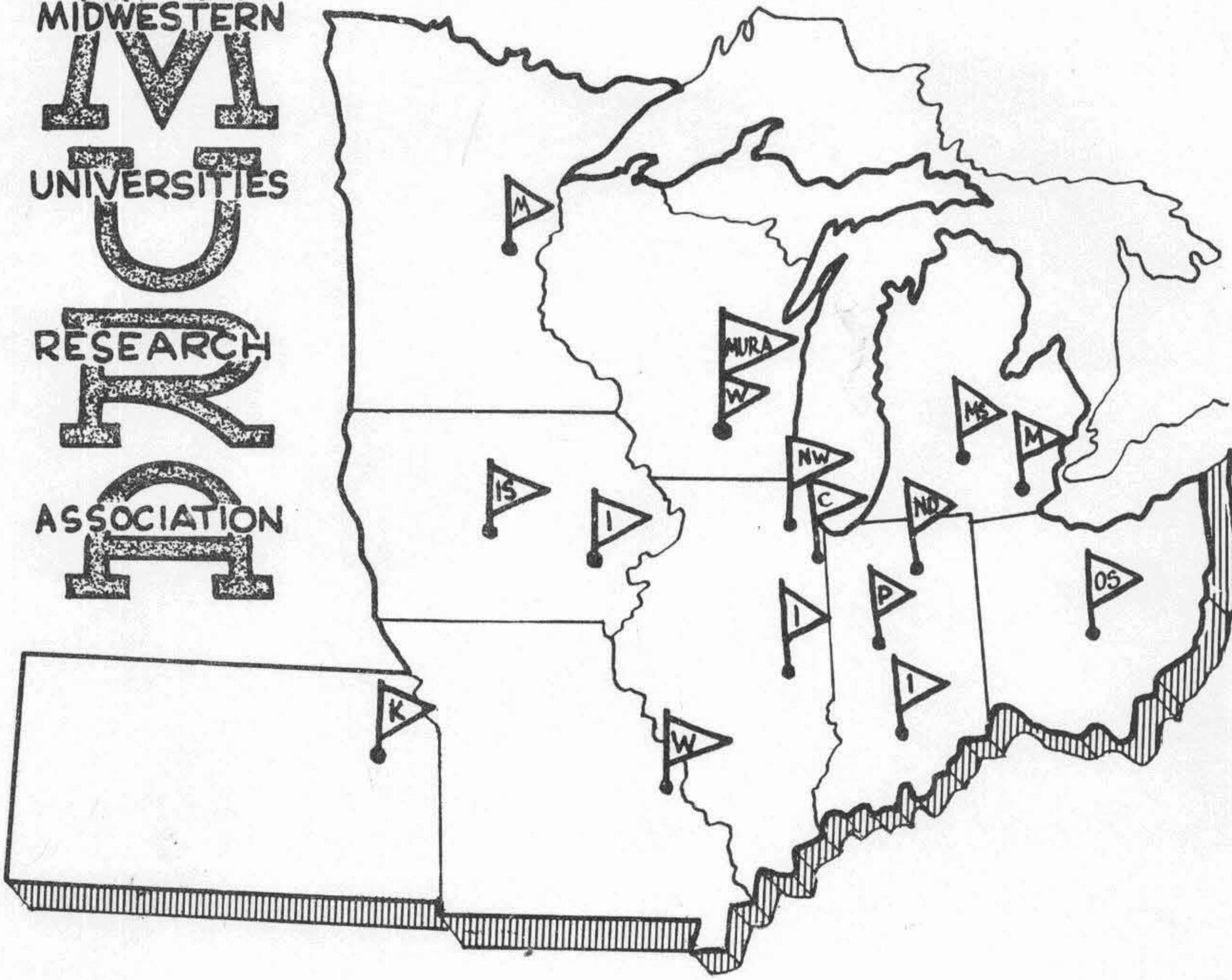
MIDWESTERN
UNIVERSITIES
RESEARCH
ASSOCIATION



The Vector Potential of the Magnetic Field in the Mark V Accelerator

REPORT NUMBER MURA-70

MURA-70
ESA(MURA-3)

The Vector Potential of the Magnetic Field in the Mark V Accelerator

Edward S. Akeley, Purdue University and the Midwestern Universities Research Association.* May 28, 1955

Abstract

Being given the magnetic field strength on the median plane, it is required to find the field on both sides of this plane in such a form that:

a) when substituted in the equations of motion of a charged particle, these equations will satisfy the theorem of Liouville, exactly.

b) the field is both Maxwelian, and assumes its prescribed values on the median plane to a certain degree of approximation.

c) its mathematical expression should be sufficiently simple, so that too much time will not be required for the computation of trajectories by a digital computer.

Thus the degree of approximation desired must be weighed against the corresponding time of computation. A series of solutions corresponding to ascending degrees of approximation has been found.

* Partially supported by the National Science Foundation

INDEX

Section 1. Two proposed expressions for the magnetic field strength in the median plane and their relation to certain complex fields, in which a certain approximation is involved.

The following expressions have been proposed for the magnetic field strength in the median plane for the Mark V, FFAG accelerator, namely

$$\left.\begin{aligned} H^{I}_{z,o} &= H_o\left(\frac{r}{r_o}\right)^{K}\left\{1 + f\sin\left[\frac{r-r_o}{\lambda} - N\theta\right]\right\} \\ H^{I}_{r,o} &= H^{I}_{\theta,o} = 0 \end{aligned}\right\} \quad (1.1)$$

$$\left.\begin{aligned} H^{II}_{z,o} &= H_o\left(\frac{r}{r_o}\right)^{K}\left\{1 + f\sin\left[\frac{1}{w}\ln\left(\frac{r}{r_o}\right) - N\theta\right]\right\} \\ H^{II}_{r,o} &= H^{II}_{\theta,o} = 0 \end{aligned}\right\} \quad (1.2)$$

where (r, θ) are polar coordinates in the median plane, and z is a rectangular coordinate perpendicular to that plane. The equation of the median plane is z=0. Also $\vec{H}_o = \vec{H}\,(r, \theta, o)$.

The above fields are the pure imaginary parts of the following complex fields:

$$\left.\begin{aligned} H^{I,c}_{z,o} &= H_o\left(\frac{r}{r_o}\right)^{K}\left\{i + f e^{i\left[\frac{r-r_o}{\lambda} - N\theta\right]}\right\} \\ H^{I,c}_{r,o} &= H^{I,c}_{\theta,o} = 0 \end{aligned}\right\} \quad (1.3)$$

$$\left.\begin{aligned} H^{II,c}_{z,o} &= H_o\left(\frac{r}{r_o}\right)^{K}\left\{i + f\left(\frac{r}{r_o}\right)^{\frac{i}{w}} e^{-iN\theta}\right\} \\ H^{II,c}_{r,o} &= H^{II,c}_{\theta,o} = 0 \end{aligned}\right\} \quad (1.4)$$

Now $\left(1+\frac{x}{n}\right)^n = e^x\left[1 - \frac{x^2}{2n} + \frac{x^3}{3n^2} - \frac{x^4}{4n^3} - - - - - -\right]$ (1.5)

so that $\left(1+\frac{x}{n}\right)^n \simeq e^x$ (1.6)

the error being less than $\frac{x^2 e^x}{2n}$

Therefore $\left(\frac{r}{r_0}\right)^K = \left(1+\frac{r-r_0}{r_0}\right)^K = \left[1+\frac{K(r-r_0)}{Kr_0}\right]^K \simeq e^{K\frac{r-r_0}{r_0}}$ (1.7)

the error being less than $e^{K\frac{(r-r_0)}{r_0}} \; K\left(\frac{r-r_0}{r_0}\right)^2$

(1.8)

and $\left(\frac{r}{r_0}\right)^{\frac{1}{\omega}} \simeq e^{\frac{r-r_0}{r_0 \omega}}$

the error being less than $e^{\frac{r-r_0}{r_0 \omega}} \; \frac{1}{\omega}\left(\frac{r-r_0}{r_0}\right)^2$

It shall be assumed in this report that these errors are sufficiently small to be neglected. However, there would be no difficulty in using additional terms of the series (1.5) in order to achieve a better approximation. The methods of this report could easily be carried out in this case.

Substituting (1.7) and (1.8) into (1.3) and (1.4), one finds

$$H_{z,0}^{I,c} = H_0\left[ie^{K\frac{r-r_0}{r_0}} + fe^{-K}e^{K\frac{r}{r_0}} + i\frac{r}{\chi} - iN\theta - i\frac{r_0}{\chi}\right] \qquad (1.9)$$

$$H_{z,0}^{II,c} = H_0\left[ie^{K\frac{r-r_0}{r_0}} + e^{-K}e^{K\frac{r}{r_0}} + \frac{ir}{r_0\omega} - iN\theta - \frac{i}{\omega}\right]$$

(1.10)

Both of these fields can be written in the form:

$$H_{z,0}^{(c)} = H_0 e^{-\kappa}\left[ie^{\alpha r} + f e^{\alpha r + i(\beta_1 r - \nu\theta - \chi)}\right] \tag{1.11}$$

which is expressible as linear combinations of the fields

$$H_{z,0} = e^{\alpha r} \quad \text{and} \quad H_{z,0} = e^{i(\beta r - \nu\theta - \chi)} \tag{1.12}$$

where $\beta = \beta_1 - i\alpha$ and where α, β_1, ν and χ are real constants

For $H_{z,0}^{I,c}$: $\alpha = \dfrac{\kappa}{r_0}$, $\beta_1 = \dfrac{1}{\lambda}$, $\nu = N$, $\chi = \dfrac{-r_0}{\lambda}$ (1.13)

For $H_{z,0}^{II,c}$: $\alpha = \dfrac{\kappa}{r_0}$, $\beta_1 = \dfrac{1}{r_0\omega}$, $\nu = N$, $\chi = -\dfrac{1}{\omega}$

It is required to find the vector potential on either side of the median plane in terms of the magnetic field strength on that plane. It is sufficient to solve this problem for the two fields (1.12). The fields (1.9) and (1.10) are linear combinations of these. Finally, the desired vector potentials corresponding to the real fields $H_{z,0}^{I}$ and $H_{z,0}^{II}$ can be found by taking the pure imaginary parts of the complex fields.

Statement of the Problem

We shall require that the expressions for these fields satisfy vector potential has been found to a certain degree of approximation, this vector potential is substituted in the differential equations of motion without any further change or approximation. It is because of this condition, that one must first find an approximate expression for the vector potential, instead of the magnetic field strength.

2) The field should be Maxwelian to as high a degree of approximation as possible, consistent with conditions (3) and (4).

3) The magnetic field should assume its proposed values (1.1) or (1.2) on the median plane to a certain degree of approximation.

4) The degree of approximation in (2) and (3) should be as high as possible, consistent with keeping the differential equations of motion sufficiently simple, so that the time of computation of a point on the orbit by a digital computer will not exceed some reasonable value.

In connection with (4), it might be noted that in the differential equations of the orbit, that A_r and A_z are their derivations occur in many places, while A_θ occurs with a much smaller frequency. The orbital equations can then be simplified if one chooses the gauge in such a way that A_r and A_z have as simple expressions as possible, and allowing A_θ to become more complicated as a consequence. This is one of the considerations that will guide us in the choice of the gauge.

Section II. Fundamental equations for the case $H_{z,0} = e^{i(\beta r - \tau\theta - \chi)}$

$H_{r,0} = H_{\theta,0} = 0$. The general method of procedure and the solutions of certain differential equations.

We shall assume that the vector potential is represented by the following expressions:

$$
\begin{aligned}
A_r &= e^{i(\beta r - \tau\theta - \chi)}\left\{ rR_1 + R_0 + \frac{R_{-1}}{r} + \frac{R_{-2}}{r^2} \right\} \\
A_\theta &= e^{i(\beta r - \tau\theta - \chi)}\left\{ \Theta_0 + \frac{\Theta_{-1}}{r} + \frac{\Theta_{-2}}{r^2} + \frac{\Theta_{-3}}{r^3} \right\} \qquad (2.1) \\
A_z &= e^{i(\beta r - \tau\theta - \chi)}\left\{ rZ_1 + Z_0 + \frac{Z_{-1}}{r} + \frac{Z_{-2}}{r^2} \right\}
\end{aligned}
$$

where R_k Θ_k, and Z_k (k) are functions of z only.

It might be noted that if certain expansions* were used, the vector potential would be expressed in a descending power series in r, the first few terms of which would be represented by (2.1). In fact, these expansions suggested the relations (2.1). The expressions (2.1) are, however, not infinite series, as they actually terminate. They will represent exactly a certain field, which approximates the actual field of the accelerator. The functions R_k, Θ_k, and Z_k(k) are to be so determined that the four conditions of section I are satisfied. This can be done in many ways, as there are different choices for the gauge, as well as for the degree of approximation.

---

* Expansions which express the magnetic field on either side of a plane surface in terms of the magnetic field on the surface, and their application to the Mark V, FFAG accelerator.
Edward S. Akeley, Purdue University, January 26, 1955
ESA(MURA)-1

The expressions in cylindrical coordinates for $\nabla \times A$ and $\nabla \times (\nabla \times A)$ are

$$
\begin{aligned}
(\nabla\times\vec{A})_r &= \frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z} \\
(\nabla\times\vec{A})_\theta &= \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \qquad (2.2)\\
(\nabla\times\vec{A})_z &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \\
\left[\nabla\times(\nabla\times\vec{A})\right]_r &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r\frac{\partial A_\theta}{\partial\theta}\right] - \frac{1}{r^2}\frac{\partial^2 A_r}{\partial\theta^2} - \frac{\partial^2 A_r}{\partial z^2} + \frac{\partial^2 A_z}{\partial z\partial r} \\
\left[\nabla\times(\nabla\times\vec{A})\right]_\theta &= -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)\right] + \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial A_r}{\partial\theta}\right] - \frac{\partial^2 A_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial^2 A_z}{\partial z\partial\theta} \qquad (2.3)\\
\left[\nabla\times(\nabla\times\vec{A})\right]_z &= \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial A_r}{\partial z}\right] - \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial A_z}{\partial r}\right] - \frac{1}{r^2}\frac{\partial^2 A_z}{\partial\theta^2} + \frac{1}{r}\frac{\partial^2 A_\theta}{\partial\theta\partial z}
\end{aligned}
$$

Substituting (2.1) into (2.2) and (2.3), we find

$$
\begin{aligned}
H_r = (\nabla\times\vec{A})_r &= e^{i(\beta r-\tau\theta-\chi)}\Big\{-\left[i\tau Z_1+\Theta_3'\right] - \frac{1}{r}\left[i\tau Z_0+\Theta_{-1}'\right] \\
&\quad - \frac{1}{r^2}\left[i\tau Z_{-1}+\Theta_{-2}'\right] - \frac{1}{r^3}\left[i\tau Z_{-2}+\Theta_{-3}'\right]\Big\} \\
H_\theta = (\nabla\times\vec{A})_\theta &= e^{i(\beta r-\tau\theta-\chi)}\Big\{r\left[R_1'-i\beta Z_1\right] + \left[R_0'-Z_1-i\beta Z_0\right] + \\
&\quad \frac{1}{r}\left[R_{-1}'-i\beta Z_{-1}\right] + \frac{1}{r^2}\left[R_{-2}'+Z_{-1}-i\beta Z_{-2}\right] + \frac{2}{r^3}Z_{-2}\Big\} \qquad (2.4)\\
H_z = (\nabla\times\vec{A})_z &= e^{i(\beta r-\tau\theta-\chi)}\Big\{i\beta\Theta_0 + i\tau R_1 + \frac{1}{r}\left[\Theta_0 + i\beta\Theta_{-1} + i\tau R_0\right] \\
&\quad + \frac{1}{r^2}\left[i\beta\Theta_{-2}+i\tau R_{-1}\right] + \frac{1}{r^3}\left[-\Theta_{-2}+i\beta\Theta_{-3}+i\tau R_{-2}\right] - \frac{2\Theta_{-3}}{r^4}\Big\}
\end{aligned}
$$

$$\left[\nabla\times(\nabla\times\vec{A})\right]_r = e^{i(\beta r-\gamma\theta-\chi)}\Big\{ r\left[-R_1''+i\beta Z_1'\right]+\left[-R_0''+Z_1'+i\beta Z_0'\right]$$
$$+\frac{1}{r}\left[\gamma^2R_1-R_{-1}''+i\beta Z_{-1}'+\beta\gamma\Theta_0\right]$$
$$+\frac{1}{r^2}\left[-i\gamma\Theta_0+i\beta Z_{-2}'-R_{-2}''+\beta\gamma\Theta_{-1}+\gamma^2R_0-Z_{-1}'\right]+\frac{1}{r^3}\left[3\gamma\Theta_{-2}+\gamma^2R_{-1}-2Z_{-2}'\right] \quad (2.5a)$$
$$+\frac{1}{r^4}\left[i\gamma\Theta_{-1}+\beta\gamma\Theta_{-3}+\gamma^2R_{-2}\right]+\frac{1}{r^5}2i\gamma\Theta_{-3}\Big\}$$

$$\left[\nabla\times(\nabla\times\vec{A})\right]_\theta = e^{i(\beta r-\gamma\theta-\chi)}\Big\{-i\gamma Z_1'-\Theta_0''+\beta^2\Theta_0+\beta\gamma R_{-1}+\frac{1}{r}$$
$$\left[-i\beta\Theta_0-i\gamma Z_0'-\Theta_{-1}''+\beta^2\Theta_{-1}+\beta\gamma R_0\right]+\frac{1}{r^2}\left[-i\gamma Z_{-1}'-\Theta_{-2}''+i\beta\Theta_{-1}+\beta^2\Theta_{-2}+i\gamma R_0+\beta\gamma R_{-1}\right]$$
$$+\frac{1}{r^3}\left[-i\gamma Z_{-2}'-\Theta_{-3}''+3i\beta\Theta_{-2}+\beta^2\Theta_{-3}+2i\gamma R_{-1}+\beta\gamma R_{-2}\right]+\frac{1}{r^4}\left[-3\Theta_{-2}+5i\beta\Theta_{-3}+3i\gamma R_{-2}\right]-\frac{8\Theta_{-3}}{r^5}\Big\} \quad (2.5b)$$

$$\left[\nabla\times(\nabla\times\vec{A})\right]_z = e^{i(\beta r-\gamma\theta-\chi)}\Big\{ r\left[i\beta R_1'+\beta^2Z_1\right]+\left[2R_1'+i\beta R_0'-3i\beta Z_1+\beta^2Z_0\right]$$
$$+\frac{1}{r}\left[R_0''+i\beta R_{-1}'-Z_1-i\beta Z_0+\beta^2Z_{-1}-i\gamma\Theta_0'+\gamma^2Z_1\right]$$
$$+\frac{1}{r^2}\left[i\beta R_{-2}'+i\beta Z_{-1}+\beta^2Z_{-2}+\gamma^2Z_0-i\gamma\Theta_{-1}'\right]$$
$$+\frac{1}{r^3}\left[-R_{-1}'-Z_{-1}+3i\beta Z_{-2}+\gamma^2Z_{-1}-i\gamma\Theta_{-1}'\right]+\frac{1}{r^4}\left[-4Z_{-2}+\gamma^2Z_{-2}-i\gamma\Theta_{-3}'\right]\Big\}$$

(2.5c)

Since we want $\nabla\times(\nabla\times\vec{A})=0$ to a certain degree of approximation, we shall require that the coefficients of the highest power of r be zero in each of the equations (2.5) or

$$-R_1''+i\beta Z_1'=0 \quad (2.6)$$
$$-i\gamma Z_1'-\Theta_0''+\beta^2\Theta_0+\beta\gamma R_{-1}=0$$
$$i\beta R_1'+\beta^2 Z_1=0$$

Now we might set $\Theta_0 = 0$. We would find only two independent equations, and we could solve for R_1 and Z_1. Such a choice would partially determine the gauge. In this choice, the A_r and A_z would be more important than A_θ. As has been previously mentioned, we are looking for solutions where the expressions for R_k and Z_k are as simple as possible. Therefore, we set $R_1 = Z_1 = 0$, (2.7)

so that $\Theta_0'' - \beta^2 \Theta_0 = 0$ (2.8)

of which the general solution is

$$\Theta_0 = \Theta_{0,c} \cosh \beta z + \Theta_{0,s} \sinh \beta z \tag{2.9}$$

where $\Theta_{0,c}$ and $\Theta_{0,s}$ are constants.

$$\text{Now} \quad H_{r,0} = H_{\theta,0} = 0, \ H_{z,0} = e^{i(\beta r - \gamma \theta - \chi)} \tag{2.10}$$

are required to be satisfied to a certain degree of approximation. Accordingly, in equations (2.4), we shall require that

$$\begin{aligned} &[i\gamma Z_1' + \Theta_0']_{z=0} = 0 \\ &[R_1' - i\beta Z_1]_{z=0} = 0 \\ &[i\beta \Theta_0]_{z=0} = 0 \end{aligned} \tag{2.11}$$

The first of these equations and (2.7) give

$$[\Theta_0']_{z=0} = 0, \text{ and } \therefore \Theta_{0,s} = 0 \tag{2.12}$$

The third equation gives

$$\Theta_{0,c} = -\frac{i}{\beta} \tag{2.13}$$

Therefore

$$\Theta_0 = -\frac{i}{\beta}\cosh\beta z \qquad (2.14)$$

Returning to equations (2.5) we shall equate to zero the coefficient of the second highest power in each, or

$$\begin{aligned} &-R_0'' + Z_1' + i\beta Z_0' = 0 \\ &-i\beta\Theta_0 - i\gamma Z_0' - \Theta_{-1}'' + \beta^2\Theta_{-1} + \beta\gamma R_0 = 0 \\ &2R_1' + i\beta R_0' - 3i\beta Z_1 + \beta^2 Z_0 = 0 \end{aligned} \qquad (2.15)$$

We shall set $R_0 = Z_0 = 0$ (2.16)

so that $\Theta_{-1}'' - \beta^2\Theta_{-1} = i\beta\Theta_0 = \cosh\beta z$ (2.17)

so that $\Theta_{-1} = \Theta_{-1,c}\cosh\beta z + \Theta_{-1,s}\sinh\beta z - \frac{z}{2\beta}\sinh\beta z$ (2.18)

We shall now require that the second terms of equation (2.5) be zero, in order to satisfy (2.10). Then

$$\begin{aligned} &\left[i\gamma Z_0 + \Theta_{-1}'\right]_{z=0} = 0 \\ &\left[R_0' - Z_1 + i\beta Z_0\right]_{z=0} = 0 \\ &\left[\Theta_0 + i\beta\Theta_{-1} + i\gamma R_0\right]_{z=0} = 0 \end{aligned} \qquad (2.19)$$

or $\left[\dot{\Theta}_{-1}\right]_{z=0} = 0$ and therefore $\Theta_{-1,s} = 0$ (2.20)

and $\left[\Theta_0 + i\beta\Theta_{-1}\right]_{z=0} = 0 = -\frac{i}{\beta} + i\beta\Theta_{-1,c}$, and therefore $_{-1,c} = \frac{1}{\beta^2}$ (2.21)

Therefore $\Theta_{-1} = \frac{1}{\beta^2}\cosh\beta z - \frac{z}{2\beta}\sinh\beta z$ (2.22)

We can continue in this manner as we go to higher and higher approximations. Or, on the other hand, we can stop wherever we wish and equate to zero, the functions R_k, Θ_k and Z_k (k) which have not been determined. The further we go, before setting the undetermined functions equal to zero, the better the approximation.

If, now, we require that the third term of each of the equations (2.5) be set equal to zero, we have

$$\left[\gamma^2 R_1 - R''_{-1} + i\beta\dot{Z}_{-1} + \beta\gamma\Theta_0 = 0\right.$$

$$-i\gamma Z_{-1} - \Theta''_{-2} + i\beta\Theta_{-1} + \beta^2\Theta_{-2} + i\gamma R_0 + \beta\gamma R_{-1} = 0 \qquad (2.23)$$

$$\dot{R}_0 + i\beta\dot{R}_{-1} - Z_1 - i\beta Z_0 + \beta^2 Z_{-1} - i\gamma\dot{\Theta}_0 + \gamma^2 Z_1 = 0$$

Making use of (2.7) and (2.16) and setting

$$R_{-1} = 0 \qquad (2.24)$$

these equations become

$$i\beta\dot{Z}_{-1} + \beta\gamma\Theta_0 = 0$$

$$-i\gamma Z_{-1} - \Theta''_{-2} + i\beta\Theta_{-1} + \beta^2\Theta_{-2} = 0 \qquad (2.25)$$

$$\beta^2 Z_{-1} - i\gamma\dot{\Theta}_0 = 0$$

The last one of these equations gives:

$$Z_{-1} = \frac{i\gamma}{\beta^2} H_0' = \frac{\gamma}{\beta^2} \sinh\beta z \tag{2.26}$$

If this is substituted in the first equation of (2.25), we find that it is satisfied. The second equation of this set becomes

$$H_{-2}'' - \beta^2 H_{-2} = i\beta H_{-1} - i\gamma Z_{-1}' = \frac{i}{\beta}(1-\gamma^2)\cosh\beta z - \frac{iz}{2}\sinh\beta z \tag{2.27}$$

$$\therefore H_{-2} = H_{-2,c}\cosh\beta z + H_{-2,s}\sinh\beta z + \frac{iz}{2\beta^2}(1-\gamma^2)\sinh\beta z - \frac{i}{8\beta}\left[z^2\cosh\beta z - \frac{z}{\beta}\sinh\beta z\right] \tag{2.28}$$

Equating to zero at $z = 0$, the third terms of equation (2.2), one finds:

$$\left[i\gamma Z_{-1} + H_{-2}'\right]_{z=0} = 0 \text{ and therefore } H_{-2,s} = 0 \tag{2.29}$$

$$\left[R_{-1}' - i\beta Z_{-1}\right]_{z=0} = 0 \text{ is satisfied}$$

$$\left[i\beta H_{-2} + i\gamma R_{-1}\right]_{z=0} = 0 \text{ and therefore } H_{-2,c} = 0 \tag{2.30}$$

We find therefore, that

$$H_{-2} = -\frac{i}{8\beta^2}(4\gamma^2 - 5)z\sinh\beta z - \frac{i}{8\beta}z^2\cosh\beta z \tag{2.31}$$

Continuing in the same way, if one equates to zero the fourth term of equations (2.5), one has:

$$-i\gamma \Theta_0 + i\beta Z'_{-2} - R_{-2} + \beta\gamma \Theta_{-1} + \gamma^2 R_0 - Z'_{-1} = 0$$

$$-i\gamma Z'_{-2} - \Theta'_{-3} + 3i\beta \Theta_{-1} + \beta^2 \Theta_{-3} + 2i\gamma R_{-1} + \beta\gamma R_{-2} = 0 \qquad (2.32)$$

$$i\beta R'_{-2} + i\beta Z_{-1} + \beta^2 Z_{-2} + \gamma^2 Z_0 - i\gamma \Theta'_{-1} = 0$$

Making use of previous results, and setting $R_{-2} = 0$, one finds

$$i\beta Z'_{-2} + \beta\gamma \Theta_{-1} - Z'_{-1} = 0$$

$$i\gamma Z'_{-2} - \Theta''_{-3} + 3i\beta \Theta_{-2} + \beta^2 \Theta_{-3} = 0 \qquad (2.33)$$

$$i\beta Z_{-1} + \beta^2 Z_{-2} - i\gamma \Theta'_{-1} = 0$$

The third equation of this set gives:

$$Z_{-2} = -\frac{i}{\beta} Z_{-1} + \frac{i\gamma}{\beta^2} \Theta'_{-1} = -\frac{i\gamma}{2\beta^2}\left[\frac{\sinh \beta z}{\beta} + z\cosh \beta z\right] \qquad (2.34)$$

The first equation of (2.33) is now found to check. From the second equation of this set, Θ_{-3} can be found if required.

We shall now tabulate the results, which have been obtained:

$$R_K = 0 \quad (K = 1, 0, -1, -2)$$

$$\Theta_0 = -\frac{i}{\beta}\cosh \beta z$$

$$\Theta_{-1} = \frac{1}{\beta^2}\left[\cosh \beta z - \frac{\beta z}{2}\sinh \beta z\right] \qquad (2.35)$$

$$\Theta_{-2} = \frac{i}{8\beta^2}\left[(5 - 4\gamma^2) z \sinh \beta z - \beta z^2 \cosh \beta z\right]$$

$$Z_{+1} = Z_0 = 0$$
$$Z_{-1} = \frac{\gamma}{\beta^2} \sinh\beta z \qquad (2.36)$$
$$Z_{-2} = \frac{-i\gamma}{2\beta^3} \left[\sinh\beta z + \beta z \cosh\beta z\right]$$

By setting $\beta = -ia, \gamma = \chi = 0$ with a real, we obtain the corresponding equations for the case $H_{z,0} = e^{ar}, H_{r,0} = H_{\theta,0} = 0$, namely

$$R_K = Z_K = 0 \; (K = 1, 0, -1, -2)$$
$$\Theta_0 = \frac{\cos az}{a} \qquad (2.37)$$
$$\Theta_{-1} = -\frac{1}{a^2}\left[\cos az + \frac{az \sin az}{2}\right]$$
$$\Theta_{-2} = \frac{1}{8a^3}\left[a^2 z^2 \cos az - 5az \sin az\right]$$

The above expressions represent successive solutions of certain differential equations, which are obtained when certain terms in the expressions (2.5) and certain of the functions appearing are both set equal to zero. Using these results, we shall obtain in the next section, a series of solutions of our problem corresponding to various degrees of approximation.

Section III: Solutions of the problem corresponding to various degrees of approximation.

The various solutions will be represented by the symbols $(Ap)_0$, $(Ap)_{1/2}$, $(Ap)_1$, $(Ap)_{3/2}$ etc., arranged in the order of increasing degree of approximation.

<u>Solution$(Ap)_0$</u>: $R_K = Z_K = 0 \; (K)$

$$\Theta_K = 0 \; (K \neq 0) \qquad (3.1)$$
$$\Theta_0 = -\frac{i}{\beta} \cosh\beta z$$

$$A_r = A_z = 0; \quad A_\theta = -\frac{i}{\beta} e^{i(\beta r - \gamma\theta - \chi)} \cosh \beta z \tag{3.2}$$

The closeness of this approximation may be found by substituting (3.1) in equations (2.2) with z = o, and into equations (2.3). We obtain:

$H_{r,0} = H_{\theta,0} = 0$ (exactly)

$H_{z,0} = e^{i(\beta r - \gamma\theta - \chi)}\left[1 - \frac{i}{\beta r}\right]$ as against its prescribed value $e^{i(\beta r - \gamma\theta - \chi)}$ (3.3)

$$\left[\nabla \times (\nabla \times \vec{A})\right]_r = e^{i(\beta r - \gamma\theta - \chi)}\left\{r[0] + [0] - \frac{i\gamma}{r}\cosh\beta z - \frac{\gamma}{\beta r^2}\cosh\beta z\right\}$$

$$\left[\nabla \times (\nabla \times \vec{A})\right]_\theta = e^{i(\beta r - \gamma\theta - \chi)}\left\{[0] - \frac{\cosh\beta r}{r} + \frac{1}{r^2}[0]\right\} \tag{3.4}$$

$$\left[\nabla \times (\nabla \times A)\right]_z = e^{i(\beta r - \gamma\theta - \chi)}\left\{r[0] + [0] - \frac{\gamma}{r}\sinh\beta z\right\}$$

This field satisfied both the conditions on the median plane, and is Maxwelian except for terms of the order $\frac{1}{|\beta|r}$ and higher. It therefore represents a very good approximation when $|\beta| r$ is large. If one leaves out of consideration the quantities $H_{z,o}$ and $\left[\nabla \times (\nabla \times \vec{A})\right]_\theta$ the approximation is even better.

For the special field

$$H_{z,0} = e^{ar}, \quad H_{r,0} = H_{\theta,0} = 0$$

the $(Ap)_o$ solution is $A_r = A_z = 0, \; A_\theta = \frac{e^{ar}}{a}\cos az$ (3.5)

Solution $(Ap)_{\frac{1}{2}}$: $R_K = Z_K = 0 \ (K)$, $Ⓗ_K = 0 \ (K \neq 0, -1)$

$$Ⓗ_0 = -\frac{i}{\beta}\cosh\beta z \qquad (3.6)$$

$$Ⓗ_{-1} = \frac{1}{\beta^2}\left[\cosh\beta z - \frac{\beta z}{2}\sinh\beta z\right]$$

$$A_r = A_z = 0$$

$$A_\theta = e^{i(\beta r - \gamma\theta - \chi)}\left\{-\frac{i}{\beta}\cosh\beta z + \frac{1}{\beta^2 r}\left[\cosh\beta z - \frac{\beta z}{2}\sinh\beta z\right]\right\} \qquad (3.7)$$

The closeness of the approximation may be found as in the case of $(Ap)_0$. Thus we find

$$H_{r,0} = H_{\theta,0} = 0 \qquad \text{(exactly)}$$

$$H_{z,0} = e^{i(\beta r - \gamma\theta - \chi)} \qquad \text{(exactly)} \qquad (3.8)$$

$$\left[\nabla \times (\nabla \times \vec{A})\right]_r = e^{i(\beta r - \gamma\theta - \chi)}\left\{r[0] + [0] - \frac{i\gamma}{r}\cosh\beta z - \frac{\gamma z}{2r^2}\sinh\beta z\right\}$$

$$\left[\nabla \times (\nabla \times \vec{A})\right]_\theta = e^{i(\beta r - \gamma\theta - \chi)}\left\{[0] + \frac{1}{r}[0] + \frac{1}{r^2}\left[\cosh\beta z\right]\right\} \qquad (3.9)$$

$$\left[\nabla \times (\nabla \times \vec{A})\right]_z = e^{i(\beta r - \gamma\theta - \chi)}\left\{r[0] + [0] - \frac{\gamma}{r}\sinh\beta z + \frac{i\gamma}{2\beta r^2}\left[-\sinh\beta z + \beta z\cosh\beta z\right]\right\}$$

The prescribed conditions on the median plane are satisfied exactly. This is Maxwelian except for terms of the order $\frac{\gamma}{|\beta|^2 r^2}$ or higher.

For the special field

$$H_{z,0} = e^{ar}; \quad H_{r,0} = H_{\theta,0} = 0$$

the $(Ap)_{\frac{1}{2}}$ solution is

$$A_r = A_z = 0$$

$$A_\theta = e^{ar}\left\{\frac{\cos az}{a} - \frac{1}{a^2 r}\left[\cos az + \frac{az\sin az}{2}\right]\right\}$$

Solution $(A_p)_1$; $R_k = O(k)$; $Z_k = 0 \; (k \neq -1)$

$$\begin{aligned} (H)_0 &= -\frac{i}{\beta} \cosh\beta z \\ (H)_{-1} &= \frac{1}{\beta^2}\left[\cosh\beta z - \frac{\beta z}{2} \sinh\beta z\right] \\ Z_{-1} &= \frac{\gamma}{\beta^2} \sinh\beta z \end{aligned} \tag{3.10}$$

$$\begin{aligned} A_r &= 0 \\ A_\theta &= e^{i(\beta r - \gamma\theta - \chi)}\left\{-\frac{i}{\beta}\cosh\beta z + \frac{1}{\beta^2 r}\left[\cosh\beta z - \frac{\beta z}{2}\sinh\beta z\right]\right\} \\ A_z &= \frac{\gamma}{\beta^2} e^{i(\beta r - \gamma\theta - \chi)} \sinh\beta z \end{aligned} \tag{3.11}$$

The closeness of the approximation:

$$\begin{aligned} &H_{r,0} = H_{\theta,0} = 0 \quad \text{(exactly)} \\ &H_{z,0} = e^{i(\beta r - \gamma\theta - \chi)} \quad \text{(exactly)} \end{aligned} \tag{3.12}$$

$$\begin{aligned} \left[\nabla\times(\nabla\times\vec{A})\right]_r &= e^{i(\beta r - \gamma\theta - \chi)}\left\{r[0] + [0] + \frac{1}{r}[0] - \frac{\gamma}{\beta r^2}\left[\cosh\beta z + \frac{\beta z}{2}\sinh\beta z\right]\right\} \\ \left[\nabla\times(\nabla\times\vec{A})\right]_\theta &= e^{i(\beta r - \gamma\theta - \chi)}\left\{[0] + \frac{1}{r}[0] - \frac{i}{\beta r^2}\left[(\gamma^2-1)\cosh\beta z + \frac{\beta z}{2}\sinh\beta z\right]\right\} \\ \left[\nabla\times(\nabla\times\vec{A})\right]_z &= e^{i(\beta r - \gamma\theta - \chi)}\left\{r[0] + [0] + \frac{1}{r}[0] + \frac{i\gamma}{2r^2}\left[\frac{\sinh\beta z}{\beta} + \frac{z\cosh\beta z}{2}\right]\right\} \end{aligned} \tag{3.13}$$

The solution takes its prescribed values on the median plane exactly and is Maxwelian except for terms of the order $\frac{\gamma^2}{|\beta^2| r^2}$ and higher.

For the special field

$$H_{z,o} = e^{ar};\; H_{r,o} = H_{\theta,o} = 0$$

the $(Ap)_1$ solution is

$$A_r = A_z = 0$$

$$A_\theta = e^{ar}\left\{\frac{\cos az}{a} - \frac{1}{a^2 r}\left[\cos az + \frac{az\sin az}{2}\right] + \frac{1}{8a^3 r^2}\left[a^2 z^2 \cos az - 5az\sin az\right]\right\}$$

Solution $(Ap)_{3/2}$: $R_K = 0$ (K) $Z_K = 0$ $(K \neq -1)$, $Ⓗ_K = 0$ $(K \neq 0, -1)$

$$Ⓗ_0 = -\frac{i}{\beta}\cosh\beta z$$

$$Ⓗ_{-1} = \frac{1}{\beta^2}\left[\cosh\beta z - \frac{\beta z}{2}\sinh\beta z\right] \qquad (3.14)$$

$$Ⓗ_{-2} = -\frac{1}{8\beta^2}\left\{(4\gamma^2 - 5) z \sinh\beta z - \beta z^2 \cosh\beta z\right\}$$

$$Z_{-1} = \frac{\gamma}{\beta^2}\sinh\beta z$$

The closeness of the approximation:

$$H_{r,o} = H_{z,o} = 0 \quad \text{(exactly)}$$

$$H_{z,o} = e^{i(\beta r - \gamma\theta - \chi)} \quad \text{(exaclty)} \qquad (3.15)$$

$$\left[\nabla\times(\nabla\times\vec{A})\right]_r = e^{i(\beta r-\gamma\theta-\chi)}\left\{r[0] + [0] + \frac{1}{r}[0] - \frac{\gamma}{\beta r^2}\left[\cosh\beta z + \frac{\beta z}{2}\sinh\beta z\right]\right\}$$

$$\left[\nabla\times(\nabla\times\vec{A})\right]_\theta = e^{i(\beta r-\gamma\theta-\chi)}\left\{[0] + \frac{1}{r}[0] + \frac{1}{r^2}[0] + \frac{3\gamma}{2\beta^2 r^3}\left[\sinh\beta z + \beta z\cosh\beta z\right]\right\} \qquad (3.16)$$

$$\left[\nabla+(\nabla\times\vec{A})\right]_\theta = e^{i(\beta r-\gamma\theta-\chi)}\left\{r[0] + [0] + \frac{1}{r}[0] + \frac{i\gamma}{2r^2}\left[\frac{\sinh\beta z}{\beta} + z\,\frac{\cosh\beta z}{2}\right]\right\}$$

This solution takes its prescribed values on the median plane exactly and is Maxwelian except for terms of the order $\frac{\gamma}{|\beta|^3 r^3}$ and higher.

For the special field: $H_{z,o} = e^{ar}$; $H_{r,o} = H_{\theta,o} = 0$

the $(A_p)_{3/2}$ solution is: $A_r = A_z = 0$

$$A_\theta = e^{ar}\left\{\frac{\cos az}{z} - \frac{1}{a^2 r}\left[\cos az + \frac{az}{2}\sin az\right] + \frac{1}{8a^3r^2}\left[a^2r^2\cos az - 5az\sin az\right]\right\}$$

Section IV. The field $H_{z,o} = Im\left[e^{ar+i(\beta_1 r - \gamma\theta - \chi)}\right] = e^{ar}\sin(\beta_1 r - \gamma\theta - \chi)$.

In order to find the pure imaginary part of certain expressions containing $e^{i(\beta r - \gamma\theta - \chi)}$ where $\beta = \beta_1 - ia$, and where a, β_1, γ, χ are real, the following definitions and relations are useful.

Definitions: $i\beta = a + i\beta_1 = Ke^{i\psi}$

$$\Omega_n = \beta_1 r - \gamma\theta - \chi - n\psi \qquad (4.1)$$

$$(\Omega_n c) = \frac{e^{i\Omega_n}}{K^n}\cosh\beta z = (\Omega_n cr) + i(\Omega_n ci)$$

$$(\Omega_n s) = \frac{e^{i\Omega_n}}{K^n}\sinh\beta z = (\Omega_n sr) + i(\Omega_n si) \qquad (4.2)$$

where $(\Omega_n cr), (\Omega_n ci), (\Omega_n sr), (\Omega_n si)$ are real, and are given by:

$$K^n(\Omega_n cr) = \cos\Omega_n \cos az\cosh\beta_1 z + \sin\Omega_n \sin az\sinh\beta_1 z$$

$$K^n(\Omega_n ci) = -\cos\Omega_n \sin az\sinh\beta_1 z + \sin\Omega_n \cos az\cosh\beta_1 z \qquad (4.3)$$

$$K^n(\Omega_n sr) = \cos\Omega_n \cos az\sinh\beta_1 z + \sin\Omega_n \sin az\cosh\beta_1 z$$

$$K^n(\Omega_n si) = -\cos\Omega_n \sin az\cosh\beta_1 z + \sin\Omega_n \cos az\sinh\beta_1 z$$

The following relations are also needed:

$$e^{i(\beta r-\gamma\theta-\chi)}\frac{\cosh\beta z}{\beta} = e^{ar}\left\{-(\Omega_1 ci) + i(\Omega_1 cr)\right\}$$

$$e^{i(\beta r-\gamma\theta-\chi)}\frac{\cosh\beta z}{\beta^2} = -e^{ar}\left\{(\Omega_2 cr) + i(\Omega_2 ci)\right\}$$

$$e^{i(\beta r-\gamma\theta-\chi)}\frac{\sinh\beta z}{\beta} = e^{ar}\left\{-(\Omega_1 si) + i(\Omega_1 sr)\right\} \tag{4.4}$$

$$e^{i(\beta r-\gamma\theta-\chi)}\frac{\sinh\beta z}{\beta^2} = -e^{ar}\left\{(\Omega_2 sr) + i(\Omega_2 si)\right\}$$

$$e^{i(\beta r-\gamma\theta-\chi)}\frac{\sinh\beta z}{\beta^3} = e^{ar}\left\{(\Omega_3 si) - i(\Omega_3 sr)\right\}$$

The following expressions for partial derivatives are useful:

$$\frac{\partial}{\partial z}(\Omega_n c) = -i(\Omega_{n-1}, s) \qquad \frac{\partial}{\partial r}\left[e^{ar}(\Omega_n c)\right] = e^{ar}(\Omega_{n-1}, c)$$

$$\frac{\partial}{\partial z}(\Omega_n s) = -i(\Omega_{n-1}, c) \qquad \frac{\partial}{\partial r}\left[e^{ar}(\Omega_n s)\right] = e^{ar}(\Omega_{n-1}, s) \tag{4.5}$$

Equating real and pure imaginary parts in these expressions, one finds:

$$\frac{\partial}{\partial z}(\Omega_n cr) = (\Omega_{n-1} si) \qquad \frac{\partial}{\partial r}\left[e^{ar}(\Omega_n cr)\right] = e^{ar}(\Omega_{n-1} cr)$$

$$\frac{\partial}{\partial z}(\Omega_n ci) = -(\Omega_{n-1} sr) \qquad \frac{\partial}{\partial r}\left[e^{ar}(\Omega_n ci)\right] = e^{ar}(\Omega_{n-1} ci)$$

$$\frac{\partial}{\partial z}(\Omega_n sr) = (\Omega_{n-1} ci) \qquad \frac{\partial}{\partial r}\left[e^{ar}(\Omega_n sr)\right] = e^{ar}(\Omega_{n-1} sr) \tag{4.6}$$

$$\frac{\partial}{\partial z}(\Omega_n si) = -(\Omega_{n-1} cr) \qquad \frac{\partial}{\partial r}\left[e^{ar}(\Omega_n si)\right] = e^{ar}(\Omega_{n-1} si)$$

These latter relations are not used in this paper, but they are particularly useful when the vector potentials are substituted in the differential equations of motion of a charged particle.

If we take the pure imaginary part of equations (2.35) and (2.36), we at once find the corresponding equations for the field $H_{z,0} = Im\left[e^{ar+i(\beta_1 r - \gamma\theta - \chi)}\right]$.

For this latter field, we obtain

$$R_k = 0 \ (k) \qquad Z_{+1} = Z_0 = 0$$

$$(H)_0 = e^{ar}(\Omega_1 ci) \qquad Z_{-1} = -\gamma e^{ar}(\Omega_{-2} si)$$

$$(H)_{-1} = -e^{ar}\left[(\Omega_2 ci) + \frac{(\Omega_1 cr)}{2}z\right] \qquad Z_{-2} = \frac{\gamma}{2}e^{ar}\left[-(\Omega_3 si) + z(\Omega_2 cr)\right]$$

$$(H)_{-2} = e^{ar}\left[\frac{4\gamma^2-5}{8}(\Omega_2 sr)z + \frac{1}{8}(\Omega_1 ci)z^2\right]$$

We can now write down at once the solutions $(Ap)_0$ $(Ap)_{\frac{1}{2}}$ etc., for this field, as follows:

Solution $(Ap)_0$: $A_r = A_z = 0$

$$A_\theta = +e^{ar}(\Omega_1 ci)$$

Solution $(Ap)_{\frac{1}{2}}$: $A_r = A_z = 0$

$$A_\theta = e^{ar}\left\{(\Omega_1 ci) - \frac{1}{r}\left[(\Omega_2 ci) + \frac{z}{2}(\Omega_1 sr)\right]\right\}$$

Solution $(Ap)_1$: $A_r = 0$

$$A_\theta = e^{ar}\left\{(\Omega_1 ci) - \frac{1}{r}\left[(\Omega_2 ci) + \frac{z}{2}(\Omega_1 sr)\right]\right\}$$

$$A_z = -\gamma\frac{e^{ar}}{r}(\Omega_2 si)$$

Solution $(A_P)_{3/2}$ $A_r = 0$

$$A_\theta = e^{ar}\Big\{(\Omega_1 ci) - \frac{1}{r}\Big[(\Omega_2 ci) + \frac{z}{2}(\Omega_1 sr)\Big] + \frac{1}{r^2}\Big[\frac{4\gamma^2-5}{8}(\Omega_2 sr)z + \frac{(\Omega_1 ci)}{8}z^2\Big]\Big\}$$

$$A_z = -\gamma\frac{e^{ar}}{r}(\Omega_2 si)$$

Section V. The vector potential for the Mark V, FFAG accelerator.

We are now able to write down solutions corresponding to the various approximations for the Mark V, FFAG accelerator, where

$$H_{r,0} = H_{\theta,0} = 0$$

$$H_{z,0} = H_0 e^{-k} I_m\left[ie^{ar} + f e^{ar + i(\beta_1 r - \gamma\theta - \chi)}\right]$$

(see equation 1.11). By using the values of a, β_1, γ and χ given by equations (1.13), we at once obtain the values of A corresponding to the proposed fields for this machine. See equations (1.1) and (1.2)

Solution $(A_P)_0$: $A_r = A_z = 0$

$$A_\theta = H_0 e^{ar-k}\Big\{\frac{\cos az}{a} + f(\Omega_1 ci)\Big\}$$

Solution $(A_P)_{\frac{1}{2}}$: $A_r = A_z = 0$

$$A_\theta = H_0 e^{ar-k}\Big\{\frac{\cos az}{a} - \frac{1}{a^2 r}\Big[\cos az + \frac{az\sin az}{2}\Big] + f\Big[(\Omega_1 ci) - \frac{1}{r}\Big((\Omega_2 ci) + \frac{z}{2}(\Omega_1 sr)\Big)\Big]\Big\}$$

Solution $(A_p)_1$: $A_r = 0$

$$A_\theta = H_0 e^{ar-k}\left\{\frac{\cos cz}{a} - \frac{1}{a^2 r}\left[\cos cz + \frac{az\sin az}{2}\right] + f\left[(\Omega_1 ci) - \frac{1}{r}\left((\Omega_2 ci) + \frac{z}{2}(\Omega_1 sr)\right)\right]\right\}$$

$$A_z = f\frac{H_0}{r} e^{ar-k}(\Omega_2 si)$$

Solution $(A_p)_{3/2}$: $A_r = 0$

$$A_\theta = H_0^{ar-k}\left\{\frac{\cos az}{a} - \frac{1}{a^2 r}\left[\cos az + \frac{az}{2}\sin az\right] + \frac{f}{8a^2 r^2}\left[a z^2 \cos az - 5z\sin az\right] + f\left((\Omega_1 ci) - \frac{1}{r}\left[(\Omega_2 ci) + \frac{z}{2}(\Omega_1 sr)\right] + \frac{1}{8r^2}\left[(4r^2 - 5)z(\Omega_2 sr)^z + (\Omega_1 ci)z^2\right]\right)\right\}$$

$$A_z = \frac{f}{r} H_0 e^{ar-k}(\Omega_2 si)$$

Which of the above solutions are used, should be decided in accordance with the conditions discussed in Section I. The better the approximation, the more the time required for the computation of each point on the orbit. Higher approximations may be found in the same way. The next one requires H_{-3} which can be found as a solution of the second of equation (2.33).