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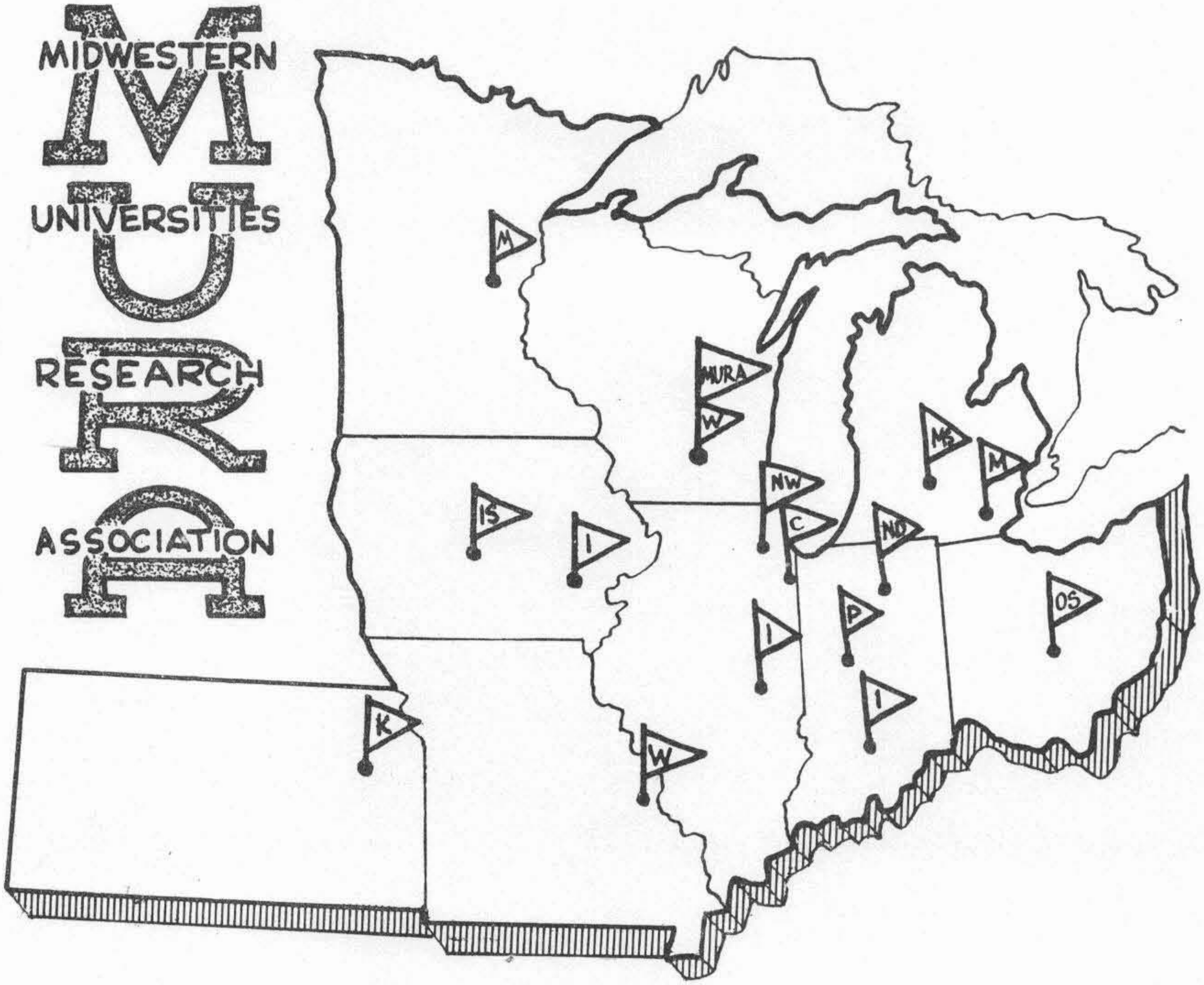

POTENTIALS AND FIELDS IN A SCALING ACCELERATOR

A. M. Sessler

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I. Introduction

Kerst has pointed out that in a scaling machine, knowledge of the magnetic scalar potential (or field components) on the plane $\theta = 0$ is sufficient to determine the fields every place in the accelerator. Thus the equation of motion may be integrated if the digital computer stores only a two dimensional grid of potential values. This method of calculation enjoys great flexibility for a completely different accelerator may be studied by merely changing the grid of potential values. The method is only limited by the requirement that the machine scale. Clearly this requirement can be relaxed, to only requiring that the deviation from scaling be small over a distance of the order of the amplitude of a stable betatron oscillation.

In using this method one is faced with the problem of determining the magnetic scalar potential on the plane $\theta = 0$, given the pole faces (i.e. equipotentials at some distance from the median plane). Such a problem is easy to solve if one ignores the $\vec{\theta}$ direction, and concerns himself with only a two dimensional Laplace Equation. Because the accelerator scales, it is, possible to find a different two dimensional equation, whose solution satisfies Laplace's Equation in three dimensions. These notes are devoted to finding this equation, as well as proving and reformulating Kerst's observation.

II. Equation for the Scalar Potential

If we designate the scalar potential by $\psi(x, y, \theta)$, then Laslett has formulated Kerst's observation as:

$$\psi(x, y, \theta) = \left(\frac{1 + x}{1 + x_o}\right)^{k + 1} \psi(x_o, y_o, o) \tag{1}$$

where:

$$\phi(x_o) = \phi(x) - N\theta$$

$$\frac{y_o}{1+x_o} = \frac{y}{1+x} \tag{2}$$

or more explicitly:

$$K \log(1+x_o) = K \log(1+x) - N\theta \tag{3}$$

clearly this is equivalent to:

$$\psi(r, y, \theta) = e^{\frac{(k+1)N\theta}{K}} \psi(u, v, o) \tag{4}$$

where:

$$u = r\, e^{-\frac{N\theta}{K}}$$

$$v = y\, e^{-\frac{N\theta}{K}} \tag{5}$$

We shall now show that this is correct, by applying the Laplacian Operator to $\psi(r, y, \theta)$ and seeing that the resulting equation is independent of θ. Incidently, this will produce a second order two dimensional differential equation for $\psi(u, v, o)$, hereafter called $\psi(u, v)$.

$$\nabla^2 \psi(r,y,\theta) = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial y^2}\right]\psi(r,y,\theta) \quad (6)$$

$$\frac{\partial^2}{\partial r^2}\psi(r,y,\theta) = e^{\frac{(k+1)N\theta}{K}} e^{-\frac{2N\theta}{K}} \frac{\partial^2 \psi}{\partial u^2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\psi(r,y,\theta) = e^{\frac{(k+1)N\theta}{K}} e^{-\frac{2N\theta}{K}} \frac{1}{u}\frac{\partial \psi}{\partial u}$$

$$\frac{\partial^2}{\partial y^2}\psi(r,y,\theta) = e^{\frac{(k+1)N\theta}{K}} e^{-\frac{2N\theta}{K}} \frac{\partial^2 \psi}{\partial v^2}$$

$$\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\psi(r,y,\theta) = e^{\frac{(k+1)N\theta}{K}} e^{-\frac{2N\theta}{K}} \left\{ \quad \right\}$$

where:

$$\left\{\frac{(k+1)^2 N^2}{K^2}\frac{\psi}{u^2} - \frac{2(k+1)N^2}{K^2}\left[\frac{1}{u}\frac{\partial \psi}{\partial u} + \frac{v}{u^2}\frac{\partial \psi}{\partial v}\right] + \right.$$
$$\left. + \frac{N^2}{K^2}\left[\frac{1}{u}\frac{\partial \psi}{\partial u} + \frac{v}{u}\frac{\partial \psi}{\partial v} + \frac{\partial^2 \psi}{\partial u^2} + \frac{v^2}{u^2}\frac{\partial^2 \psi}{\partial v^2} + \frac{2v}{u}\frac{\partial^2 \psi}{\partial u \partial v}\right]\right\} \quad (7)$$

Combining Eq. 6 and Eq. 7, one obtains:

$$0 = \psi_{uu} + \frac{1}{u}\psi_u + \psi_{vv} + \frac{(k+1)^2 N^2}{K^2}\frac{1}{u^2}\psi +$$
$$- \frac{2(k+1)N^2}{K^2}\left[\frac{1}{u}\psi_u + \frac{v}{u}\psi_v\right] +$$
$$+ \frac{N^2}{K^2}\left[\psi_{uu} + \frac{1}{u}\psi_u + \frac{2v}{u}\psi_{uv} + \frac{v}{u^2}\psi_v + \frac{v^2}{u^2}\psi_{vv}\right] \quad (8)$$

III Formulas for the fields

Using;

$$H_r = \frac{\partial \psi(r, y, \theta)}{\partial r} \qquad H_y = \frac{\partial \psi(r, y, \theta)}{\partial y}$$

$$H_\theta = \frac{1}{r} \frac{\partial \psi(r, y, \theta)}{\partial \theta} \tag{9}$$

one obtains the following formulas for the field components:

$$H_y = e^{\frac{k\theta}{K}} \frac{\partial \psi}{\partial v}$$

$$H_r = e^{\frac{k\theta}{K}} \frac{\partial \psi}{\partial u}$$

$$H_\theta = e^{\frac{k\theta}{K}} \frac{1}{u} \left\{ \frac{(k+1)N}{K} \psi - \left(\frac{N}{K}\right) \left[u \frac{\partial \psi}{\partial u} + v \frac{\partial \psi}{\partial v} \right] \right\}$$

For any particular pole shape, one should solve Eq. 8 for ψ. Then one has a choice of storing ψ; or ψ and ψ_u and ψ_v; or H_y and H_r and H_θ. This choice depends on the properties of the digital computer.