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STUDY OF THE EFFECTS OF DISPLACED SECTORS WITH APPROXIMATE PHASE PLANE TRANSFORMATIONS

Andrew M. Sessler

Ohio State University and
Midwestern Universities Research Association*
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I. Introduction

The construction of approximate phase plane transformations which are exactly area preserving, directly from the differential equation of motion has been described in two previous reports (MURA-A.M.S. 2 and 3, hereafter called I and II). These transformations are defined in terms of new variables (Birkhoff Variables) in which the betatron oscillations are described in the phase plane by motion on circles. It is the purpose of this investigation to study the effect of displaced sectors, in the new variables, by means of transformations. This was undertaken in a detailed manner simply as an exploratory calculation prior to the coding of the Illiac for similar problems. For this reason we did not study a transformation obtained via perturbation theory, but rather took Powell's approximate transformation (MURA-R.W./JLP-5) and worked with it in Birkhoff Variables.

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This allows us to compare results with those obtained on the Illiac by Powell, while the method and formulas are exactly the same as for a transformation obtained by perturbation theory.

We shall describe the method by treating one particular problem in detail. Namely we consider the one dimensional, non-linear C.L.S. machine for a working point specified by $n_1 = -n_2 = 16$, $e_1 = - e_2 = 1000$. In a 50 sector machine, two sectors will be displaced by amounts $\Delta x = .001$, and following Powell the displacement is from the center of a positive sector to the center of a positive sector.

II. The Transformation to Birkhoff Variables

We start with the Powell Transformation:

$$
\begin{aligned}
x &= \bar{a}\, x_0 + \bar{b}\, y_0 + \alpha \left\{ A\, x_0 + B\, y_0 \right\}^3 \\
y &= \bar{c}\, x_0 + \bar{d}\, y_0 + \beta \left\{ A\, x_0 + B\, y_0 \right\}^3
\end{aligned}
\qquad (1)
$$

where:

$\bar{a} = \bar{d} = .83373$

$\bar{b} = .61841$

$\bar{c} = - .4930294$

$\alpha = - .3709657$

$\beta = - 1.100000$

$A = 2.965234$

$B = 1.00000$

This may be written in the form:

$$x = \cos\sigma_0\, x_0 + \beta_0 \sin\sigma_0\, y_0 + a\, x_0^3 + b y_0^3 + c x_0^2 y_0 + d\, x_0 y_0^2$$
$$\hfill (2)$$
$$y = -\frac{1}{\beta_0} \sin\sigma_0 x_0 + \cos\sigma_0\, y_0 + a' x_0^3 + b' y_0^3 + c' x_0^2\, y_0 + d' x_0\, y_0^2$$

where:

$\sigma_0 = .5847$

$\beta_0 = 1.11996$

$a = -9.67187$	$a' = -28.67937$
$b = -.370966$	$b' = -1.10000$
$c = -9.78527$	$c' = -29.015622$
$d = -3.30000$	$d' = -9.785272$

We now follow the proceedure outlined in I, and transform to variables Q and P defined by :

$$Q = \frac{x}{2} - \frac{\beta_0}{2}\, i\, y \qquad x = Q + P$$
$$\hfill (3)$$
$$P = \frac{x}{2} + \frac{\beta_0}{2}\, i\, y \qquad y = \frac{i}{\beta_0}\left[Q - P\right]$$

One obtains from (2):

$$Q = e^{i\sigma_0}\, Q_0 + A Q_0^3 + B P_0^3 + C Q_0^2\, P_0 + D Q_0\, P_0^2$$
$$\hfill (4)$$
$$P = e^{-i\sigma_0}\, P_0 + A' Q_0^3 + B' P_0^3 + C' Q_0^2\, P_0 + D' Q_0\, P_0^2$$

where:

$$A = \left[1/2\, a - \frac{b'}{2\beta_0^2} + \frac{c'}{2} - \frac{d}{2\beta_0^2}\right] + i\left[-\frac{\beta_0}{2}\, a' - \frac{b}{2\beta_0^3} + \frac{c}{2\beta_0} + \frac{d'}{2\beta_0}\right] \tag{5}$$

$$B = \left[a/2 + \frac{b'}{2\beta_0^2} - \frac{c'}{2} - \frac{d}{2\beta_0^2}\right] + i\left[-\frac{\beta_0}{2}\, a' + \frac{b}{2\beta_0^3} - \frac{c}{2\beta_0} + \frac{d'}{2\beta_0}\right]$$

$$C = \left[3/2\, a + \frac{3b'}{2\beta_0^2} + \frac{c'}{2} + \frac{d}{2\beta_0^2}\right] + i\left[-\frac{3\beta_0}{2}\, a' + \frac{3b}{2\beta_0^3} + \frac{c}{2\beta_0} - \frac{d'}{2\beta_0}\right]$$

$$D = \left[3/2\, a - \frac{3b'}{2\beta_0^2} - \frac{c'}{2} + \frac{d}{2\beta_0^2}\right] + i\left[-\frac{3\beta_0}{2}\, a' - \frac{3b}{2\beta_0^3} - \frac{c}{2\beta_0} - \frac{d'}{2\beta_0}\right]$$

$A' = B^*$ $\qquad B' = A^*$ $\qquad C' = D^*$ $\qquad D' = C^*$

Numerical evaluation yields:

$$A = -17.590 + 7.4547\,i$$

$$B = 10.549 + 15.928\,i$$

$$C = -31.647 + 47.783\,i$$

$$D = 0 + 57.313\,i$$

One can check his calculations at this point by requiring that the Jacobian be unity through quadratic terms, or that:

$$\begin{aligned} B' &= -1/3\, e^{-i2\sigma_0} D \\ C' &= -3e^{-i2\sigma_0} A \\ D' &= -e^{-i2\sigma_0} C \end{aligned} \qquad (6)$$

A further change of variables is now made to coordinates $\bar{Q}$ and $\bar{P}$ defined by:

$$\begin{aligned} \bar{Q} &= Q + \bar{\alpha} Q^3 + \bar{\beta} P^3 + \bar{\gamma} Q^2P + \bar{\delta} QP^2 \\ \bar{P} &= P + \bar{\alpha}' Q^3 + \bar{\beta}' P^3 + \bar{\gamma}' Q^2P + \bar{\delta}' QP^2 \end{aligned} \qquad (7)$$

where the coefficients are defined in I and II, and are given numerically by:

$\bar{\alpha} = -17.304$	$\bar{\alpha}' = (\bar{\beta})^*$
$\bar{\beta} = 10.377$	$\bar{\beta}' = (\bar{\alpha})^*$
$\bar{\gamma} = 0$	$\bar{\gamma}' = (\bar{\delta})^*$
$\bar{\delta} = 51.940$	$\bar{\delta}' = 0$

Now define:
$$u = \overline{Q} + \overline{P}$$
$$v = i\left[\overline{Q} - \overline{P}\right] \tag{8}$$

which yields, for the transformation (1), correct to lowest non-trivial order:

$$u = \cos\sigma\, u_0 + \sin\sigma\, v_0$$
$$v = -\sin\sigma\, u_0 + \cos\sigma\, v_0 \tag{9}$$

where $\cos\sigma = \cos\sigma_0 + \frac{e}{2}\left[1 - e^{-i2\sigma_0}\right]\overline{P}_0\,\overline{Q}_0$ (10)

and $\overline{P}_0\,\overline{Q}_0 = u^2 + v^2$, so that:

$$\sigma = .5851 + 14.329\left[u^2 + v^2\right] \tag{11}$$

Finally we obtain the transformation relating x,y with u,v:

$$u = x + p_1x^3 + q_1y^3 + r_1x^2y + s_1xy^2$$
$$v = \beta_0 y + p_2x^3 + q_2y^3 + r_2x^2y + s_2xy^2 \tag{12}$$

where:

$$p_1 = 1/8\left\{(\bar{\alpha} + \bar{\alpha}') + (\bar{\beta} + \bar{\beta}') + \bar{\gamma}' + \bar{\delta}\right\} \tag{13}$$
$$q_1 = 1/8\left\{(\bar{\alpha} + \bar{\alpha}') - (\bar{\beta} + \bar{\beta}') - \bar{\gamma}' + \bar{\delta}\right\}\beta_0^3\, i$$
$$r_1 = 1/8\left\{-3(\bar{\alpha} + \bar{\alpha}') + 3(\bar{\beta} + \bar{\beta}') - \bar{\gamma}' + \bar{\delta}\right\}\beta_0\, i$$
$$s_1 = 1/8\left\{-3(\bar{\alpha} + \bar{\alpha}') - 3(\bar{\beta} + \bar{\beta}') + \bar{\gamma}' + \bar{\delta}\right\}\beta_0^2$$
$$p_2 = i/8\left\{(\bar{\alpha} - \bar{\alpha}') + (\bar{\beta} - \bar{\beta}') - \bar{\gamma}' + \bar{\delta}\right\}$$
$$q_2 = 1/8\left\{-(\bar{\alpha} - \bar{\alpha}') + (\bar{\beta} - \bar{\beta}') - \bar{\gamma}' - \bar{\delta}\right\}\beta_0^3$$

$$r_2 = 1/8\left\{3(\bar{\alpha}-\bar{\alpha}') - 3(\bar{\beta} - \bar{\beta}') - \bar{\gamma}' - \bar{\delta}\right\}\beta_0$$

$$s_2 = i/8\left\{-3(\bar{\alpha} - \bar{\alpha}') - 3(\bar{\beta} - \bar{\beta}') - \bar{\gamma}' + \bar{\delta}\right\}\beta_0^2$$

Numerically these are:

$p_1 = 11.253$ $\qquad$ $p_2 = 0$

$q_1 = 0$ $\qquad$ $q_2 = -8.5193$

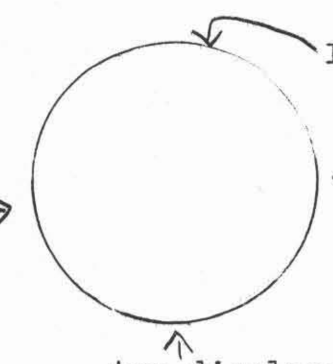
$r_1 = 0$ $\qquad$ $r_2 = -37.794$

$s_1 = 18.180$ $\qquad$ $s_2 = 0$

It should be noted that in either x,y or u,v variables all quantities are real. Also, the relation (12) between these variables involves only real numbers. The introduction of complex numbers is only to facilitate calculations.

III. Transformations With Displaced Sectors

We want to calculate the effect of two displaced (center to center) sectors in a fifty sector machine, which is arranged so:



Within the acurracy of this calculation we may replace the transformations through the displaced sectors by appropriate linear transformations. This simplification, (which is not required by the method) yields for a displaced sector the transformation:

$$
\begin{aligned}
x &= \bar{a}\,(x_0 - \Delta x) + \bar{b}\, y_0 \\
y &= \bar{c}\,(x_0 - \Delta x) + \bar{a}\, y_0
\end{aligned}
\qquad (14)
$$

where Δx is the magnitude of the displacement.

This may be transformed to u - v variables where it becomes:

$$
\begin{aligned}
u &= \cos\sigma_0\, u_0 + \sin\sigma_0\, v_0 - \bar{a}(\Delta x) \\
v &= -\sin\sigma_0\, u_0 + \cos\sigma_0\, v_0 - \beta_0\, \bar{c}(\Delta x)
\end{aligned}
\qquad (15)
$$

We now fold together two transformations of the form of Eq. 15, obtaining as the transformation through the displaced sectors:

$$
\begin{aligned}
u &= \cos 2\sigma_0\, u_0 + \sin 2\sigma_0\, v_0 - \left[\cos\sigma_0 + \cos 2\sigma_0\right]\Delta x \\
v &= -\sin 2\sigma_0\, u_0 + \cos 2\sigma_0\, v_0 + \left[\sin\sigma_0 + \sin 2\sigma_0\right]\Delta x
\end{aligned}
\qquad (16)
$$

Ignoring terms of third order, in the displaced sectors transformation, this becomes equivalent to using, the following three transformations to go once around the machine:

$$
\mathrm{I} \quad \begin{cases}
u = (\cos 25\sigma)\, u_0 + (\sin 25\sigma)\, v_0 \\
v = (-\sin 25\sigma)\, u_0 + (\cos 25\sigma)\, v_0
\end{cases}
$$

$$\text{II.} \quad \begin{cases} u = u_0 - \left[\cos \sigma_0 + \cos 2\sigma_0\right] \Delta x \\ v = v_0 + \left[\sin \sigma_0 + \sin 2\sigma_0\right] \Delta x \end{cases} \qquad (17)$$

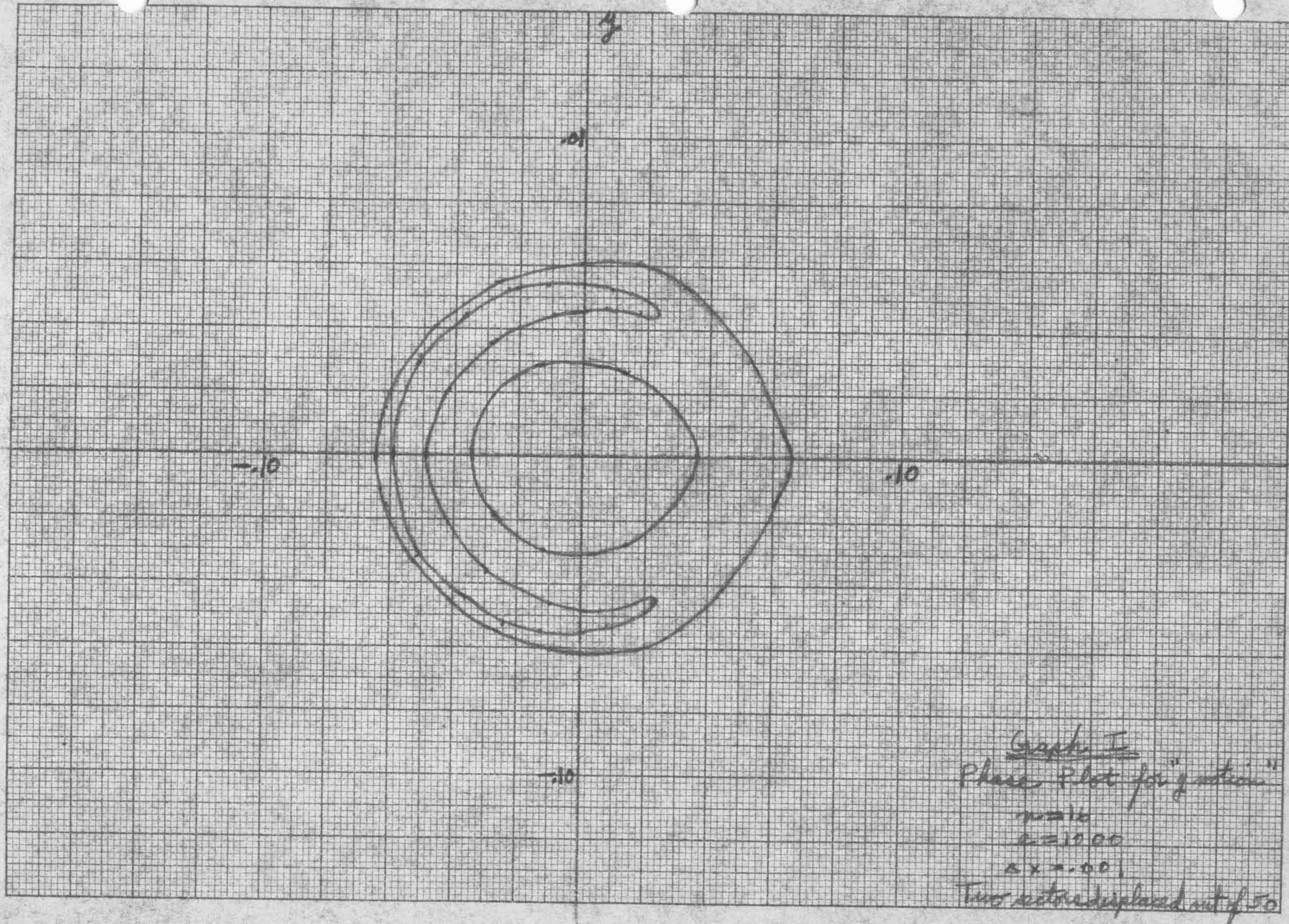
$$\text{III.} \quad \begin{cases} u = (\cos 25\sigma)\, u_0 + (\sin 25\sigma)\, v_0 \\ v = (-\sin 25\sigma)\, u_0 + (\cos 25\sigma)\, v_0 \end{cases}$$

where σ is given by Eq. 11.

Thus we have two quasi-linear transformations, plus transformation II which inserts the effect of the displacement.

These transformations are very amenable to rapid calculation, and in particular it is even feasible to perform desk calculations. On graph I we have indicated the phase plots for a few sets of initial values. In this work $\Delta x = -.001$. The results may be seen to agree with the work of Powell (MURA-R.W./J.L.P. -5 -Fig. 3) within a few percent. The difference is presumably due to the approximation made in Eq. 14, as well as round off, and truncation errors in the numerical work.

The calculations were performed by Mr. K. Fowler and Mr. G. Mohan.



Graph I

Phase Plot for "y motion"

$n = 16$

$\epsilon = 1000$

$\Delta x = .001$

Two sectors displaced out of 50