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THE CONSTRUCTION OF APPROXIMATE PHASE PLANE TRANSFORMATIONS FOR A.G. SYNCHROTRONS, II

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I. INTRODUCTION

The study of stability of particle trajectories in A.G. Synchrotrons may be very much simplified if one has phase plane transformations available. These transformations are in general approximate solutions to the betatron oscillation equations, relating the final displacements and velocities to their values at a similar point in a previous sector. The transformations are used on high speed computers and iterated, typically, many hundreds of times. This places certain restrictions on the approximate nature of the transformations. In particular, if one desires the phase plane plots to remain stable then the transformations must be <u>exactly</u> area preserving. Also, to have the transformations prove more convenient than the integration of the differential equation, they must be either of algebraic, or of simple functional form.

The requirement of exact area preservation precludes the possibility of using most simple perturbation theories to

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generate transformations. Powell and Wright (MURA-R.W./J.L.P. -5) investigated the most general transformation of simple algebraic form, which was exactly area preserving. The arbitrary constants in this transformation were obtained by comparison with the numerical solution to the differential equation. This only proved feasible for simple equations, and their work was restricted to a few cases only. As pointed out in (MURA -A.M.S. -2, called I hereafter) the work of Birkhoff may be used to somewhat simplify this procedure . However, it still remained true that the arbitrary constants must be determined by comparison with solutions to the differential equation.

In this report a method is described which allows one to obtain approximate transformations, in the neighborhood of an equilibrium orbit, directly from the differential equation. These transformations are exactly area preserving, and may be obtained analytically from the differential equation. The method is based on the work of Birkhoff (Vol. II of Collected Mathematical Papers, Am. Math. Soc., 1950), as reformulated by Jauch (MURA - J.M.J. -1) , and outlined in detail in I.

The method proceeds by first obtaining an approximate solution to the differential equation by means of perturbation theory. This solution is an expansion of the exact solution in powers of the amplitude of the oscillation. It is terminated after a few terms, and consequently describes the solution arbitrarily accurately for small enough amplitude.

It is not, however, exactly area preserving. It is then shown that such a transformation may be approximately transformed into Birkhoff's normal form, the procedure being such that one can continue this process thus obtaining a transformation which formally approximates the exact transformation to arbitrarily high accuracy, for sufficiently small amplitude of oscillation. As described in I a transformation in Birkhoff variables is exactly area preserving, if terminated after any degree of approximation. The resulting transformation is simple in form, and convenient for computer calculation.

In Section II we describe the method by applying it to the case of the one dimensional non-linear C.L.S. machine. We include formulas which make it relatively easy to construct transformations for any values of the field gradients, and non-linearities.

In Section III these formulas are evaluated numerically for a working point which has been studied by numerical integration, as well as the Powell Transformation.

In Section IV the theory is extended to the case of non-linear two dimensional motion, which is uncoupled in linear approximation.

These methods may readily be used on complicated equations, and in particular will be first used in connection with the Mark V. It is hoped to describe this in a future report.

II. One Dimensional Non-Linear C. L. S. Equation

We shall describe the method by studying the equation with coefficients of period 2π, defined by:

$$\ddot{x} + w^2 x + e/3\, x^3 = 0$$

$$\text{where:} \quad w^2 = \begin{cases} w_+^2 & 0 \le t \le \pi \\ -w_-^2 & \pi < t < 2\pi \end{cases} \qquad e = \begin{cases} e & 0 \le t \le \pi \\ -e & \pi < t < 2\pi \end{cases} \tag{1}$$

We first obtain the solution to the equation in positive and negative sectors, valid to terms linear in e. This is obtained by letting

$$x = x_0 + e x_1 \tag{2}$$

where:

$$\begin{aligned} \ddot{x}_0 + w^2 x_0 &= 0 \\ \ddot{x}_1 + w^2 x_1 &= -e/3\, x_0^3 \end{aligned} \tag{3}$$

One obtains the solutions:

$$x = A \sin w_+ t + B \cos w_+ t + e \Big\{ -\frac{t}{8w_+} \left[B^3 + A^2 B\right] \sin w_+ t + \frac{t}{8w_+} \left[A^3 + AB^2\right] \cos w_+ t + \frac{1}{96 w_+^2} \left[B^3 - 3A^2 B\right] \cos 3w_+ t + \frac{1}{96 w_+^2} \left[-A^3 + 3AB^2\right] \sin 3w_+ t \Big\}$$

$$x = C \sinh w_- t + D \cosh w_- t + e\left\{\frac{t}{8w_-}\left[D^3 - C^2 D\right] \sinh w_- t + \right.$$
$$+ \frac{t}{8w_-}\left[- C^3 + C D^2\right] \cosh w_- t + \frac{1}{96w_-^2}\left[C^3 + 3CD^2\right] \sinh 3w_- t$$
$$\left. + \frac{1}{96w_-^2}\left[D^3 + 3C^2 D\right] \cosh 3w_- t\right\} \tag{4}$$

where A,B,C,D, are arbitrary constants.

If we express each of these solutions in terms of the values of $x(t)$ and $\dot{x}(t)$ for $t = 0$, we obtain for positive sectors:

$$x(t) = \frac{\dot{x}(0)}{w_+} \sin w_+ t + x(0) \cos w_+ t + e\left\{a_1 x^3(0)\right.$$
$$\left. + a_2 \dot{x}^3(0) + a_3 x^2(0) \dot{x}(0) + a_4 x(0) \dot{x}^2(0)\right\} \tag{5}$$

$$\dot{x}(t) = \dot{x}(0) \cos w_+ t - x(0) w_+ \sin w_+ t + e\left\{b_1 x^3(0) + b_2 \dot{x}^3(0)\right.$$
$$\left. + b_3 x^2(0) \dot{x}(0) + b_4 x(0) \dot{x}^2(0)\right\}$$

where: $a_1 = -\frac{1}{96w_+^2} \cos w_+ t - \frac{1}{8w_+} t \sin w_+ t + \frac{1}{96w_+^2} \cos 3w_+ t$

$$a_2 = -\frac{3}{32w_+^5} \sin w_+ t + \frac{t}{8w_+^4} \cos w_+ t - \frac{1}{96w_+^5} \sin 3w_+ t$$

$$a_3 = -\frac{7}{32w_+^3} \sin w_+ t + \frac{1}{32w_+^3} \sin 3w_+ t + \frac{1}{8w_+^2} t \cos w_+ t$$

$$a_4 = \frac{1}{32w_+^4} \cos w_+ t - \frac{t}{8w_+^3} \sin w_+ t - \frac{1}{32w_+^4} \cos 3w_+ t$$

$$b_1 = -\frac{11}{96w_+} \sin w_+ t - \frac{t}{8} \cos w_+ t - \frac{1}{32w_+} \sin 3w_+ t$$

$$b_2 = \frac{1}{32w_+^4} \cos w_+ t - \frac{1}{8w_+^3} t \sin w_+ t - \frac{1}{32w_+^4} \cos 3w_+ t$$

$$b_3 = -\frac{3}{32w_+^2} \cos w_+ t - \frac{1}{8w_+} t \sin w_+ t + \frac{3}{32w_+^2} \cos 3w_+ t \tag{6}$$

$$b_4 = -\frac{5}{32w_+^3} \sin w_+ t - \frac{1}{8w_+^2} t \cos w_+ t + \frac{3}{32w_+^2} \sin 3 w_+ t$$

For convenience we also give the result for negative sectors which is :

$$x(t) = \frac{\dot{x}(0)}{w_-} \sinh w_- t + x(0) \cosh w_- t + e\left\{c_1 x^3(0) + c_2 \dot{x}^3(0) + c_3 x^2(0)\,\dot{x}(0) + c_4 x(0)\,\dot{x}^2(0)\right\} \tag{7}$$

$$\dot{x}(t) = \dot{x}(0) \cosh w_- t + x(0) w_- \sinh w_- t + e\left\{d_1 x^3(0) + d_2 \dot{x}^3(0) + d_3 x^2(0)\dot{x}(0) + d_4 x(0)\dot{x}^2(0)\right\}$$

where:

$$c_1 = \frac{t}{8w_-} \sinh w_- t + \frac{1}{96w_-^2} \cosh 3w_- t - \frac{\cosh w_- t}{96w_-^2}$$

$$c_2 = -\frac{t}{8w_-^4} \cosh w_- t + \frac{1}{96w_-^5} \sinh 3w_- t + \frac{1}{8w_-^5} \sinh w_- t - \frac{\sinh w_- t}{32w_-^5}$$

$$c_3 = \frac{t}{8w_-^2} \cosh w_- t + \frac{1}{32w_-^3} \sinh 3w_- t - \frac{7}{32w_-^3} \sinh w_- t$$

$$c_4 = \frac{-t}{8w_-^3} \sinh w_- t + \frac{\cosh 3w_- t}{32\, w_-^4} - \frac{\cosh w_- t}{32\, w_-^4}$$

$$d_1 = -\frac{11}{96w_-} \sinh w_- t + \frac{t}{8} \cosh w_- t + \frac{1}{32w_-} \sinh 3w_- t$$

$$
\begin{aligned}
d_2 &= -\frac{t}{8w_-^3}\sinh w_-t + \frac{1}{32w_-^4}\cosh 3w_-t - \frac{\cosh w_-t}{32\, w_-^4} \\
d_3 &= \frac{t}{8w_-}\sinh w_-t + \frac{3}{32w_-^2}\cosh 3w_-t - \frac{3\cosh w_-t}{32\, w_-^2} \qquad (8)\\
d_4 &= -\frac{5}{32w_-^3}\sinh w_-t - \frac{t}{8w_-^2}\cosh w_-t + \frac{3}{32\, w_-^3}\sinh 3w_-t
\end{aligned}
$$

One can readily check that these solutions which are valid to terms linear in e, yield a Jacobian which is unity plus terms of order e^2. Thus the Jacobian is not exactly unity, although it is correct through terms linear in e.

We now wish to obtain a transformation from the center of a positive sector to the center of the next positive sector. This involves folding together three transformations of the above form. For this purpose the following formula is convenient. If one iterates two transformations of the form:

$$
\begin{aligned}
x(1) &= \alpha_1 x(0) + \beta_1 \dot{x}(0) + e\Big\{a_1 x^3(0) + a_2\dot{x}^3(0) + a_3 x^2(0)\dot{x}(0) \\
&\quad + a_4 x(0)\dot{x}^2(0)\Big\} \\
\dot{x}(1) &= \gamma_1 x(0) + \delta_1 \dot{x}(0) + e\Big\{b_1 x^3(0) + b_2\dot{x}^3(0) + b_3 x^2(0)\dot{x}(0) \\
&\quad + b_4 x(0)\dot{x}^2(0)\Big\} \qquad (9)\\
x(2) &= \alpha_2 x(1) + \beta_2 \dot{x}(1) + e\Big\{c_1 x^3(1) + c_2\dot{x}^3(1) + c_3 x^2(1)\dot{x}(1) \\
&\quad + c_4 x(1)\dot{x}^2(1)\Big\} \\
\dot{x}(2) &= \gamma_2 x(2) + \delta_2 \dot{x}(2) + e\Big\{d_1 x3(1) + d_2\dot{x}^3(1) + \\
&\quad + d_3 x^2(1)\dot{x}(1) + d_4 x(1)\dot{x}^2(1)\Big\}
\end{aligned}
$$

to obtain a transformation, valid through terms linear in e, of the form:

$$
\begin{aligned}
x(2) &= \alpha\, x(0) + \beta\, \dot{x}(0) + e\Big\{ a x^3(0) + b\dot{x}^3(0) + c x^2(0)\, \dot{x}(0) \\
&\qquad + d x(0)\, \dot{x}^2(0) \Big\} \qquad\qquad (10) \\
\dot{x}(2) &= \gamma\, x(0) + \delta\, \dot{x}(0) + e\Big\{ a' x^3(0) + b' \dot{x}^3(0) + c' x^2(0) \dot{x}(0) \\
&\qquad + d' x(0)\, \dot{x}^2(0) \Big\}
\end{aligned}
$$

then the coefficients are given by :

$$
\begin{aligned}
\alpha &= \alpha_1 \alpha_2 + \gamma_1 \beta_2 = \cos \sigma_0 + \alpha_0 \sin \sigma_0 \qquad\qquad (11) \\
\beta &= \beta_1 \alpha_2 + \delta_1 \beta_2 = \beta_0 \sin \sigma_0 \\
\gamma &= \alpha_1 \gamma_2 + \gamma_1 \delta_2 = -\left(\frac{1 + \alpha_0^2}{\beta_0} \right) \sin \sigma_0 \\
\delta &= \beta_1 \gamma_2 + \delta_1 \delta_2 = \cos \sigma_0 - \alpha_0 \sin \sigma_0
\end{aligned}
$$

$$a = \alpha_2 a_1 + \beta_2 b_1 + c_1 \alpha_1^3 + c_2 \gamma_1^3 + c_3 \alpha_1^2 \gamma_1 + c_4 \alpha_1 \gamma_1^2$$

$$b = \alpha_2 a_2 + \beta_2 b_2 + c_1 \beta_1^3 + c_2 \delta_1^3 + c_3 \beta_1^2 \delta_1 + c_4 \beta_1 \delta_1^2$$

$$
\begin{aligned}
c &= \alpha_2 a_3 + \beta_2 b_3 + c_1 3\alpha_1^2 \beta_1 + c_2 \gamma_1^2 \delta_1 + c_3 \alpha_1^2 \delta_1 + \\
&\quad + c_3 2 \alpha_1 \beta_1 \gamma_1 + c_4 \gamma_1^2 \beta_1 + c_4 2 \gamma_1 \delta_1 \alpha_1
\end{aligned}
$$

$$
\begin{aligned}
d &= \alpha_2 a_4 + \beta_2 b_4 + c_1 3 \alpha_1 \beta_1^2 + c_2 \gamma_1 \delta_1^2 + c_3 \beta_1^2 \gamma_1 \\
&\quad + c_3 2 \alpha_1 \beta_1 \delta_1 + c_4 \delta_1^2 \alpha_1 + c_4 2 \gamma_1 \delta_1 \beta_1
\end{aligned}
$$

and a', b', c', d' are given by similar formulas where

$$\begin{aligned} \alpha_2 &\rightarrow \gamma_2 \\ \beta_2 &\rightarrow \delta_2 \\ c_i &\rightarrow d_i \end{aligned} \tag{12}$$

We may thus obtain a transformation through one sector of the form of eq. 10, which although not exactly area preserving, is area preserving through terms linear in e. We now follow the procedure outlined in I, reducing this to normal form by means of the transformations:

$$\begin{aligned} Q &= \frac{1 - i\alpha_0}{2} x - \frac{\beta_0 i}{2} \dot{x} \\ P &= \frac{1 + i\alpha_0}{2} x + \frac{\beta_0 i}{2} \dot{x} \end{aligned} \tag{13}$$

and

$$\begin{aligned} \overline{Q} &= Q + \overline{\alpha} Q^3 + \overline{\beta} P^3 + \overline{\gamma} Q^2 P + \overline{\delta} QP^2 \\ \overline{P} &= P + \overline{\alpha}' Q^3 + \overline{\beta}' P^3 + \overline{\gamma}' Q^2 P + \overline{\delta}' QP^2 \end{aligned} \tag{14}$$

and

$$\begin{aligned} u &= \overline{Q} + \overline{P} \\ v &= i\left[\overline{Q} - \overline{P}\right] \end{aligned} \tag{15}$$

where the result of transformation (13) may be written as:

$$\begin{aligned} Q &= e^{i\sigma_0} Q_0 + \left(\frac{1 - i\alpha_0}{2}\right)(\overline{x})_3 - \frac{\beta_0 i}{2} (\dot{\overline{x}})_3 \\ P &= e^{-i\sigma_0} P_0 + \left(\frac{1 + i\alpha_0}{2}\right)(\overline{x})_3 + \frac{\beta_0 i}{2} (\dot{\overline{x}})_3 \end{aligned} \tag{16}$$

$$\text{where } (\bar{x})_3 = e\left\{ax^3(0) + b\dot{x}^3(0) + cx^2(0)\dot{x}(0) + dx(0)\dot{x}^2(0)\right\} \tag{17}$$

$$(\dot{\bar{x}})_3 = e\left\{a'x^3(0) + b'\dot{x}^3(0) + c'x^2(0)\dot{x}(0) + d'x(0)\dot{x}^2(0)\right\}$$

and

$$x(0) = Q_0 - P_0$$
$$\dot{x}(0) = \left[\frac{i - \alpha_0}{\beta_0}\right]Q_0 - \left[\frac{i + \alpha_0}{\beta_0}\right]P_0 \tag{18}$$

If we call the result of this substitution:

$$Q = e^{i\sigma_0} Q_0 + AQ_0^3 + BP_0^3 + CQ_0^2 P_0 + DQ_0P_0^2$$
$$\tag{19}$$
$$P = e^{-i\sigma_0} P_0 + A'Q_0^3 + B'P_0^3 + C'Q_0^2 P_0 + D'Q_0P_0^2$$

then:

$$\bar{\alpha} = \left[\frac{eA}{e^{i\sigma_0} - e^{i3\sigma_0}}\right] \qquad \bar{\alpha}' = \left[\frac{eA'}{e^{-i\sigma_0} - e^{i3\sigma_0}}\right]$$
$$\bar{\beta} = \left[\frac{eB}{e^{i\sigma_0} - e^{-i3\sigma_0}}\right] \qquad \bar{\beta}' = \left[\frac{eB'}{e^{-i\sigma_0} - e^{-i3\sigma_0}}\right]$$
$$\bar{\delta} = \left[\frac{eD}{e^{i\sigma_0} - e^{-i\sigma_0}}\right] \qquad \bar{\gamma}' = \left[\frac{eC'}{e^{-i\sigma_0} - e^{i\sigma_0}}\right] \tag{20}$$

$\bar{\delta}'$ & $\bar{\gamma}$ are arbitrary

Finally one obtains in the variables u and v an expression of the form:

$$u = u_0\cos\sigma + v_0\sin\sigma + \Phi_n$$
$$v = -u_0\sin\sigma + v_0\cos\sigma + \Psi_n \tag{21}$$

where Φ_n and Ψ_n are of 4th order and

$$\cos\sigma = \cos\sigma_0 + \frac{Ce}{2}\left[1 - e^{-12\sigma_0}\right]\overline{P}_0\,\overline{Q}_0 \tag{22}$$

The invariant curves are given by $\overline{P_0 Q_0}$ = constant, and one may readily express this in terms of x_0 and $\dot{x}_0$. The above is only possible if $D' = -Ce^{-2i\sigma_0}$ which is quaranteed by the fact that Eq. 10 is area preserving through terms linear in e.

The comments made in I concerning convergence, are still applicable.

The expression for C is :

$$\begin{aligned} C = \Bigg[& \frac{3b\alpha_0^2}{\beta_0^2} + \frac{3}{2}\frac{b}{\beta_0^2} + \frac{c}{2\beta_0} - \frac{3}{2}\frac{\alpha_0 d}{\beta_0^2} - \frac{3}{2}\alpha_0\, a \\ & + \frac{3}{2}\frac{\alpha_0^4 b}{\beta_0^2} + \frac{3}{2}\frac{\alpha_0^2}{\beta_0}c - \frac{3}{2}\frac{\alpha_0^3}{\beta_0^2}d - \frac{3}{2}\beta_0 a' + \frac{3}{2}\frac{\alpha_0^3}{\beta_0}b' + \\ & + \frac{3}{2}\frac{\alpha_0}{\beta_0}b' + \frac{3}{2}\alpha_0 c' - \frac{d'}{2\beta_0} - \frac{3\alpha_0^2}{2\beta_0}d' \Bigg] i + \\ & + \Bigg[\frac{3}{2}a - \frac{\alpha_0}{\beta_0}c + \frac{d}{2\beta_0^2} + \frac{\alpha_0^2}{2\beta_0^2}d + \frac{3\alpha_0^2}{2\beta_0}b' + \\ & + \frac{3b'}{2\beta_0} + \frac{c'}{2} - \frac{\alpha_0}{\beta_0}d' \Bigg] \end{aligned} \tag{23}$$

Of course, for the example considered $\alpha_0 = 0$, but we have given all expressions for the general case of $\alpha_0 \neq 0$.

The advantage of the variables u and v is that the transformation in these variables may be approximated to any order by an area preserving transformation. In the variables x and $\acute{x}$ one would have to add an infinite number of terms (in general) to Eq. 10 in order to keep the correct linear terms in e and yet make it exactly area preserving. This leads to a transformation which is not convenient for computer use. The alternative of modifying the linear terms makes it difficult to relate the transformation to the differential equation. By transforming from an algebraic to a trigonometric form, one can keep the terms linear in e unmodified and yet obtain exact area preservation with a finite number of terms. A transformation of the form of Eq. 21 is still amenable to rapid calculation, although of course, not as rapid as a transformation of Powell's form. It should be noted that the tedious transformation from x,$\acute{x}$ to u,v variables is only made twice in any calculation. The main body of the calculation proceeding in u, v variables.

III Numerical Example

We carried through the above analysis numerically for the case specified by

$$n_1 = -n_2 = 16;\ w_+ = -w_- = 4$$
$$e_1 = -e_2 = 1000$$
$$\sigma_0 = .5847$$
$$\beta_0 = 1.11998$$

This yielded:

$a_1 = -.0023973$	$b_1 = -.03524$
$a_2 = -.00001866$	$b_2 = -.0005663$
$a_3 = -.0002945$	$b_3 = -.006595$
$a_4 = -.00000048$	$b_4 = -.00001866$
$c_1 = .01473$	$d_1 = .16015$
$c_2 = .00002113$	$d_2 = .0004668$
$c_3 = .00388$	$d_3 = .059132$
$c_4 = .0004668$	$d_4 = .008792$
$\alpha = .8338$	$\gamma = -.4929$
$\beta = .6184$	$\bar{\delta} = \alpha$
$a = -.01122$	$a' = -.02386$
$b = .0000113$	$b' = -.004975$
$c = .002649$	$c' = -.020979$
$d = .0009706$	$d' = -.018784$

<u>Yielding:</u>

$$C = -.03288 + .04966\, i$$

$$\Delta\sigma = -\frac{Ce(1-e^{-i2\sigma_0})}{8\sin\sigma_0}\, x^2(0) \qquad (\text{if } \dot{x}(0) = 0)$$

$$\Delta\sigma = 14.9x^2(0)$$

This should be compared to the best fit to the integration of the differential equation, which is $\Delta\sigma = 14.2\, x^2(0)$. The difference is believed due to small numerical errors in the calculation reported here. The calculation is very long and it was not felt to be of sufficient interest to repeat it and so attempt to obtain better agreement.

A study of the numerical solution to the differential equation indicates that the parabolic dependence of σ on $x(0)$ is good even for amplitudes where the amount of cubic term is 20%, ie, $1/3\ \frac{e}{n}\ x^2(0) = .2$. However, the motion becomes unstable for amplitudes $x(0)x.17$, whereas the above work indicates no such instability, instability only occuring at amplitudes where $|\cos\sigma| \geq 1$, which occurs for much larger values of $x(0)$. The instability observed in the numerical integration is presumably due to a lack of convergence of the Birkhoff form.

IV Two Dimensional Theory

In this section we show that a perturbation theory solution to two coupled equations may be reduced in a systematic way to Birkhoff form. The theory of this section follows trivially from the work of Jauch (MURA-J.M.J.-1), and consequently we shall only outline the method, and state the results.

We wish to consider two non-linear equations which are uncoupled in linear approximation. An example (our work is not of course limited to this case) arises in the study of the simplified non-linear C.L.S. machine:

$$\ddot{x} = \left[w_x^{\ 2} + e \ (xy^2 - 1/3 \ x^3) \right] \quad U(t)$$
$$y = \left[-w_y^2 + e \ (x^2 y - 1/3 \ y^3) \right] \quad U(t) \qquad (24)$$

where: $U(t)$ is periodic of period 2π

and: $U(t) = \begin{cases} 1 & 0 \leq t < \pi \\ -1 & \pi \leq t < 2\pi \end{cases}$

We shall assume that the coupled equations can be solved by perturbation theory and that the transformation thus generated may be reduced to normal form as far as the linear terms are concerned. This last follows from the assumption of uncoupled linear motion. The generating function (F) describing this transformation (T) from $x_0, \dot{x}_0, y_0, \dot{y}_0 \longrightarrow x, \dot{x}, y, \dot{y}$ may be assumed to have the following form, if the transformation T is the exact transformation expressed as a power series in the amplitudes of the motion:

$$F(\dot{x}, \dot{y}, x_0, y_0) = (\rho_0)_x \dot{x} \, x_0 - (\rho_0)_y \dot{y} \, y_0 + G \qquad (25)$$

where:

$$\dot{x}_0 = \frac{\partial F}{\partial x_0} \qquad \dot{y}_0 = \frac{\partial F}{\partial y_0}$$
$$x = \frac{\partial F}{\partial \dot{x}} \qquad y = \frac{\partial F}{\partial \dot{y}} \qquad (26)$$

and $G(\dot{x},\dot{y},x_0,y_0) = G_3(\dot{x},\dot{y},x_0,y_0) + G_4(\dot{x},\dot{y},x_0,y_0) + \cdots \quad (27)$

where G_n is homogeneous of degree n.

We now wish to change variables from $x, \dot{x}, y, \acute{y}$ to new variables Q_x, P_x, Q_y, P_y, in such a way as to reduce F to normal form. This we attempt by a series of transformations generated by $K(P_x, x, P_y, y)$, where: (28)

$$K(P_x, x, P_y, y) = P_x \cdot x + P_y \cdot y + K_3(P_x, x, P_y, y) + K_4(P_x, x, P_y, y) + \ldots$$

where K_n is homogeneous of degree n.

and:

$$\dot{x} = \frac{\partial K}{\partial x} \qquad Q_x = \frac{\partial K}{\partial P_x}$$

$$\acute{y} = \frac{\partial K}{\partial y} \qquad Q_y = \frac{\partial K}{\partial P_y} \tag{29}$$

By reasoning exactly analogous to that of Jauch (and subject to the same restrictions) one can show that K may be chosen so that this can be accomplished. One finally obtains:

$$F(P_x, P_y, Q_{x_0}, Q_{y_0}) = (\rho_0)_x P_x Q_{x_0} + (\rho_0)_y P_y Q_{y_0} +$$

$$- (f_2)_x (P_x Q_{x_0})^2 + (f_2)_y (P_y Q_{y_0})^2 + \tag{30}$$

$$-(f_2)_{xy} P_x Q_{x_0} P_y Q_{y_0} + \ldots$$

where $(f_2)_x$, $(f_2)_y$, $(f_2)_{xy}$ are constants.

The reduction to Birkhoff Variables now proceeds as before, yielding

$$
\begin{aligned}
u_x &= \cos \sigma_x \, u_{x_0} + \sin \sigma_x \, v_{x_0} \\
v_x &= -\sin \sigma_x \, u_{x_0} + \cos \sigma_x \, v_{x_0} \\
u_y &= \cos \sigma_y \, u_{y_0} + \sin \sigma_y \, v_{y_0} \\
v_y &= -\sin \sigma_y \, u_{y_0} + \cos \sigma_y \, v_{y_0}
\end{aligned}
\tag{31}
$$

where:

$$
\begin{aligned}
\cos \sigma_x &= 1/2 \left[\rho_x + \rho_x^{-1} \right] \\
\cos \sigma_y &= 1/2 \left[\rho_y + \rho_y^{-1} \right]
\end{aligned}
\tag{32}
$$

and:

$$
\begin{aligned}
\rho_x = (\rho_0)_x \Big[1 &+ 2 \, (f_2)_x \, \rho_{0x}^{-2} \, (P_{x_0} \; Q_{x_0}) + \\
&+ (f_2)_{xy} \, (\rho_{ox} \, \rho_{oy})^{-1} \, (P_{y_0} \; Q_{y_0}) + \ldots \Big] \\
\rho_y = (\rho_0)_y \Big[1 &+ 2 \, (f_2)_y \, \rho_{0y}^{-2} \, (P_{y_0} \; Q_{y_0}) + \\
&+ (f_2)_{xy} \, (\rho_{0x} \, \rho_{oy})^{-1} \, (P_{y_0} \; Q_{x_0}) + \ldots\ldots \Big]
\end{aligned}
\tag{33}
$$

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