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A method for determining the magnetic field in a region in terms of its values on a plane surface.

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Abstract

In a previous report,* a method was presented for finding the magnetic field strength and its vector potential in a region in terms of the magnetic field strength on a plane surface in this region. This method was applied to find certain very simple solutions of this problem, from which the magnetic field and its vector potential for the Mark V FFAG machine could be obtained to a certain degree of approximation. In this report, methods will be developed to obtain this field to a much higher degree of approximation, using the expansions of the previous report.

We shall introduce cylindrical coordinates (r, θ, z) and shall suppose that the magnetic field strength $\vec{H}$ in the median plane is given by

$$H_z(r,\theta,o) = H_{z,o} = \sum_{\sigma=-\mu_{H_{z,o}}}^{\infty} H_{z,o}^{o,\sigma}\, r^{-\sigma} e^{i(\beta r-\gamma\theta)} \qquad (1)$$

$$H_{r,o} = H_{\theta,o} = o$$

where $H_{z,o}^{o,\sigma}$ (σ), β and γ are real or complex constants. It is required to find $\vec{H}(r,\theta,z)$ and $\vec{A}(r,\theta,z)$ where $\nabla\times\vec{H} = o$, $\nabla\cdot\vec{H} = o$, $\vec{H} = \nabla\times\vec{A}$ and $O = \nabla\cdot\vec{A}$ (2)

The notations and results of the previous report will be used here. Certain equations and theorems which were stated for the case of rectangular coordinates must be extended to the case of cylindrical coordinates. This will be left to the reader. For instance, it follows that $(A_{r,o}, A_{\theta,o}, A_{z,o})$ are only required to satisfy the equation

$$H_{z,o} = (\nabla\times\vec{H})_{z,z=o} = \frac{1}{r}\frac{\partial(rA_{\theta,o})}{\partial r} - \frac{1}{r}\frac{\partial A_{r_o}}{\partial\theta} \qquad (3)$$

but otherwise may be chosen arbitrarily. One could, for instance, make:

$$A_{z,o} = A_{r,o} = o \quad\text{and}\quad rA_{\theta,o} = \int dr\, r H_{z,o} \qquad (4)$$

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However, the following choice will be used here:

$$A_{z,o} = A_{\theta,o} = o, \quad A_{r,o} = \frac{-i}{\gamma} \sum_{\sigma=-\mu_{H_{i,o}}} H_{z,o}^{o,\sigma} \quad r^{-\sigma+1} \quad e^{i(\beta r-\gamma\theta)} \tag{5}$$

In the treatment of the general case considered here, (5) is much simpler to work with than (4). However, in the special application to the field of the Mark V FFAG machine, one is not essentially more complicated than the other. We shall need

$$\nabla_t \cdot \vec{A}_o = \frac{1}{r} \frac{\partial}{\partial r} (rA_{r,o}) + \frac{1}{r} \frac{\partial A_{\theta,o}}{\partial \theta} =$$

$$= \sum_{\sigma=-\mu_{H_{i,o}}}^{\infty} H_{z,o}^{o,\sigma} \left\{ \frac{-i}{\gamma} (2-\sigma)r^{-\sigma} + \frac{\beta}{\gamma} r^{-\sigma+1} \right\} \tag{6}$$

We shall define two operators:

$$C_\Delta = \sum_{m=o}^{\infty} (-1)^m \frac{Z^{2m}}{2m!} (\Delta_t)^m \tag{7}$$

$$S_\Delta = \sum_{m=o}^{\infty} (-1)^{m+1} \frac{Z^{2m+1}}{(2m+1)!} (\Delta_t)^m \tag{8}$$

and shall write down the expansions* which shall be required here for the special case, where $H_{r,o} = H_{\theta,o} = o$ and $A_{z,o} = o$.

$$\Phi = \sum_{m=o}^{\infty} (-1)^{m+1} (\Delta_t^{\ m} H_{z,o}) \frac{Z^{2m+1}}{(2m+1)!} = S_\Delta(H_{z,o}) \tag{9}$$

$$A_r = \sum_{m=o}^{\infty} (-1)^m (\Delta_t^{\ m} A_{r,o}) \frac{Z^{2m}}{2m!} = C_\Delta(A_{r,o}) \tag{10}$$

$$A_\theta = \sum_{m=o}^{\infty} (-1)^m (\Delta_t^{\ m} A_{\theta,o}) \frac{Z^{2m}}{2m!} = C_\Delta(A_{\theta,o}) \tag{11}$$

$$A_z = \sum_{m=o}^{\infty} (-1)^{m+1} (\Delta_t^{\ m} \cdot \vec{A}_o) \frac{Z^{2m+1}}{(2m+1)!} = S_\Delta(\nabla_t \cdot \vec{A}_o) \tag{12}$$

The expansions for H_r, H_θ, and H_z have not been given here, becuase these

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* Loc. Cit. pages 2 and 3.

functions can be obtained by differentiating ϕ . (5)

In order to solve the problem, it is only necessary to substitute (1) and (6) in (9), (10), (11), and (12) and evaluate. This will be made possible by the following developments. Introduce the functions:

$$G^{(o)} = \sum_{\sigma=\mu}^{\infty} G^{o,\sigma} r^{-\sigma} e^{i(\beta r-\gamma\theta)} \tag{13}$$

where μ is some integer positive or negative or zero.

and define $G^{(m)} = \nabla_t^{(m)} G^{(o)}$ (14)

Expand $G^{(m)}$ in the series:

$$G^{(m)} = \sum_{\sigma=-\mu}^{\infty} G^{m,\sigma} e^{i(\beta r-\gamma\theta)} \tag{15}$$

which defines $G^{m,\sigma}$ (m,σ). If one substitutes (15) in (14) and equates coefficients of $r^{-m} e^{i(\beta r-\gamma\theta)}$, one obtains the relations:

$$G^{m,\sigma} = o \qquad (\sigma < -\mu)$$

$$G^{m,\sigma} = -\beta^2 G^{m-1,\sigma} + i\beta(3-2\sigma)G^{m-1,\sigma-1} + \left[(\sigma-2)^2-\gamma^2\right] G^{m-1,\sigma-2} \tag{16}$$

$$(\sigma = -\mu,\ -\mu+1,\ -\mu+2,\ -\mu+3, - -$$

On letting σ take the values $-\mu$, $-\mu+1$, $-\mu+2$, etc., one obtains

$$G^{m,-\mu} = (-\beta^2)G^{m-1,-\mu} \tag{17}$$

$$G^{m,-\mu+1} = -\beta^2 G^{m-1,-\mu+1} + i\beta(1+2\mu)G^{m-1,-\mu} \tag{18}$$

$$G^{m,-\mu+2} = -\beta^2 G^{m-1,-\mu+2} + i\beta(2\mu-1)G^{m-1,-\mu+1} + (\mu^2-\gamma^2)G^{m-1,-\mu} \tag{19}$$

$$G^{m,-\mu+3} = -\beta^2 G^{m-1,-\mu+3} + i\beta(2\mu-3)G^{m-1,-\mu+2} + (\mu-1)^2-\gamma^2 \quad G^{m-1,-\mu+1} \tag{20}$$

Etc. - - - -

The first equation in the above sequence may be solved. This solution may then be substituted in the second, which may then be solved. This process may be continued indefinitely. The solutions of the first four equations are*:

$$G^{m,-\mu} = G^{o,-\mu}(-\beta^2)^m \tag{21}$$

$$G^{m,-\mu+1} = G^{o,-\mu+1}(-\beta^2)^m + G^{o,-\mu} i\beta(1+2\mu)m(-\beta^2)^{m-1} \tag{22}$$

$$G^{m,-\mu+2} = G^{o,-\mu+2}(-\beta^2)^m + G^{o,-\mu+1} i\beta(2\mu-1)m(-\beta^2)^{m-1} + G^{o,-\mu}\left\{\frac{\beta^2}{2}(1-4\mu^2)m(m-1)(-\beta^2)^{m-2} + (\mu^2-\gamma^2)m(-\beta^2)^{m-1}\right\} \tag{23}$$

$$\begin{aligned} G^{m,-\mu+3} = {} & G^{o,-\mu+3}(-\beta^2)^m + G^{o,-\mu+2} i\beta(2\mu-3)m(-\beta^2)^{m-1} \\ & + G^{o,-\mu+1}\left\{-\frac{\beta^2}{2}(2\mu-1)(2\mu-3)m(m-1)(-\beta^2)^{m-2} + (1-\mu)^2-\gamma^2\ m(-\beta^2)^{m-1}\right\} \\ & + G^{o,-\mu}\left\{\frac{i\beta^3}{6}(1-4\mu^2)(2\mu-3)m(m-1)(m-2)(-\beta^2)^{m-3} + \frac{i\beta}{2}(2\mu-3)(\mu^2-\gamma^2)\right. \\ & \left. m(m-1)(-\beta^2)^{m-2} + \frac{i\beta}{2}(1+2\mu)m(m-1)(-\beta^2)^{m-2}\right\} \end{aligned} \tag{24}$$

The above equations may be written in the form

$$G^{m,-\mu+n} = \sum_{l=o}^{m} T_l^m \; G^{o,-\mu+1} \qquad (m,n) \tag{25}$$

Where the T_l^m are functions only of β, γ, and μ_G.

If one substitutes (13) in (7) and (8) one finds:

$$C_\Delta(G^o) = \sum_{m=o}^{\infty} (-1)^m \frac{Z^{2m}}{2m!} \sum_{\sigma=-\mu_G} G^{m,\sigma} r^{-\sigma} e^{i(\beta r-\gamma\theta)} = \sum_{\sigma=-\mu_G}^{\infty} \frac{e^{i(\beta r-\gamma\theta)}}{r^\sigma} [CG]^\sigma \tag{26}$$

$$S_\Delta(G^o) = \sum_{\sigma=-\mu_G}^{\infty} \frac{e^{i(\beta r-\gamma\theta)}}{r^\sigma} [SG]^\sigma \tag{27}$$

Where

$$[CG]^\sigma = \sum_m (-1)^m \frac{Z^{2m}}{2m!} G^{m,\sigma} \quad (\sigma) \tag{28}$$

$$[SG]^\sigma = \sum_m (-1)^{m+1} \frac{Z^{2m+1}}{(2m+1)!} G^{m,\sigma} \quad (\sigma) \tag{29}$$

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* It should not be difficult to find the general expression for $G^{m,-\mu+n}$. It has not been attempted as the first two equations of this sequence are probably sufficient for the accelerator.

The $[CG]^{\sigma}$ and $[SG]^{\sigma}$ (σ) are infinite series, which can easily be summed. One makes use of the formulas:

$$\sum_{m=o}^{\infty} (\beta^2)^m \frac{Z^{2m}}{2m!} = \text{Cosh } \beta z \tag{30}$$

$$\sum_{m=o}^{\infty} m(m-1)- - -(m-m+1)(\beta^2)^{m-n} \frac{Z^{2m}}{2m!} = \frac{\partial^n}{\partial(\beta^2)^n} \cosh \beta z \quad (n=1,2,3- -) \tag{31}$$

$$\sum_{m=o}^{\infty} (\beta^2)^m \frac{Z^{2m+1}}{(2m+1)!} = \frac{\sinh \beta z}{\beta} \tag{32}$$

$$\sum_{m=o}^{\infty} m(m-1) - - (m-n+1)(\beta^2)^{m-n} \frac{Z^{2m+1}}{(2m+1)!} = \frac{\partial^n}{\partial(\beta^2)^n} \frac{\sinh \beta z}{\beta} \quad (n=1,2,- -) \tag{33}$$

when substituting (21), (22),(23), etc., in (28) and finds:

$$[CG]^{-\mu} = G^{o,-\mu} \cosh \beta z \tag{34}$$

$$[CG]^{-\mu+1} = G^{o,-\mu+1} \cosh \beta z - G^{o,-\mu} i\beta(1+2\mu) \frac{\partial}{\partial(\beta^2)} \left[\cosh \beta z\right] \tag{35}$$

$$[CG]^{-\mu+2} = G^{o,-\mu+2} \cosh \beta z - G^{o,-\mu+1} i\beta(2\mu-1) \frac{\partial}{\partial(\beta^2)} \left[\cosh \beta z\right]$$
$$+ G^{o,-\mu} \left\{ \frac{\beta^2}{2}(1-4\mu^2) \frac{\partial^2}{\partial(\beta^2)^2} \left[\cosh \beta z\right] - (\mu^2-\gamma^2) \frac{\partial}{\partial(\beta^2)} \cosh \beta z \right\} \tag{36}$$

$$[CG]^{-\mu+3} = G^{o,-\mu+3} \cosh \beta z - G^{o,-\mu+2}\, i\beta(2\mu-3) \frac{\partial}{\partial(\beta^2)} \left[\cosh \beta z\right]$$
$$+ G^{o,-\mu+1} \left\{ - \frac{\beta^2}{2}(2\mu-1)(2\mu-3) \frac{\partial^2}{\partial(\beta^2)^2} \left[\cosh\beta z\right] - \left[(1-\mu)^2-\gamma^2\right] \frac{\partial}{\partial(\beta^2)} \left[\cosh\beta z\right] \right\}$$
$$+ G^{o,-\mu} \left\{ - \frac{i\beta^3}{6}(1-4\mu^2)(2\mu-3) \frac{\partial^3}{\partial(\beta^2)^3} \left[\cosh\beta z\right] + \frac{i\beta}{2}(2\mu-3)(\mu^2-\gamma^2) \right.$$
$$\left. \frac{\partial^2}{\partial(\beta^2)^2} \left[\cosh\beta z\right] + \frac{i\beta}{2}(1+2\mu) \frac{\partial^2}{\partial(\beta^2)^2} \cosh\beta z \right\} \tag{37}$$

The corresponding $[SG]^\sigma$ are found by replacing cosh βz in the expressions for $[CG]^\sigma$ by $\frac{\sinh \beta z}{\beta}$. It might also be noted that

$$[CG]^\sigma = \frac{\partial}{\partial z}[SG]^\sigma \qquad (\sigma) \tag{38}$$

Substituting (28) and (29) in (9), (10), (11) and (12), one finds:

$$\Phi = S_\Delta(H_{z,o}) = \sum_{\sigma=-\mu_{H_{z,o}}}^{\infty} \frac{e^{i(\beta r-\gamma\theta)}}{r^\sigma}\left[SH_{z,o}\right]^\sigma \tag{39}$$

$$A_r = C_\Delta(A_{r,o}) = \sum_{\sigma=-\mu_{A_{r,o}}}^{\infty} \frac{e^{i(\beta r-\gamma\theta)}}{r^\sigma}\left[CA_{r,o}\right]^\sigma \tag{40}$$

$$A_\theta = C_\Delta(A_{\theta,o}) = \sum_{\sigma=-\mu_{A_{\theta,o}}}^{\infty} \frac{e^{i(\beta r-\gamma\theta)}}{r^\sigma}\left[CA_{\theta,o}\right]^\sigma \tag{41}$$

$$A_z = S_\Delta(\ _t\cdot A_o) = \sum_{\sigma=-\mu_{\nabla\cdot\vec{A}_o}}^{\infty} \frac{e^{i(\beta r-\gamma\theta)}}{r^\sigma}\left[S\nabla_t \cdot A_o\right]^\sigma \tag{42}$$

We are now ready to find explicit expressions for and A when H_o is given by (1). From (5) and (6), one finds:

$$(A_{r,o})^{o,\sigma} = -\frac{i}{\gamma} H_{z,o}^{o,\sigma+1} \qquad (\sigma) \tag{43}$$

$$(\nabla_t\cdot A_o)^{o,\sigma} = \frac{\beta}{\gamma} H_{z,o}^{o,\sigma+1} - \frac{i}{\gamma}(2-\sigma)H_{z,o}^{o,\sigma} \qquad (\sigma) \tag{44}$$

and

$$\left[CA_{r,o}\right]^\sigma = -\frac{i}{\gamma}\left[CH_{z,o}\right]^{\sigma+1} \qquad (\sigma) \tag{45}$$

$$\left[S\ _t\cdot\vec{A}_o\right]^\sigma = \frac{\beta}{\gamma}\left[SH_{z,o}\right]^{\sigma+1} - \frac{i}{\gamma}(2-\sigma)\left[SH_{z,o}\right]^\sigma \qquad (\sigma) \tag{46}$$

Making use of the equations (34)-(37), and remembering that the coefficients of $G^{o,\sigma}$ in these equations are functions of β, γ and μ only, we can at once write down the expressions for $\left[CH_{z,o}\right]^{\sigma}$ from those for $\left[CG\right]^{\sigma}$ simply by replacing the μ in (34)-(37) by $\mu_{H_{z,o}}$ and the G's by H's. In the same way one finds the $\left[SH_{z,o}\right]^{\sigma}$. When the $\left[CH_{z,o}\right]^{\sigma}$ and $\left[SH_{z,o}\right]^{\sigma}$ are substituted in (45) and (46), and these latter are substituted in (39)-(43), the problem is solved.

The first two terms of these series will now be written down. In these formula μ will be written in place of $\mu_{H_{z,o}}$.

$$\Phi = e^{i(\beta r-\gamma\theta)}\left\{ r^{\mu} H_{z,o}^{o,-\mu} \frac{\sinh\beta z}{\beta} + r^{\mu-1}\left[H_{z,o}^{o,-\mu+1} \frac{\sinh\beta z}{\beta} - H_{z,o}^{o,-\mu} i\beta(1+2\mu) \frac{\partial}{\partial(\beta^2)} \frac{\sinh\beta z}{\beta}\right] + \text{- -}\right\}$$

$$A_r = -\frac{i}{\gamma} e^{i(\beta r-\gamma\theta)}\left\{ r^{\mu+1} H_{z,o}^{o,-\mu} \cosh\beta z + r^{\mu}\left[H_{z,o}^{o,-\mu+1} \cosh\beta z - H_{z,o}^{o,-\mu} i\beta(1+2\mu) \frac{\partial}{\partial(\beta^2)} \cosh\beta z + \text{- - - -} \right] + \text{- - -} \right\}$$

$$A_\theta = o$$

$$A_z = e^{i(\beta r-\gamma\theta)}\left\{ r^{\mu+1} \frac{\beta}{\gamma} H_{z,o}^{o,-\mu} \frac{\sinh\beta z}{\beta} + r^{\mu}\left[\frac{\beta}{\gamma} H_{z,o}^{o,-\mu+1} \frac{\sinh\beta z}{\beta} + H_{z,o}^{o,-\mu}\left(- \frac{i\beta}{\gamma}(1+2\mu) \frac{\partial}{\partial(\beta^2)} \frac{\sinh\beta z}{\beta} - \frac{i}{\gamma}(1+\mu) \frac{\sinh\beta z}{\beta}\right] + \text{- - -} \right\}$$

Note: The direct application of these developments to the Mark V FFAG machine will be the subject of a subsequent report.