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A FIXED FIELD-ALTERNATING GRADIENT ACCELERATOR WITH SPIRALLY RIDGED POLES*

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The high momentum spread allowed in Symon's[1] ring magnet, where the alternating gradient is provided by field reversal in alternate magnet segments, allows acceleration from a low momentum to high momentum within a relatively small radial span, without change in the magnetic field excitation. The disadvantage of this design is that the resulting accelerator has a circumference about five times that of an equivalent accelerator of the same maximum field. The spirally ridged pole magnet described here is an attempt to retain the desirable features of Symon's design (D.C. magnet excitation, high momentum content, large injection aperture) and at the same time to reduce the circumference by eliminating the regions of reverse field.

This is accomplished by constructing a magnet pole contour of high positive n (high field at the outer radius) and superimposing on it spiral ridges to provide the alternate gradient focussing. Thus the field in the median plane might be described as:

$$H_Y = H_0\left(\frac{r}{r_0}\right)^k \left[1 + f \sin\left(\frac{r-r_0}{\lambda} - N\theta\right)\right] \qquad (1)$$

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1. MURA-KRS 6.

where $\lambda = 2\pi \lambda\!\!\!^{-}$ is the radial separation of ridges and N is the number of ridges passed per revolution.

The vertical focussing may be thought of as provided by the alternating gradient resulting from passage from one side to the other side of the spiral ridges, while radial focussing and momentum compaction are mainly provided by the exponential k governing the average field. The pitch of the spirals is made small so that particles in circular orbits cross them at small angles. A particle in a circular orbit then feels a sinusoidally varying gradient. The effect of this gradient on vertical focussing is then similar to that in a conventional A.G. design.

Two examples of how these righes might be used will be given. The first example (A) gives the case of a vertical field which is sharply peaked in space so the particle experiences an abruptly reversing gradient as it passes over the ridge in the field.

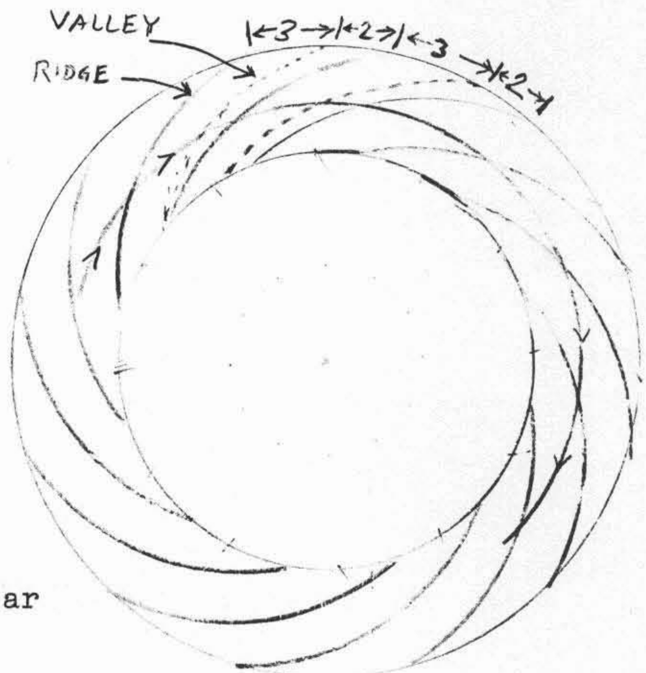
The second example (B) developes the approximate Mathieu differential equations in the more gently ridged sinusoidally varying field as in (1), and applies them to the case of converting a 466 Mev. synchrocyclotron to a fixed frequency cyclotron.

These analyses are very incomplete and preliminary, but they are being distributed immediately to the MURA technical group for information and assistance in considering these possibilities,

(A) Sharply ridged field:

We know that in a Mark Ib FFAG magnet with alternating gradients provided by reversing the fields in the radially defocussing sectors so that $\frac{\partial H_1}{\partial r} = -\frac{\partial H_2}{\partial r}$ we can have stability if $S_2/S_1 = 2/3$ is the ratio of arc lengths. The cos σ formula is written for arc lengths with $H\rho$ continuous from sector to sector for our cases, and it gives $\sigma_x = 5\pi/6$, $\sigma_Y = \pi/6$. x is radial.

Thus if a particle approaches a ridge top for 2 units of arc length in gradient $\frac{\partial H_2}{\partial r}$ which is negative and then crosses and travels for 3 units of arc length while it goes from ridge to valley with $\frac{\partial H_1}{\partial r} = -\frac{\partial H_2}{\partial r}$, then we should have focussing conditions similar to those in the Mark Ib FFAG magnet with reversed fields.

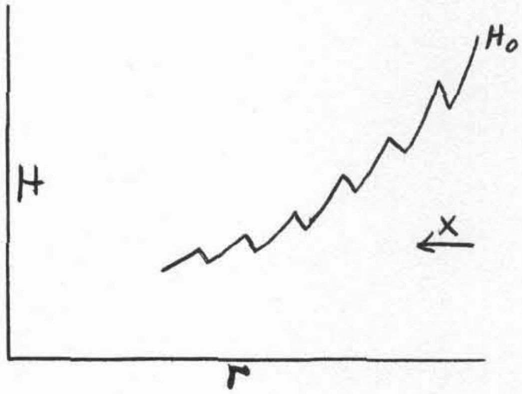


However, now we do not reverse the field, instead the particle follows a scalloped path through the ridged field. This type of magnet evolved from an attempt to arrange concentric rings of AG magnets with weak field magnets inside and strong field magnets outside so that the beam could be passed outward to the higher fields continuously as it is accellerating.

Our differential equation is:

$$\frac{d^2Y}{dS^2} \pm \frac{n}{\rho^2} Y = 0 \quad \text{OR} \quad \frac{d^2Y}{dS^2} \pm \frac{1}{H\rho}\left|\frac{\partial H}{\partial r}\right| Y = 0 \qquad (2)$$

where $H\rho$ is constant although H changes along the path. For calculating the field variation in a narrow ring magnet we take $\frac{1}{H}\left|\frac{\partial H}{\partial r}\right|$ constant. If a field can be shaped to give this, and if from the outside radius the field goes down to $H = H_0 e^{-\frac{mx}{R}}$ for 3 units ~~of~~ radial and then rises like $e^{+\frac{mx}{R}}$ for two radial units of x, we have a drop $H/H_o = e^{-\frac{m}{5R}\lambda}$ for each ridge of width λ so for a ratio of $H/H_o = 1/150$ for injection into a 20 Bev accelerator we have



$H_i/H_o = e^{-5}$ and $\frac{m}{5}\frac{\lambda}{R}M = 5$ where M is the number of ridges across the magnet. If we must choose $m = 500$, then $\frac{M\lambda}{R} = .05$. And since $R \cong 10^4$ cm, the radial aperture $M\lambda \cong 500.$ cm.

However, the amount of flux is

$$\Phi = 2\pi R_0 \int_0^{M\lambda} \overline{H_0}\, e^{-x/\ell}\, dx = 2\pi R_0 \overline{H_0}\, \ell$$

where $\overline{H_0}$ is the field averaged around the circumference at $R_o + \ell = \frac{5R}{m} = \frac{M\lambda}{5}$, a fifth of the radial aperture. So the effective width of the magnet is about 100.cm at flux density $\overline{H_0}$. Thus whereas Symon's Mark I magnet has several times the circumference of a unidirectional field magnet, the spirally ridged field has several times the radial width of an A.G. magnet of the type contemplated by Brookhaven and CERN. The leakage flux is not a big fraction of the total flux in this new magnet.

If the field is to change by a factor of 1.2 between successive peaks, then $1.2^{M} = 150$.

$M = 27$ ridges on the magnet cross section.

A field of this shape may require current carrying copper on its pole faces to provide the field ridges and to allow the gap at low field to be only about 10 times the gap at the high field radius. The factor 10 is the adiabatic damping of vertical oscillations expected during acceleration.

A 25 Bev accelerator might have the following parameters.

$M = 25$ ridges across pole

$N = 50$ sectors (ridges) around pole

$M\lambda = 5$ meters radial aperture

$R_0 = 100$ meters

$m = 500$

$H_0/H = 150$

The spirals would make an angle $\phi = \frac{\lambda N}{2\pi R_0} \cong 1^\circ$.

If higher values of m could be used, the extreme radial aperture of this type of machine could be decreased. The field would be difficult to design so that σ_x and σ_y remained constant throughout acceleration, and the field would have non-linearities. Much theoretical knowledge and computational experience with non-linearities would be needed to be confident of these fields, or working model tests, such as those contemplated on the Illinois 80 Mev betatron, would be needed.

(B) Sinusoidal ridges and the example of the cyclotron:

Using $x = r - r_0$ in equation (1) expand about r_0 as a unit circle

$$H_Y(x,\theta) = H_0(1+kx)\left[1 + f\frac{x}{\lambda}\cos N(\theta-\theta_0) - f\sin N(\theta-\theta_0)\right]$$

$$H_Y(x,\theta) = H_0\Big[1 - f\sin N(\theta-\theta_0) + \left\{k + \frac{f}{\lambda}\cos N(\theta-\theta_0) - kf\sin N(\theta-\theta_0)\right\}x + \left\{f\frac{k}{\lambda}\cos N(\theta-\theta_0)\right\}x^2\Big]$$

Let $k\lambda = \tan\delta$ since $k\lambda$ is smaller than 1 ordinarily.

$$H_Y(x,\theta) = H_0\Big[1 - f\sin N(\theta-\theta_0) + \left\{k + \frac{f}{\lambda}\sqrt{k^2\lambda^2+1}\cos N(\theta-\theta_0+\delta)\right\}x + \left\{f\frac{k}{\lambda}\cos N(\theta-\theta_0)\right\}x^2\Big] \quad (3)$$

Comparing with Powell's MAC JLP-1 where

$$H_Y = 1 + \frac{\partial\psi}{\partial Y} = \Big\{1 + \beta - mx - (a+\phi'')Y - \frac{1}{4}(6b+\beta'')x^2 - \frac{1}{2}(6c+\alpha'')xY + \frac{1}{4}(6b-\beta'')Y^2 + \frac{1}{2}(m''+4e)x^3 - \frac{1}{2}Dx^2Y + \frac{1}{4}(m''-4e)xY^2 + \frac{1}{6}(D+[a+\phi''])Y^3 + \ldots$$

We have

$$\beta = -f \sin N(\theta - \theta_0) \qquad (4)$$

$$-m = k + \frac{f}{\lambda}\sqrt{k^2\lambda^2+1}\ \cos N(\theta-\theta_0+\delta) \qquad (5)$$

$$\left\{f \frac{k}{\lambda} \cos N(\theta-\theta_0)\right\}x^2 + F(\theta)Y^2 \equiv \frac{6}{4}b(Y^2-x^2) - \frac{1}{4}\beta''(x^2+Y^2) \qquad (6)$$

Using (4) + β'' from (4) we find

$$b = -f\left[\frac{N^2}{6}\sin N(\theta-\theta_0) + \frac{4}{6}\frac{k}{\lambda}\cos N(\theta-\theta_0)\right]$$

and $$F(\theta) = -f\left[\frac{N^2}{2}\sin N(\theta-\theta_0) + \frac{k}{\lambda}\cos N(\theta-\theta_0)\right] \qquad (7)$$

so in (3) $H_Y(x,\theta)$

should have added to it

$$-f\left[\frac{N^2}{2}\sin N(\theta-\theta_0) + \frac{k}{\lambda}\cos N(\theta-\theta_0)\right]Y^2 \qquad (8)$$

There are no xy terms because $\alpha'' = 0 + C$ may be chosen zero. Thus we see the presence of a quadratic force in both x and y which is caused by the spiralling angle of the ridges.

In the linear approximation our equations of motion are (MAC JLP-1)

$$x'' + \left[1 + k + \frac{f}{\lambda}\sqrt{k^2\lambda^2+1}\ \cos N(\theta-\theta_0+\delta)\right]x = f \sin N(\theta-\theta_0) \qquad (9)$$

$$y'' - \left[k + \frac{f}{\lambda}\sqrt{k^2\lambda^2+1}\ \cos N(\theta-\theta_0+\delta)\right]Y = 0 \qquad (10)$$

These are Mathieu differential equations which will be treated at this point without the inhomogeneity in (9). If we use the form $\frac{d^2 Z}{d\tau^2} + \left\{ a + 16\, q \cos 2\tau \right\} Z = 0$ then (11)

the figure shows the stability limits with

$$a_x = \frac{4}{N^2}(1+k) \qquad \text{for x D.E.}$$

$$a_y = -\frac{4}{N^2} k \qquad \text{for the y D.E.} \tag{12}$$

$$q = \frac{f}{4N^2} \frac{\sqrt{k^2 \lambda^2 + 1}}{\lambda} \qquad \text{for both D.E.'s}$$

Taking the case of a 466 Mev synchrocyclotron which we would like to convert to a constant frequency machine to increase the current we have

$$\omega = \frac{eH}{m_0 c}\sqrt{1-\beta^2} = \frac{eH}{m_0 c}\sqrt{1-\rho^2\omega^2/c^2} \tag{13}$$

to get the radial dependence of field differentiate

$$\frac{\rho}{H}\frac{dH}{d\rho} = \frac{\rho^2\omega^2/c^2}{1-\rho^2\omega^2/c^2} = \frac{\beta^2}{1-\beta^2} = \frac{1-\left(\frac{m_0 c^2}{E_T}\right)^2}{\left(\frac{m_0 c^2}{E_T}\right)^2} = n \tag{14}$$

So n = 1.25 at the outer edge of the pole

try a_y = -.102

q = .07

Then $a_x = -a_y\left(\frac{1+k}{k}\right) = .183$

and by (12) we have

$N = \sqrt{\frac{4k}{-a_y}}$ = 7 ridges (sectors) around circumference.

$$f = 4qN^2 \frac{\lambda}{\sqrt{k^2\lambda^2+1}} \cong 13.8\,\lambda$$

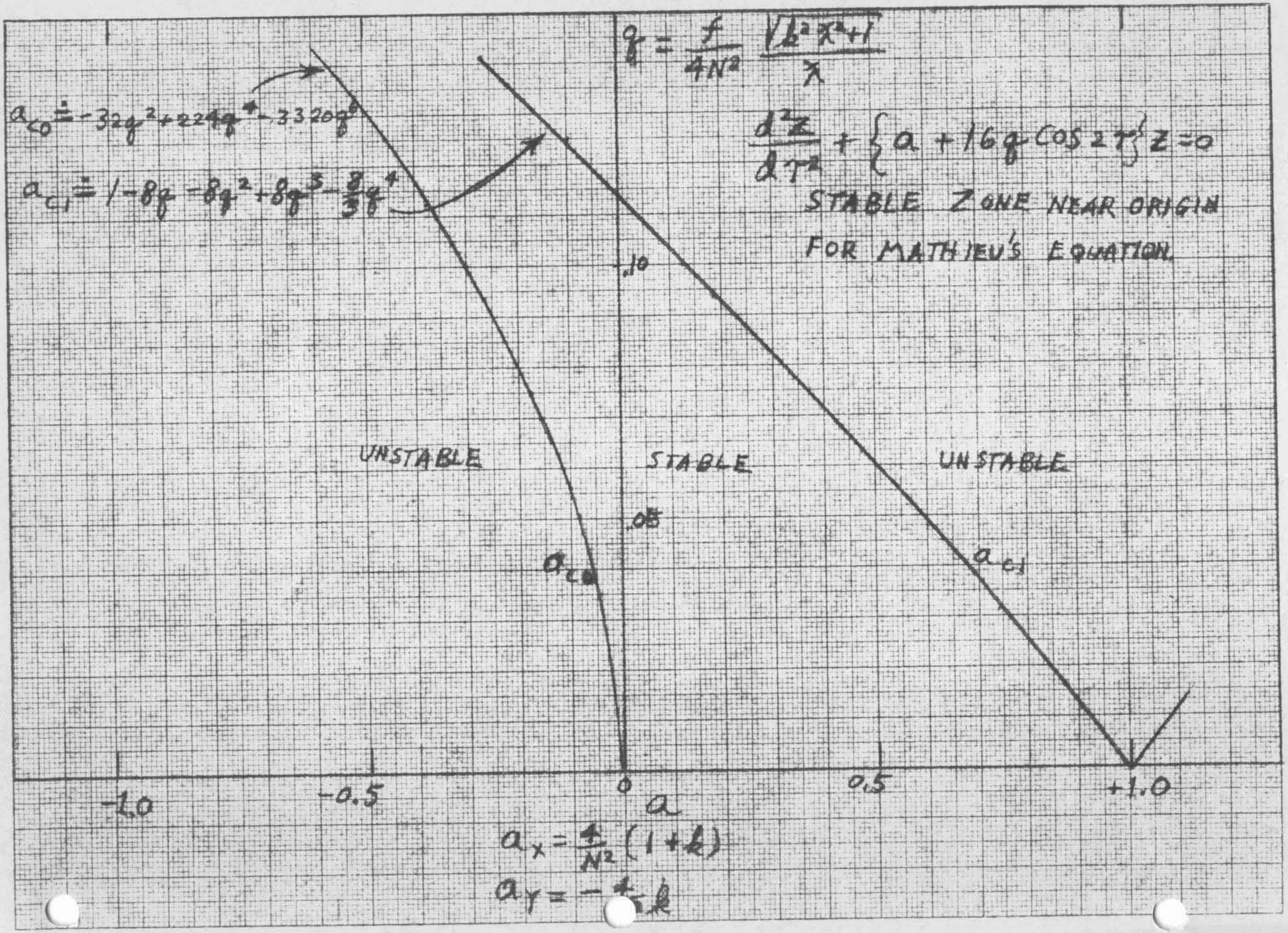
Take λ = .01

This gives $f = \pm$ 13.8% variation of the field above the average field and below the average field. The radial separation of the ridges is λ = .0628 of the outer radius since r_0 is a unit circle.

With k = 1.25 we would be on a radial half integral resonance, if the radial focussing resulted only from the $(r/r_0)^k$ shape of the field. The factors contributing to radial focussing are not yet determined. The inhomogeneous term on the right of equation (9) must be considered.

It may be possible to use negative momentum compaction with these poles and thus to eliminate the transition energy. The high energy orbit would be on the inside of the magnet.

Application of FFAG poles of this type to a betatron for purposes of intensity would require k very large to provide a hole for the central flux.


$q = \frac{f}{4N^2} \frac{\sqrt{k^2\lambda^2+1}}{\lambda}$
$\frac{d^2 z}{d\tau^2} + \{a + 16q \cos 2\tau\} z = 0$
STABLE ZONE NEAR ORIGIN
FOR MATHIEU'S EQUATION
$a_{c0} \doteq -32q^2 + 224q^4 - 3320q^6$
$a_{c1} \doteq 1 - 8q - 8q^2 + 8q^3 - \frac{8}{3}q^4$
UNSTABLE
STABLE
UNSTABLE
a_{c0}
a_{c1}
.10
.05
-1.0
-0.5
0
0.5
+1.0
a
$a_x = \frac{4}{N^2}(1+k)$
$a_y = -\frac{4}{N^2}k$