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THE CONSTRUCTION OF APPROXIMATE PHASE PLANE TRANSFORMATIONS FOR A.G. SYNCHROTRONS, I.

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I. INTRODUCTION

In the design of A.G. Synchrotrons it is necessary to study the stability of particle trajectories. This may be accomplished by extensive computational programs involving the numerical integration of the equations for oscillations about an equilibrium orbit. As pointed out by Powell and Wright (MURA- R.W./J.L.P. -5), the computation program can be very much reduced if one has available phase plane transformations. A transformation is simply an approximate solution to the equations of oscillation which relates the displacement and velocity at some point, to the displacement and velocity at some previous point. Powell and Wright have considered in detail the one-dimensional non-linear problem defined by the equation:

$$\ddot{x} = \left[w^2 x + 1/3\ e\ x^3\right] U(s) \qquad (1)$$

Where U(s) is a periodic function, of period 2π and

$$U(s) = \begin{cases} 1 & 0 < s < \pi \\ -1 & \pi < s < 2\pi \end{cases}$$

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They have formed an area preserving algebraic transformation of simple form, which they have used to calculate the betatron oscillation wave length as a function of initial conditions. By comparing this with a number of numerical solutions of the equation, they are able to fix the arbitrary parameter in the transformation. Unfortunately this involves a great deal of computation.

As part of a general program to obtain transformations analytically, we have been investigating the extensive work of Birkhoff (See the numerous papers in Vol. II of Collected Mathematical Papers, Am. Math Soc., 1950) on transformations in one dimensional problems. This material has been reformulated by Jauch (MURA-J.M.J.-1), into a very convenient form. One of the results of Birkhoff's work is that a general area preserving transformation may always be formally reduced to a normal form in which the betatron wavelength may be immediately obtained as a function of initial conditions. This result allows one to reduce a transformation of Powell's form to normal form and thus fix the arbitrary parameters by comparison with one numerical solution of the differential equation. This constitutes a considerable saving in computing time over the methods employed by Powell. In particular, given one numerical solution for each operating point, one can easily obtain transformations through a full sector, and from a point of symmetry, either on or off the diagonal of the necktie. This is done for a few cases in Sections III and IV.

The method can readily be applied to obtain transformations from non-symmetric points, or even through 1/2 sectors. In this case the additional parameters in Powell's form may be obtained by locating fixed points. For the study of bumps it is expected that such transformations will be extremely useful and more realistic than the use of physically unreasonable displaced full sector transformations. The construction of 1/2 sector transformations in now underway and will be described in a future report.

A study of bumps by use of Birkhoff Variables is probably easier than by the method outlined here. A survey study of this is now being undertaken with the use of desk computers. However, for many cases the equations have already been solved numerically, so the techniques outlined here may be used immediately to obtain useful transformations. Also the Illiac has been coded to use transformations of Powell's form.

We are now investigating the use of Perturbation Theory in order to obtain transformations directly from the differential equation. These transformations will employ Birkhoff Variables and be described in a future report.

II GENERAL THEORY

Consider an equation of the form:

$$\ddot{x} = F(x, t) \tag{2}$$

where $F(x, t)$ is periodic of period 2π. Let T be the transformation (or solution) which relates $x(t)$ and $\dot{x}(t)$ with the values $x(0)$ and $\dot{x}(0)$. One can easily show that T is area preserving in phase space. Formally we may write T as a power series in $x(0)$ and $\dot{x}(0)$, thus:

$$\begin{aligned} x = x(2\pi) &= \sum_{i,j=0}^{\infty} a_{ij} \left[x(0)\right]^i \left[\dot{x}(0)\right]^j \\ \dot{x} = \dot{x}(2\pi) &= \sum_{i,j=0}^{\infty} b_{ij} \left[x(0)\right]^i \left[\dot{x}(0)\right]^j \end{aligned} \tag{3}$$

Birkhoff has shown that one may formally effect a change of variables such that in the new variables u and v the transformation T takes the form:

$$\begin{aligned} u &= u_0 \cos\sigma + v_0 \sin\sigma + \Phi_n \\ v &= -u_0 \sin\sigma + v_0 \cos\sigma + \Psi_n \end{aligned} \tag{4}$$

where Φ_n and Ψ_n are power series in u_0, v_0 which have a leading term of arbitrary high order n, and $\sigma = \sigma(u_0^2 + v_0^2)$. Thus if one is interested in transformations which are valid near an equilibrium orbit one may ignore the terms Φ_n and Ψ_n.

It will be observed that the remaining transformation is exactly area preserving, and furthermore of a particularly simple form. For example the variation of betatron wavelength with amplitude may be obtained simply by inspection. It is suggested that a calculation proceed by change of variables from x, x' to u, v. Then the effect of bumps, etc., is calculated by the computer in u, v variables. This is particularly simple due to the quasi-linear form of the transformation. The information so obtained may then be transformed back into $x\ x'$ variables.

It should be noted that the convergence of the transformation (4) is not established. In particular it will not converge at or beyond the separatrix. It is not even clear that it converges within any neighborhood of the equilibrium orbit. We shall not attack the question of convergence. The change of variables from (x, x') to (u,v) has been discussed by Jauch.

The program is to take Powell's Transformation and put it into the form (3). This is then reduced to normal form by a series of successive approximations. Thus one obtains variables P and Q such that :

$$P = \mathcal{S}^{-1}(P_0 \cdot Q_0)\ P_0 \qquad (5)$$

$$Q = \mathcal{J}\ (P_0 \cdot Q_0)\ Q_0$$

where $\rho(P_0 \cdot Q_0) = \sum_{n=0}^{\infty} \rho_n (P_0 \cdot Q_0)^n$

The series for ρ is now terminated after a few terms. One further change of variables defined by:

$$u = Q + P \qquad\qquad (6)$$
$$v = i\,(Q - P)$$

brings the transformation into the form (4), where:

$$\cos\sigma = 1/2 \left[\rho + \rho^{-1}\right] \qquad\qquad (7)$$

and $\rho = \rho\left[u^2 + v^2\right]$

III. Powell's Transformation at a Point of Symmetry

For a point of symmetry Powell's area preserving transformation may be written as:

$$x = \alpha x_0 + \beta y_0 + \beta k \left\{(1 + \alpha)x_0 + \beta y_0\right\}^3 \qquad\qquad (8)$$
$$y = \gamma x_0 + \alpha y_0 + (1+\alpha)k \left\{(1 + \alpha)x_0 + \beta y_0\right\}^3$$

where: $\alpha = \cos\sigma_0$
$\beta = \beta_0 \sin\sigma_0$
$\gamma = -\frac{1}{\beta_0}\sin\sigma_0$

and ($\alpha_0 = 0, \beta_0, \sigma_0$) define the transformation for the linear problem, The parameter k is arbitrary.

We may now effect a transformation which reduced the linear terms to normal form. This may be done by means of:

$$\begin{cases} Q = x/2 - \dfrac{\beta_0 i}{2} y \\ P = x/2 + \dfrac{\beta_0 i}{2} y \end{cases} \qquad \begin{cases} x = Q + P \\ y = \dfrac{i}{\beta_0}\left[Q - P\right] \end{cases} \tag{9}$$

Which when applied to (8), written in the form:

$$\begin{aligned} x &= \cos\sigma_0\, x_0 + \beta_0 \sin\sigma_0\, y_0 + a x^3(0) + b y^3(0) + c x^2(0) y(0) + \\ &\quad + d x(0) y^2(0) \end{aligned} \tag{10}$$

$$\begin{aligned} y &= -\frac{1}{\beta_0} \sin\sigma_0\, x_0 + \cos\sigma_0\, y_0 + a' x^3(0) + b' y^3(0) + c' x^2(0) y(0) + \\ &\quad + d' x(0) y^2(0) \end{aligned}$$

yields:

$$\begin{aligned} Q &= e^{i\sigma_0} Q_0 + A Q_0^3 + B P_0^3 + C Q_0^2 P_0 + D Q_0 P_0^2 \\ P &= e^{-i\sigma_0} P_0 + A' Q_0^3 + B' P_0^3 + C' Q_0^2 P_0 + D' Q_0 P_0^2 \end{aligned} \tag{11}$$

where $A, B \ldots C', D'$ may be easily obtained by algebraic manipulation.

We now wish to change variables once again, so that the transformation through cubic terms is in normal form. For this purpose introduce variables $\overline{Q}$ and $\overline{P}$ defined by:

$$\overline{Q} = Q + \overline{\alpha} Q^3 + \overline{\beta} P^3 + \overline{\gamma} Q^2 P + \overline{\delta} Q P^2 \tag{12}$$

$$\bar{P} = P + \bar{\alpha}' Q^3 + \bar{\beta}' P^3 + \bar{\gamma}' Q^2 P + \bar{\delta}' Q P^2$$

where:

$$\bar{\alpha} = \frac{A}{e^{i\sigma_0} - e^{i3\sigma_0}} \qquad \bar{\alpha}' = \frac{A'}{e^{-i\sigma_0} - e^{i3\sigma_0}}$$

$$\bar{\beta} = \frac{B}{e^{i\sigma_0} - e^{-i3\sigma_0}} \qquad \bar{\beta}' = \frac{B'}{e^{-i\sigma_0} - e^{-i3\sigma_0}} \qquad (13)$$

$$\bar{\delta} = \frac{D}{e^{i\sigma_0} - e^{-i\sigma_0}} \qquad \bar{\gamma}' = \frac{C'}{e^{-i\sigma_0} - e^{i\sigma_0}}$$

$\bar{\delta}'$ and $\bar{\gamma}$ are arbitrary

Then:

$$\bar{Q} = \bar{Q}_0 \left\{ e^{i\sigma_0} + C(\bar{P}_0 \bar{Q}_0) \right\}$$

$$\bar{P} = \bar{P}_0 \left\{ e^{-i\sigma_0} + D'(\bar{P}_0 \bar{Q}_0) \right\} \qquad (14)$$

and now choosing:

$$u = Q + P$$

$$v = i\left[Q - P\right] \qquad (15)$$

reduces the transformation to form (4), with:

$$\cos\sigma = \cos\sigma_0 + \frac{C}{2}\left[1 - e^{-i2\sigma_0}\right] \bar{P}_0 \bar{Q}_0 + \ldots.. \qquad (16)$$

To first approximation, one may replace $\overline{P_0}\ \overline{Q_0}$ in Eq. (16) with:

$$\overline{P}_0\ \overline{Q_0} \approx P_0\ Q_0 = +\frac{1}{4}\ x^2\ (0) + \frac{\beta_0^2}{4}\ y^2\ (0) \tag{17}$$

In order for the above to work it must be true that $D' = -Ce^{-2i\sigma_0}$, which condition is quaranteed by the fact that the transformation Eq. (8) is area preserving.

The parameter k is determined by comparing the expression (Eq. 16) for σ, with a numerical integration of the differential equation. Only C is needed for this purpose, the expression for which is:

$$C = \left[3/2\ a + \frac{3b'}{2\beta_0^2} + \frac{c'}{2} + \frac{d}{2\beta_0^2} \right] + i \left[\frac{-3a'\beta_0}{2} + \frac{3b}{2\beta_0^3} + \frac{c}{2\beta_0} - \frac{d'}{2\beta_0} \right] \tag{18}$$

Although the analysis has involved the introduction of complex numbers, it is computationally easier than working in a purely real representation, which would have necessitated solution of numerous simultaneous linear equations. The resulting expression for $\cos\sigma$ (eq. 16) is always real.

<u>IV Numerical Results for Full Sector Transformations</u>

The method was first applied to a point on the diagonal, specified by:

$n_1 = -n_2 = 16$ $\qquad$ $\alpha = .834$

$e_1 = -e_2 = 1000$ $\qquad$ $\beta = .618$

$\sigma_0 = .5847$ $\qquad$ $\gamma = -.493$

$\beta_0 = 1.120$

Upon reduction to normal form and comparison with the results of the differential equation for σ, k was found to agree with Powell's value to better than 1%.

We then used the method to obtain the k in Powell's Transformation for two points off the diagonal. In each case the numerical solution has a σ which could be fitted accurately with a parabola, for amplitudes as large as $x_0 = 0.10$.

	Case I:		Case II:
	$n_1 = 15.7683$		$n_1 = 15.8074$
	$n_2 = -15.6985$		$n_2 = -15.6678$
	$e_1 = 1000$		$e_1 = 1000$
	$e_2 = -1000$		$e_2 = -1000$
	$\sigma_0 = .58905$		$\sigma_0 = .59690$
	$\beta_0 = 1.12776$		$\beta_0 = 1.12327$
	$\sigma = \sigma_0 + 14.16x_0^2$		$\sigma = \sigma_0 + 13.96x_0^2$
	$k = -2.498$		$k = -2.481$

It will be noted that all of the above examples are for "y-motion". The method is similar for "x-motion", although it is not true that the only change is in the sign of k, when the operating point is off diagonal.

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