy accelerators needed a large aperture. The results presented in this article show that an increase of the aperture up to the commonly-used dimensions is not enough to prevent beam losses.

To prevent an increase of the effective beam dimensions, one can decrease the tolerances on the magnetic lens location, the precision of the magnetic field gradients, and the stability of the lens power supplies. As follows from this article, a small decrease of the tolerance decreases the beam radius significantly. Thus, for example, a two-fold increase of the precision of the magnetic field parameters decreases θ as $\theta^{1/4}$ when the beam perturbations are large. But it is technically difficult to make closer tolerances, for example the tolerance of setting of the median planes of the lenses. Moreover, it is possible in principle to reduce periodically to zero the gradual increase of the transverse oscillations, under the action of random errors, using correcting equipment located along the accelerator. In the last section of this article, some correction problems are discussed.

5. CORRECTION

It is possible to use correcting equipment to prevent an increase of the effective radius of the beam. The correcting equipment need not be automatic nor fast-acting. Of all the errors, only the instabilities of the lens power supplies require fast correction. They can, however, be reduced to negligible magnitudes that cease to influence the particle motion. The other errors are either constant or vary very slowly. Therefore, the correction also may be slow.

There are two types of correcting equipment. The first regulates the position of the centre of gravity of the beam, that is, it eliminates coherent transverse oscillations. Equipment of the second type decreases the beam radius and the emittance, i.e., the areas of the projections of a four-dimensional ellipsoid on the phase planes (x, x') and (y, y') and thus transforms its phase volume.

It is convenient to control the beam centre position with the help of magnetic dipole. Usually these magnets' lengths are small enough and they change the beam trajectory inclination without changing the beam position. The analysis performed in Ref. 4 shows that if the magnets are placed along the channel at distances corresponding to an odd number of quarter wavelength of transverse oscillations then the change of the beam centre trajectory inclination k_i by magnet which is placed before the ith section can be given as

$$k_i = -\frac{1}{a_{12i-1}} d_{i-1} - \frac{1}{a_{12\,i}} d_{i+1}$$

where a_{12i-1} and a_{12i} are the elements of $(i-1)$th and ith section matrices and d_{i-1} and d_{i+1} are readings of the probes which are placed before $(i-1)$th and after ith sections correspondingly. As can be seen from this equation the correction action of each magnet is independent on the correction action of others and full correction can be performed between two beam pulses due to information obtained during one pulse.

It is possible, in principle, to reduce the beam radius in an error-containing channel to the value in an errorless channel by transforming the four-dimensional beam phase volume at a certain point in an accelerator. Therefore, it is necessary to possess in the correcting section the elements which change the form of the projection of the phase volume on the (x, x') and (y, y') planes and also the elements which correct the oscillations in the two transverse planes and thus change the areas of the projections. We can use for the first of these existing channel quadrupole lenses. These lenses must be supplied with an independent control of the magnetic field gradients. For the second type of elements it is possible to use quadrupole lenses whose median planes are rotated by forty-five degrees with respect to the regular channel lens. The total number of the variable parameters of the correcting section cannot be less than nine, according to the number of coefficients of the four-dimensional central ellipsoid equation. But the complete system of parameters which provides the necessary transformation of the beam phase volume is not yet known.

Let us keep the areas of the projections of the phase volume on the (x, x') and (y, y') planes constant, and limit ourselves only to reducing their form to the forms of projections of a consistent ellipsoid. Then the correcting section can consist of the four regular lenses of a focusing channel. Let us estimate what is to be obtained from this setup. The errors in setting of the median planes lead to an increase in the beam radius which is described by formula (3.10) and an increase in the beam emittance which is described by formulae (3.11) and (3.12). If the form of the beam emittance coincides with the form of the consistent ellipsoid emittance, the value of the beam increase becomes

$$\bar{\theta}_{\mathrm{E}} = \overline{(\mathrm{Sp}\, E_{11,1}} - 1)^{n/4}. \tag{5.1}$$

Comparison of formulae (3.10) and (5.1) shows that up to magnitudes of order two, $\bar{\theta}_{\mathrm{E}} - 1$ is approximately half of $\bar{\theta} - 1$. Let us assume that the errors in the gradients contribute the same increase in the beam radius as do the errors in the median planes. Then the equipment under consideration eliminates 75 per cent of the whole increase in the beam dimension and is thus of sufficient efficiency. We can add that owing to nonlinear increase of θ, even a small decrease in the dimension of a beam of intermediate energy effectively reduces the dimension of the exciting beam.

Magnetic field gradients of correcting lenses can be obtained using an iterative process. A result of this procedure is that the maximum beam radius $R_{\max}$ is minimized by signals from a few beam probes which are located behind the correcting lenses over a length equal to at least half a wavelength of a transverse oscillation.

The iterative process can be described as follows. By changing field gradients G_i in each of four quadrupole lenses chosen for correction and measuring the corresponding beam maximum radius change we obtain partial derivatives $\partial R_{\max}/\partial G_i$ of the function $R_{\max}(G_1, G_2, G_3, G_4)$. The function gradient direction in the initial point can be obtained with the help of these derivatives. After that the gradients G_i are being changed so that the movement from the initial to the final point should go along the line coinciding with the direction of the function gradient. To achieve such movement each lens gradient should be changed by

$$\Delta G_i = -\frac{\partial R_{\max}/\partial G_i}{\sum_{i=1}^{4} (\partial R_{\max}/\partial G_i)^2}(R_{\max} - R_0)$$

where R_0 is the minimum value of function $R_{\max}$. The beam radius in an ideal channel can be assumed for this value. After changing G_i a new value of $R_{\max}$ is calculated and then all operations are repeated.

The iterated process mentioned above has been simulated by means of a computer. Several channel alignments have been tested and in all cases there was rather rapid convergence. In order to achieve a satisfactory result from 5 to 8 iterations were needed.

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