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Contributions to the Unified Theory of FFAG Fields

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I. Introduction

Terwilliger has developed a unified theory of all FFAG fields (unpublished) based on an expansion of the orbit equations about a reference circle, keeping only linear terms. In view of Laslett's discovery that second order terms contribute focussing effects comparable to the first order focussing, it seems desirable to try to develop the unified theory in terms of an expansion of the orbit equations about the actual equilibrium orbit. We therefore propose to assume a set of equilibrium orbits given in terms of suitable parameters, and to determine from these parameters the character of the associated radial and vertical betatron oscillations as well as the magnetic field pattern required.

II. Geometry of the Equilibrium Orbits.

We will assume that a set of closed equilibrium orbits lying in the median plane is given. The geometrical properties of each orbit, and the relations between orbits will be periodic in the azimuthal angle θ with period $2\pi/N$. Each orbit will be specified by its equivalent radius $\underline{r}$ defined by

$$L = 2\pi \underline{r} \tag{1}$$

where L is the length of the orbit. In general, $\underline{r}$ will be somewhat larger than the mean radius $\bar{r}$. We define an azimuthal parameter $\underline{\theta}$ by the equation

$$s = \underline{\theta}\,\underline{r} \tag{2}$$

where s is the distance measured along the orbit from some reference point. The reference points shall be so chosen that they lie along an orthogonal trajectory to the set of orbits. The parameter $\underline{\theta}$ will be equal to the azimuthal angle θ plus a small periodic function with period $2\pi/N$.

Each orbit will now be specified by a periodic parameter $\mu(\underline{\theta}, \underline{r})$ defined by

$$\rho(\underline{\theta}, \underline{r}) = \frac{\underline{r}}{\mu(\underline{\theta}, \underline{r})} \tag{3}$$

where ρ is the radius of curvature. Specification of $\mu(\underline{\theta}, \underline{r})$, together with the requirement that the center of the orbit lie at the origin in the median plane, completely determines the orbit $\underline{r}$. Choice of the parameter μ is restricted only by the requirement that it be periodic with mean value

$$\bar{\mu} = \frac{1}{2\pi}\int_0^{2\pi} \mu \, d\underline{\theta} = \frac{1}{2\pi}\int_0^{L} \frac{ds}{\rho} = 1 \tag{4}$$

We will need also a parameters η, ϵ relating the increment $d\underline{\theta}$ in $\underline{\theta}$ along an orthogonal trajectory and the perpendicular distance dx between two nearby orbits, to the increment $d\underline{r}$ in the parameter $\underline{r}$:

$$dx = \eta \, d\underline{r}$$
$$d\underline{\theta} = \epsilon \frac{d\underline{r}}{\underline{r}} \tag{5}$$

The area between the two orbits is

$$dA = \int_{s=0}^{L} dx\, ds = \underline{r}\, d\underline{r} \int_0^{2\pi} \eta \, d\underline{\theta} = 2\pi \bar{\eta}\, \underline{r}\, d\underline{r} \tag{6}$$

If we define a mean radius $\bar{r}$ by

$$A = \pi \bar{r}^2 \tag{7}$$

then we have from Eq. (6),

$$\bar{\eta} = \frac{\bar{r}\, d\bar{r}}{\underline{r}\, d\underline{r}} \doteq 1 \tag{8}$$

so that $\bar{\eta}$ is very nearly 1 unless the scallops in the orbit are very great.

It is shown in the appendix that

$$\epsilon = \int_0^{\underline{\theta}} (\mu\eta - 1)\,d\underline{\theta}\ , \quad \overline{\mu\eta} = 1 \tag{9}$$

and that η satisfies the differentio-integral equation

$$\frac{d\eta}{d\underline{\theta}} = \mu\int_0^{\underline{\theta}} (1-\mu\eta)\,d\underline{\theta} - \int_0^{\underline{\theta}} \underline{r}\,\mu_{\underline{r}}\,d\underline{\theta} + C \tag{10}$$

where the constant C is chosen so that the right member has zero mean value. Since the integrands on the right have periods $2\pi/N$, it is clear that the periodic part of η is of order $1/N^2$, relative to the periodic part of μ or $\partial\mu/\partial\underline{r}$ (whichever is greater). If for example, we set

$$\mu = 1 + a\cos(N\underline{\theta} - \phi(\underline{r})) \tag{11}$$

$$\eta = 1 + \delta + b\cos(N\underline{\theta} - \phi) + c\sin(N\underline{\theta} - \phi) \tag{12}$$

and substitute in Eqs (9) and (10), requiring that Eq. (10) be satisfied to terms in sin $N\underline{\theta}$, cos $N\underline{\theta}$, we find

$$b = \frac{a/N^2}{1 - \frac{1}{N^2} + \frac{a^2}{2N^2}} \doteq \frac{a}{N^2} \tag{13}$$

$$c = \frac{a/N^2}{1 - \frac{1}{N^2} + \frac{a}{N^2}}\,\underline{r}\frac{d\phi}{d\underline{r}} \doteq \frac{a}{N^2}\,\underline{r}\frac{d\phi}{d\underline{r}} \tag{14}$$

$$\delta = -\frac{1}{2}ab \doteq -\frac{a^2}{2N^2}$$

(15)

$$\epsilon \doteq \frac{a}{N}\left[\sin\phi + \sin(N\underline{\theta} - \phi)\right] + \frac{c}{N}\left[\cos\phi - \cos(N\underline{\theta} - \phi)\right] + \frac{ac}{4N}\left[\cos 2\phi - \cos 2(N\underline{\theta} - \phi)\right]$$

For N $\geq$ 10, the coefficients b and δ are negligible. For Mark I machines, C = 0, while for Mark V, $\underline{r}\ d\phi/d\underline{r}$ may be of order N^2, so that C is of the same order as a.

In machines in which all orbits are geometrically similar, μ, η and ϵ must depend only on $\underline{\theta}$ except for a possible phase shift, so that

$$\mu(\underline{\theta}, \underline{r}) = \mu(N\underline{\theta} - \phi(\underline{r})) \tag{16}$$

and in order for the same to be true of η, ϵ, by Eq. (10), we must require that

$$\underline{r}\frac{d\phi}{d\underline{r}}\mu'(N\underline{\theta} - \phi) = \frac{2\pi}{\lambda}\mu'(N\underline{\theta} - \phi) \tag{17}$$

where λ is a constant (radial distance between ridges in units of $\underline{r}$). Hence

$$\phi(\underline{r}) = \frac{2\pi}{\lambda}\ln\frac{\underline{r}}{\underline{r}_0} \tag{18}$$

Machines satisfying Eqs (16) and (18) will be referred to as machines which scale.

III. Betatron Oscillations.

If a particle of momentum p moves in an equilibrium orbit $\underline{r}$, then we have, by Eq. (3)

$$pc = eH\rho = \frac{eH\underline{r}}{\mu} \tag{19}$$

so that

$$H(\underline{r}, \theta) = \frac{pc}{e\underline{r}}\mu(\theta, \underline{r}) \tag{20}$$

The magnetic field is thus given in terms of the coordinates $\underline{r}$, $\underline{\theta}$.

If we differentiate Eq. (19) with respect to X, where X is measured perpendicular to the orbit, we have

$$H\frac{\partial\rho}{\partial x} + \rho\frac{\partial H}{\partial x} = \frac{c}{e}\frac{\partial p}{\partial x} \tag{21}$$

The field index is therefore

$$n = -\frac{\rho}{H}\frac{\partial H}{\partial x} = \frac{\partial \rho}{\partial x} - \frac{\rho}{p}\frac{\partial p}{\partial x} \tag{22}$$

$$= \frac{\partial \rho}{\partial x} - \rho\frac{\partial \ln p}{\partial x}$$

Making use of Eqs. (3), and (5), we find

$$n = -\frac{k\mu + \varkappa + \aleph}{\eta\mu^2} \tag{23}$$

where

$$k = \underline{r}\frac{d \ln p}{d\underline{r}} - 1 \tag{24}$$

$$\varkappa = \underline{r}\frac{\partial \mu}{\partial \underline{r}}, \quad \aleph = \epsilon\frac{\partial \mu}{\partial \underline{\theta}} \tag{25}$$

According to Courant and Snyder (EDC/HSS-1, p.4), the linearized equations for betatron oscillations about an equilibrium orbit are

$$\frac{d^2x}{ds^2} + \frac{1-n}{\rho^2}x = 0 \tag{26}$$

$$\frac{d^2z}{ds^2} + \frac{n}{\rho^2}z = 0 \tag{27}$$

which become ~~become known~~ by Eqs (2) and (3),

$$\frac{d^2x}{d\underline{\theta}^2} + \mu^2(1-n)x = 0 \tag{28}$$

$$\frac{d^2z}{d\underline{\theta}^2} + \mu^2 nz = 0 \tag{29}$$

The character of the betatron oscillations is therefore determined by the functions $\mu^2(\theta, r)$ and

$$\mu^2 n = -\frac{1}{\eta}(k\mu + K + \mathcal{H}) \tag{30}$$

If the orbits scale, then, by Eqs. (16) and (18), μ is a function only of $(N\theta-\phi)$:

$$K = -\frac{2\pi}{\lambda}\mu' \ , \quad \mu' = \frac{\partial\mu}{\partial(N\theta)} \tag{31}$$

Thus $\mu^2 n$ will be a function of $(N\theta-\phi)$ only, and the betatron oscillations will also scale provided k is constant, so that

$$p = p_0\left(\frac{r}{r_0}\right)^{k+1} \tag{32}$$

and

$$H = H_0\left(\frac{r}{r_0}\right)^k \mu\left(N\theta - \frac{2\pi}{\lambda}\ln\frac{r}{r_0}\right) \tag{33}$$

(For a Mark I, $\lambda = \infty$, $K = 0$).

IV. Approximate Treatment of Betatron Oscillations.

Since $(1-\mu\eta)$ has the period $2\pi/N$, zero mean, and is of order 1, the first term in Eq. (10) will contribute a periodic term in η of order μ/N^2, which is negligible for N $\gtrsim$ 10. The integrand in the second term in Eq. (10) may be of the order of several hundred in a Mark I or II machine in which the flutter factor changes appreciably in a small fraction of the radius, or in the Mark V (spiral ridge) machine proposed by D. W. Kerst. We therefore write Eq. (10), making use of Eq. (25), in the approximate form

$$\frac{d\eta}{d\theta} \doteq -\int^{\theta} K\, d\theta \tag{34}$$

where the constant of integration is to be chosen so that $d\eta/d\theta$ has zero mean. (Note that since $\overline{\mu} = 1$ for all r, K is necessarily periodic with zero mean.) We have, therefore,

$$\eta = \bar{\eta} - \kappa_2 \tag{35}$$

$$\kappa_2 = \int^{\theta}\!\!\int^{\theta} \kappa \, d\theta \, d\theta \tag{36}$$

the constants of integration being chosen so that the double integral is periodic with zero mean. If we write

$$\mu = 1 + \bar{\bar{\mu}}(r, \theta) \tag{37}$$

where $\bar{\bar{\mu}}$ is periodic with zero mean, then by Eq. (9),

$$\bar{\eta} = 1 + \overline{\kappa_{2}\bar{\bar{\mu}}} \tag{38}$$

The second term is negligible unless the flutter factor (amplitude of $\bar{\bar{\mu}}$) is changing rapidly with r.

We now expand $1/\eta$ in powers of κ_2 in Eq. (30):

$$\mu^2 n = -\frac{k\mu + \kappa + \nu}{\eta} = -\frac{k\mu + \kappa + \nu}{\bar{\eta}}\left[1 + \frac{\kappa_2}{\bar{\eta}} + \left(\frac{\kappa_2}{\bar{\eta}}\right)^2 + \left(\frac{\kappa_2}{\bar{\eta}}\right)^3 + \cdots\right] \tag{39}$$

If we separate the right side into constant terms and periodic terms of zero mean, we get

$$\begin{aligned}\mu^2 n = &-\left[\frac{k + \bar{\nu}}{\bar{\eta}} + \frac{\overline{\kappa\kappa_2}}{\bar{\eta}^2} + \frac{\overline{(k\bar{\bar{\mu}} + \nu)\kappa_2}}{\bar{\eta}^2} + \frac{k\overline{\kappa_2^2} + \overline{(k\bar{\bar{\mu}} + \kappa + \nu)\kappa_2^2}}{\bar{\eta}^3} + \cdots\right] \\ &-\left\{\frac{k\bar{\bar{\mu}} + \kappa + \bar{\bar{\nu}}}{\bar{\eta}} + \frac{\overline{\overline{\kappa\kappa_2}} + k\overline{\overline{\mu\kappa_2}} + \overline{\overline{\nu\kappa_2}}}{\bar{\eta}^2} + \cdots\right\}\end{aligned} \tag{40}$$

where, for any quantity a (θ), we define

$$\bar{\bar{a}} = a - \bar{a} \tag{41}$$

The second term in the square brackets can be rewritten by integrating by parts:

$$\overline{\kappa\kappa_2} = \frac{1}{2\pi}\int_0^{2\pi}\kappa\int^{\theta}\int^{\theta}\kappa\,d\underline{\theta}\,d\underline{\theta} = -\frac{1}{2\pi}\int_0^{2\pi}\left[\int^{\theta}\kappa\,d\underline{\theta}\right]^2 d\underline{\theta}$$
$$= -\overline{\kappa_1^2} \tag{42}$$

In all machines so far proposed κ_2 is not greater in magnitude than 1/10, so that we may neglect higher order terms in κ_2, and Eq (40) becomes, if we also set $\bar{\eta} = 1$,

$$\mu^2 n = -k - \bar{\lambda} + \overline{\kappa_1^2} - k\bar{\bar{\mu}} - \kappa - \bar{\bar{\lambda}} \tag{43}$$

~~Current~~ Correct to 1% in the constant terms, and 10% in the periodic terms.

Equations (28) and (29) to this approximation become

$$\frac{d^2x}{d\underline{\theta}^2} + \left[1 + k + \bar{\lambda} - \overline{\kappa_1^2} + \overline{\bar{\bar{\mu}}^2} + (k+2)\bar{\bar{\mu}} + \overline{\bar{\bar{\mu}}^2} + \kappa + \bar{\bar{\lambda}}\right]x = 0 \tag{44}$$

$$\frac{d^2z}{d\underline{\theta}^2} + \left[-k - \bar{\lambda} + \overline{\kappa_1^2} - k\bar{\bar{\mu}} - \kappa - \bar{\bar{\lambda}}\right]z = 0 \tag{45}$$

The corresponding smooth approximation equations are (KRS(MURA)-4).

$$\frac{d^2X}{d\underline{\theta}^2} + \left[1 + k + \bar{\lambda} + \overline{\bar{\bar{\mu}}^2} + k^2\overline{\mu_1^2} + 2k\overline{\mu_1\kappa_1}\right]X = 0$$
$$\frac{d^2Z}{d\underline{\theta}^2} + \left[-k - \bar{\lambda} + 2\overline{\kappa_1^2} + k^2\overline{\mu_1^2} + 2k\overline{\mu_1\kappa_1}\right]Z = 0 \tag{46}$$

where we have neglected terms of order $(1 + k)/N^2$, (The last two terms in brackets in both equations are of order k^2/N^2 and are also small unless k is of order N^2.)

The subscript 1 indicates the integral of the periodic part of a function:

$$a_1(\underline{\theta}) = \int^{\theta}\bar{\bar{a}}\,d\underline{\theta} \tag{47}$$

the constant of integration to be chosen so that a_1 has zero mean. In general, a_1 is of order a/N.

We note that the term in $\overline{\kappa_1^2}$ drops out of the radial focussing equation and appears doubled in the vertical focussing equation. This is the effect discovered by Laslett for the Mark V machine in second approximation. The present treatment shows that (as long as κ_2^2 is negligible) the Laslett effect is to be expected in any machine in which the main focussing effects come from the terms in κ. The present derivation involves no approximation so far as the equilibrium orbit is concerned. It is curious to note that according to the present derivation, the Laslett effect arises out of a coincidence in phase between the term κ_1 arising from differences in amplitude and phase of the scallops in adjacent orbits, and the term κ_2 in η arising from the variation in perpendicular distance between orbits as a function of $\underline{\theta}$. The effect is therefore due to the fact that where κ is positive, the orbits are farther apart, and where κ is negative, they are closer together, so that the positive gradients are smaller and the negative gradients larger. The alternating gradient term κ is still present in the radial equation (44) and is in fact larger in magnitude than the mean term $-\overline{\kappa_1^2}$ by a factor of the order of N^2. However the focussing effect of an AG term is less by a factor $1/N^2$ than that of a constant term, hence the cancellation in the smooth equation (46).

The number of betatron wavelengths per revolution given by Eqs. (46) and (47) for radial and vertical motion are given by

$$\nu_x^2 = 1 + k + \bar{\nu} + \overline{\bar{\bar{\mu}}^2} + k^2\overline{\mu_1^2} + 2k\overline{\mu_1\kappa_1} \qquad (48)$$

$$\nu_z^2 = -k - \overline{\varkappa} + 2\overline{\varkappa_1^2} + k^2\overline{\mu_1^2} + 2k\overline{\mu_1\varkappa_1} \tag{49}$$

In Mark I and Mark II machines, $\varkappa$ is zero or small, and the terms in $\overline{\mu_1^2}$ are the dominant AG focussing terms. We can then solve Eqs. (49) and (50) for k, $\overline{\mu_1^2}$ as follows:

$$2k = \nu_x^2 - \nu_z^2 - 1 + 2\overline{\varkappa_1^2} - 2\overline{\varkappa} - \overline{\overline{\mu}}^2 \tag{50}$$

$$2k^2\overline{\mu_1^2} = \nu_x^2 + \nu_z^2 - 1 - 2\overline{\varkappa_1^2} - 4k\overline{\mu_1\varkappa_1} - \overline{\overline{\mu}}^2 \tag{51}$$

It is clear from Eq. (50) that if ν_x, ν_z are to be independent of r, k must be very nearly so, since only the small terms can depend on r. It is also clear from Eq. (51) that $\overline{\overline{\mu}}$ must be of the order $N\nu_x/k\sqrt{2}$; k can not be larger than $\nu_x^2/2$, if $\nu_z \ll \nu_x$; and since N must be at least 4ν in order for the smooth approximation to be valid, we have $\overline{\overline{\mu}} \sim 6$. An accurate treatment of Mark I in a typical case yields flutter amplitudes of 4 to 6 in field. In the Mark V machine, $\varkappa$ is of order N^2 and is the dominant AG focussing term. In this case, we can write

$$k = \nu_x^2 - 1 - 2k\overline{\mu_1\varkappa_1} - k^2\overline{\mu_1^2} - \overline{\overline{\mu}}^2 - \overline{\varkappa} \tag{53}$$

$$2\overline{\varkappa_1^2} = \nu_x^2 + \nu_z^2 - 1 - 4k\overline{\mu_1\varkappa_1} - 2k^2\overline{\mu_1^2} - \overline{\overline{\mu}}^2 \tag{54}$$

Here again, k must be nearly independent of r, if ν_x is. If $\nu_z \ll \nu_x$, then $\varkappa$ must be of order $N\nu_x/\sqrt{2}$; k can not be larger than ν_x^2, and if we set $N = 4\nu_x$, we have $\varkappa \sim 3k$.

V. Approximate Magnetic Field Patterns.

Let us consider a scalloped equilibrium orbit given in terms of polar coordinates r, θ by

$$r = F[1 + a\,g(N\theta - \phi)] \tag{55}$$

where g (ξ) is a periodic function of ξ with period 2π, zero mean, and unit amplitude, and a and ϕ may be functions of $\underline{r}$. The factor a is of order $1/N^2$. We then have

$$ds = \sqrt{dr^2 + r^2 d\theta^2} = F d\theta \sqrt{1 + 2ag + N^2 a^2 g'^2 + a^2 g^2} \tag{56}$$

$$ds \doteq F d\theta \left[1 + ag + \tfrac{1}{2} N^2 a^2 g'^2\right] \tag{57}$$

to within .01% if N $\gtrsim$ 10. The bracketed factor is 1 to within 1%. The length of the orbit is

$$L = \int_0^L ds = F \int_0^{2\pi} \left[1 + ag + \tfrac{1}{2} N^2 a^2 g'^2\right] d\theta \tag{58}$$

$$= 2\pi F \left[1 + \tfrac{1}{2} N^2 a^2 \overline{g'^2}\right]$$

so that, by Eq. (1)

$$\underline{r} = F\left[1 + \tfrac{1}{2} N^2 a^2 \overline{g'^2}\right] \tag{59}$$

where the bracket is 1 to within 1%. By eq. (2), we have

$$d\underline{\theta} = d\theta \left[\frac{1 + ag + \frac{1}{2} N^2 a^2 g'^2}{1 + \frac{1}{2} N^2 a^2 \overline{g'^2}}\right]$$

$$\doteq d\theta \tag{60}$$

to within 1%, so that

$$\theta = \underline{\theta} + \theta_0(\underline{r}) \tag{61}$$

where $\theta_0(\underline{r})$ is the equation of an orthogonal trajectory to the orbits. For a Mark I machine, a radius through the center of either a focussing or defocussing sector (or in general through a point where g has a maximum or minium) will be an orthogonal trajectory, and θ_0 will be constant. For a Mark V, the orthogonal trajectory will oscillate very slightly about a radius, being slightly convex to the left in a region of high field and to the right in a region of low field if the ridges spiral outward toward the right. It is probably a good approximation in any case to set $\theta = \underline{\theta}$. Eqs. (61) and (55) give the transformation from coordinates $\underline{r}, \underline{\theta}$ to r, θ.

The radius of curvature is given by

$$\frac{1}{\rho} = \frac{d\psi}{ds} \tag{62}$$

where, for polar coordinates,

$$d\psi = \left[1 - \frac{d}{d\theta}\left(\frac{1}{r}\frac{dr}{d\theta}\right)\right] d\theta \tag{63}$$

and

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \doteq r\,d\theta \tag{64}$$

within 1%, so that

$$\frac{1}{\rho} = \frac{1}{r}\left[1 - \frac{d}{d\theta}\left(\frac{1}{r}\frac{dr}{d\theta}\right)\right] \tag{65}$$

If we substitute from Eq. (55), we have, neglecting terms of order $1/N^2$,

$$\mu = \frac{\underline{r}}{\rho} = 1 - N^2 a g''(N\theta - \phi) \tag{66}$$

Since the flutter factor $N^2 a$ is ordinarily between $\frac{1}{4}$ and 4, we see that a is of order $1/N^2$. The magnetic field can now be computed from Eq. (20) along the orbit, and we can substitute

$$\underline{r} = r\left[1 + a g(N\theta - \phi)\right]^{-1} \tag{67}$$

to get H in terms of r, θ. If k is constant, the magnetic field is

$$H = H_0\left(\frac{r}{r_0}\right)^k \left[1 + a g(N\theta - \phi)\right]^{-k}\left[1 - N^2 a g''(N\theta - \phi)\right] \tag{68}$$

$$\doteq H_0\left(\frac{r}{r_0}\right)^k \left[1 - kag - N^2 a g'' + \frac{k(k-1)}{2} a^2 g^2 + \cdots\right] \tag{69}$$

Since $k \lesssim N^2/4$, the higher order terms will be small, though not always negligible. For a sinusoidal flutter, $g'' = -g$, and Eq. (69) becomes

$$H \doteq H_0 \left(\frac{r}{r_0}\right)^k \left[1 + (N^2 - k) a g + \frac{k(k-1)}{2} a^2 g^2 + \cdots\right] \tag{70}$$

so that the required flutter along a circle is somewhat less than along the orbit.

Appendix. Derivation of formulas for η, ϵ.

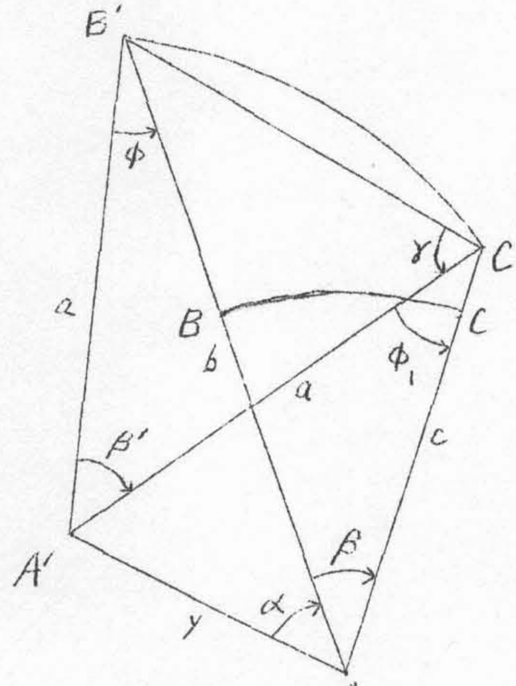

Fig. 1

Consider the diagram, Fig. 1, in which BC is an arc of the equilibrium orbit $\underline{r}$ having center of curvature at A, and B'C' is an arc of the equilibrium orbit $\underline{r} + \delta\underline{r}$ having center of curvature at A'. (The labels on the diagram Fig. 1, are for purposes of this discussion only and have no relation to the meanings of the same symbols in general FFAG theory.) Primed symbols refer to the orbit $\underline{r} + \delta\underline{r}$. We have, in terms of the orbit parameters,

$$\beta = \mu\, d\underline{\theta} \tag{1}$$

$$\beta' = \mu' d\underline{\theta}' \tag{2}$$

$$\overline{AB} = \overline{AC} = \underline{r}/\mu \tag{3}$$

$$\overline{A'B'} = \overline{A'C'} = a = \underline{r}'/\mu' \tag{4}$$

$$b = \overline{AB'} = \underline{r}/\mu + \eta\, \delta \underline{r} \tag{5}$$

$$c = \overline{AC'} = \underline{r}/\mu + (\eta + \eta_{\underline{\theta}}\, d\underline{\theta})\,\delta \underline{r} \tag{6}$$

$$\phi_1 = \phi + \phi_{\underline{\theta}}\, d\underline{\theta} \tag{7}$$

Where the last three equations are true in the limit $\delta \underline{r} \rightarrow 0$, $d\underline{\theta} \rightarrow 0$. We will pass first to the limit $d\underline{\theta} \rightarrow 0$ and then to $\delta \underline{r} \rightarrow 0$, in order to obtain a differential equation for η.

We note first the following geometrical relationships.

$$2\gamma = \pi + \phi - \phi_1 - \beta \tag{8}$$

$$\beta' = \pi - 2\gamma = \beta + \phi_1 - \phi \tag{9}$$

$$y^2 = a^2 + b^2 - 2ab\cos\phi = (a-b)^2 + 4ab\sin^2\frac{\phi}{2} \tag{10}$$

$$\sin\alpha = \frac{a\sin\phi}{y} \tag{11}$$

$$\sin\phi_1 = \frac{y\sin(\alpha+\beta)}{a} \tag{12}$$

$$c/\sin(\gamma-\phi) = b/\sin(\gamma+\phi_1) \tag{13}$$

If in Eq. (12), we let $d\underline{\theta} \rightarrow 0$, we have, in view of Eqs. (1), (7),

$$\sin\phi + \phi_{\underline{\theta}}\, d\underline{\theta}\cos\phi = \frac{y}{a}\sin\alpha + \mu\, d\underline{\theta}\,\frac{y}{a}\cos\alpha \tag{14}$$

$$\begin{aligned}\phi_{\underline{\theta}} = \frac{\partial\phi}{\partial\underline{\theta}} &= \frac{\mu y}{a}\,\frac{\cos\alpha}{\cos\phi} \\ &= \frac{\mu}{a\cos\phi}\sqrt{y^2 - a^2\sin^2\phi}\end{aligned} \tag{15}$$

We now set

$$\underline{\theta}' = \underline{\theta} + \delta\underline{\theta} \tag{16}$$

$$\delta\underline{\theta} = \epsilon\,\frac{\delta \underline{r}}{\underline{r}} \tag{17}$$

so that, as $\delta \underline{r} \rightarrow 0$, we have, by Eqs. (4), (5), and (10), since ϕ is of order $\delta \underline{r}$,

$$a = \frac{\underline{r}'}{\mu'} \doteq \frac{\underline{r}}{\mu} + \frac{\mu - \underline{r}\mu_{\underline{r}} - \epsilon\mu_{\underline{\theta}}}{\mu^2}\,\delta \underline{r} \tag{18}$$

$$y^2 \doteq \frac{\left[\mu - \underline{r}\mu_{\underline{r}} - \epsilon\mu_{\underline{\theta}} - \mu^2\eta\right]^2}{\mu^4}(\delta \underline{r})^2 + \frac{\underline{r}^2}{\mu^2}\phi^2 \tag{19}$$

$$a^2 \sin^2\phi \doteq \frac{r^2}{\mu^2}\phi^2 \tag{20}$$

Equation (15) then becomes, to first order in $\delta \underline{r}$,

$$\frac{\partial\phi}{\partial\underline{\theta}} = \left[-\mu + \underline{r}\mu_{\underline{r}} + \epsilon\mu_{\underline{\theta}} + \mu^2\eta\right]\frac{\delta\underline{r}}{\underline{r}} \tag{21}$$

where the sign of the square root is determined by the fact that $\cos\alpha$ is positive for large values of η.

If we substitute Eqs (16) and (17) in Eq. (2), and let $d\underline{\theta} \rightarrow 0$, making use of Eqs. (1) and (9), we have

$$\begin{aligned} \phi_1 - \phi &= \mu' d\underline{\theta}' - \mu\, d\underline{\theta} \\ &= (\mu' - \mu) d\underline{\theta} + \mu'\epsilon_{\underline{\theta}}\frac{\delta\underline{r}}{\underline{r}} d\underline{\theta} \end{aligned} \tag{22}$$

We now let $\delta\underline{r} \rightarrow 0$, and use Eq. (21);

$$\left[-\mu + \underline{r}\mu_{\underline{r}} + \epsilon\mu_{\underline{\theta}} + \mu^2\eta\right]\frac{\delta\underline{r}}{\underline{r}} d\underline{\theta} = \left[\underline{r}\mu_{\underline{r}} + \epsilon\mu_{\underline{\theta}}\right]\frac{\delta\underline{r}}{\underline{r}} d\underline{\theta} + \mu\epsilon_{\underline{\theta}}\frac{\delta\underline{r}}{\underline{r}} d\underline{\theta},$$

$$\epsilon_{\underline{\theta}} = \frac{\partial\epsilon}{\partial\underline{\theta}} = \eta\mu - 1 \tag{23}$$

According to Eqs. (16) and (17), ϵ is a measure of the rate of change of the parameter $\underline{\theta}$ as we move out along an orthogonal trajectory to the system of orbits. If we assume that the points $\underline{\theta} = 0$ on all orbits lie along an orthogonal trajectory, we can integrate Eq. (23) to obtain.

$$\epsilon = \int_0^{\theta} (\eta\mu - 1)\, d\underline{\theta} \tag{24}$$

Since $\epsilon(\underline{r}, \underline{\theta})$ must be periodic in $\underline{\theta}$, we have the result

$$\overline{\eta\mu} = \frac{1}{2\pi}\int_0^{2\pi} \eta\mu\, d\underline{\theta} = 1 \tag{25}$$

We now substitute Eqs. (5) and (6) in (13) using (8):

$$\frac{r}{\mu} + \eta\,\delta\underline{r} + \eta_{\underline{\theta}}\, d\underline{\theta}\,\delta\underline{r} = \left(\frac{r}{\mu} + \eta\,\delta\underline{r}\right) \frac{\cos\left(\phi + \frac{\phi_1 - \phi + \beta}{2}\right)}{\cos\left(\phi + \frac{\phi_1 - \phi - \beta}{2}\right)} \tag{26}$$

A $d\underline{\theta} \rightarrow 0$, this becomes

$$\eta_{\underline{\theta}}\, d\theta\,\delta\underline{r} = -\left(\frac{r}{\mu} + \eta\,\delta\underline{r}\right)\mu\, d\underline{\theta}\,\tan\phi \tag{27}$$

We now let $\delta\underline{r} \rightarrow 0$:

$$\eta_{\underline{\theta}} = \frac{\partial\eta}{\partial\underline{\theta}} = -\frac{r\,\phi}{\delta\underline{r}} \tag{28}$$

Equation (2) can be rewritten with the help of Eq. (24):

$$\frac{\partial\phi}{\partial\underline{\theta}} = \left[\frac{\partial}{\partial\underline{\theta}}(\mu\epsilon) + r\mu_r\right]\frac{\delta r}{\underline{r}} \tag{29}$$

This can be integrated to yield

$$\phi = \left[\mu\epsilon + \int^{\theta} \underline{r}\,\mu_{\underline{r}}\, d\underline{\theta} + C\right]\frac{\delta r}{\underline{r}} \tag{30}$$

When the integration constant is clearly to be chosen so that

$$\bar{\phi} = \frac{1}{2\pi}\int_0^{2\pi} \phi\, d\underline{\theta} = 0 \tag{31}$$

The angle ϕ is of some geometrical interest in itself as a measure of the amount by which nearby orbits are out of parallel. We may substitute Eq. (30) in Eq. (28), to obtain a differential equation for η along any orbit $\underline{r}$:

$$\frac{\partial\eta}{\partial\underline{\theta}} = \mu\int_0^{\theta}(1 - \eta\mu)\, d\underline{\theta} - \int_0^{\theta} \underline{r}\,\mu_{\underline{r}}\, d\underline{\theta} + C \tag{32}$$

where C is to be so chosen that the right member has average value zero. When Eq. (32) is integrated, the integration constant must be chosen to satisfy Eq. (25), or, alternatively, Eq. (8) of Section II.