V. MICROWAVE ELECTRONICS*

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A. TRANSVERSE FREE WAVES ON ELECTRON BEAMS

The transverse waves that propagate on a one-dimensional electron beam in a finite magnetic field are being studied. The general dielectric tensor for an unbounded, relativistic electron beam is presented and a detailed discussion of the waves along the magnetic field is given. The results are useful in understanding the instabilities that arise when such a beam passes through a plasma. (See Section VIII-C of this report.)

1. General Dielectric Tensor

The system to be investigated is shown in Fig. V-1. We assume that the electrons are cold and are neutralized by ions of infinite mass. A plane wave of the form $\exp[j(\omega t-\bar{k}\cdot\bar{r})]$ is assumed, and the vector $\bar{k}$ is chosen to lie in the x-z plane. If we use the relativistic force equation and Maxwell's equations to solve for the current in terms of the electric field, we can define a dielectric tensor for the medium and write Maxwell's equations as

$$\bar{k}\times\bar{E} = \omega\mu_o\bar{H} \tag{1}$$

$$\bar{k}\times\bar{H} = -\omega\epsilon_o\bar{\bar{K}}\cdot\bar{E} \tag{2}$$

Here,

$$\bar{\bar{K}} = \begin{bmatrix} 1-\dfrac{\omega_{pt}^2\omega_r^2}{\omega^2\left(\omega_r^2-\omega_c^2\right)} & -j\dfrac{\omega_{pt}^2\omega_r\omega_c}{\omega^2\left(\omega_r^2-\omega_c^2\right)} & -\dfrac{\omega_{pt}^2k_xv_o\omega_r}{\omega^2\left(\omega_r^2-\omega_c^2\right)} \\ j\dfrac{\omega_{pt}^2\omega_r\omega_c}{\omega^2\left(\omega_r^2-\omega_c^2\right)} & 1-\dfrac{\omega_{pt}^2\omega_r^2}{\omega^2\left(\omega_r^2-\omega_c^2\right)} & j\dfrac{\omega_{pt}^2k_xv_o\omega_c}{\omega^2\left(\omega_r^2-\omega_c^2\right)} \\ -\dfrac{\omega_{pt}^2k_xv_o\omega_r}{\omega^2\left(\omega_r^2-\omega_c^2\right)} & -j\dfrac{\omega_{pt}^2k_xv_o\omega_c}{\omega^2\left(\omega_r^2-\omega_c^2\right)} & 1-\dfrac{\omega_{p\ell}^2}{\omega_r^2}-\dfrac{\omega_{pt}^2k_x^2v_o^2}{\omega^2\left(\omega_r^2-\omega_c^2\right)} \end{bmatrix} \tag{3}$$

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$$\left.\begin{aligned} \omega_r &= \omega - k_z v_o \\ \omega_c &= \frac{|e|\ B_o}{m_t} \\ \omega_{pt}^2 &= \frac{e\rho_o}{\epsilon_o m_t} \\ \omega_{p\ell}^2 &= \frac{e\rho_o}{\epsilon_o m_\ell} \end{aligned}\right\} \tag{4}$$

and ρ_o is the charge density of the beam as measured by a stationary observer. The transverse and longitudinal masses are defined by

$$m_t = \gamma_o m_o$$

$$m_\ell = \gamma_o^3 m_o,$$

where

$$\gamma_o = \left(1 - v_o^2/c^2\right)^{-1/2},$$

and m_o is the rest mass. The dielectric tensor can easily be generalized to include a multiparticle system by summing over all species in the usual manner.

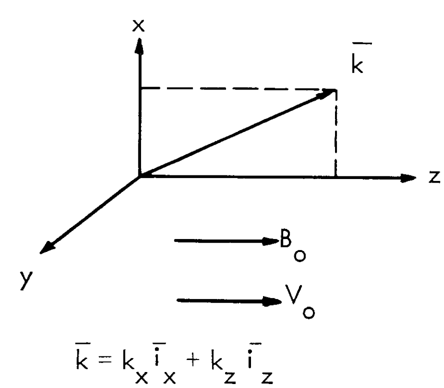

Fig. V-1. Electron-beam system.

2. Waves along the Magnetic Field

For the waves along the magnetic field, $k_x = 0$ and the dielectric tensor assumes the form

$$\bar{\bar{K}} = \begin{bmatrix} K_\perp & -K_x & 0 \\ K_x & K_\perp & 0 \\ 0 & 0 & K_\parallel \end{bmatrix}, \tag{5}$$

where the elements are easily obtained from the general tensor. The waves are of two types (in the rest of this report, $k_z = k$):

(a) Longitudinal oscillations with $\bar{H} = 0$ and $\bar{E} = E_z \bar{i}_z$; $K_\parallel = 0$. These are the well-known space-charge waves and will not be discussed further in this investigation.

(b) Transverse waves that are right or left circularly polarized with

$$k^2 = \frac{\omega^2}{c^2} K_\ell \quad \text{for left polarization (clockwise)} \tag{6}$$

$$k^2 = \frac{\omega^2}{c^2} K_r \quad \text{for right polarization (counterclockwise)} \tag{7}$$

where

$$K_\ell = K_\perp - jK_x = 1 - \frac{\omega_{pt}^2 \omega_r}{\omega^2(\omega_r + \omega_c)} \tag{8}$$

$$K_r = K_\perp + jK_x = 1 - \frac{\omega_{pt}^2 \omega_r}{\omega^2(\omega_r - \omega_c)}. \tag{9}$$

The solutions of the dispersion equations are complicated because of their cubic nature. The propagating modes are most easily obtained by a transformation into a reference frame moving with the beam because the dispersion equation is then quadratic (Fig. V-2). The transformation of (ω, k) is

$$\omega = \gamma_o(\omega' + k'v_o) \tag{10}$$

$$k = \gamma_o\left(k' + \omega' v_o/c^2\right). \tag{11}$$

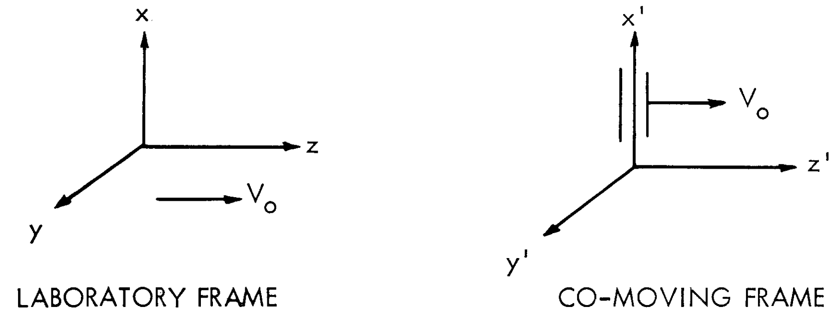

Fig. V-2. Co-moving reference frames.

In the primed reference frame, the dispersion equations are

$$k'^2 = \frac{\omega'^2}{c^2}\left[1 - \frac{\omega_p^2}{\omega'(\omega'+\omega_{co})}\right] \quad \text{(left)} \tag{12}$$

$$k'^2 = \frac{\omega'^2}{c^2}\left[1 - \frac{\omega_p^2}{\omega'(\omega'-\omega_{co})}\right] \quad \text{(right)} \tag{13}$$

where

$$\left.\begin{aligned} \omega_p^2 &= \frac{e\rho_o'}{\epsilon_o m_o} = \omega_{pt}^2 \\ \omega_{co} &= \frac{eB_o}{m_o} = \gamma_o \omega_c \end{aligned}\right\} \tag{14}$$

and ρ_o' is the charge density as measured in the co-moving reference frame. It should be noted that Eqs. 12 and 13 are just the dispersion relations for the left and right circularly polarized waves in an electron plasma. The dispersion of the right polarized waves can be obtained by letting $\omega \to -\omega$ in the left polarized case; thus a plot of ω-k for positive and negative frequencies for the left polarized waves will cover both cases in the primed reference frame.

If we have reasonably low densities and high magnetic fields ($\omega_p \ll \omega_{co}$), the dispersion equation is of the form sketched in Fig. V-3. The transformation to the laboratory frame gives the dispersion curve shown in Fig. V-4. We notice that the left polarized wave has a low-frequency backward wave under

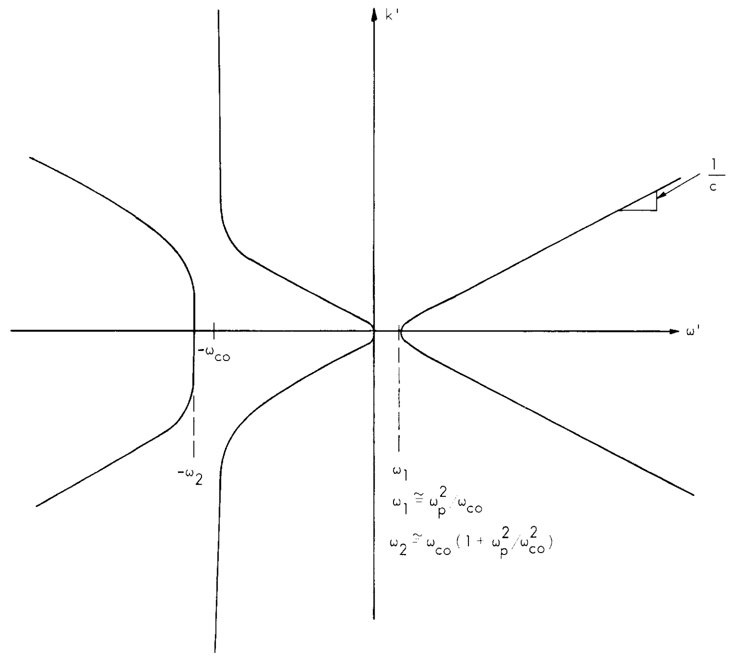

Fig. V-3. Dispersion of left polarized wave in co-moving frame ($\omega_p \ll \omega_{co}$).

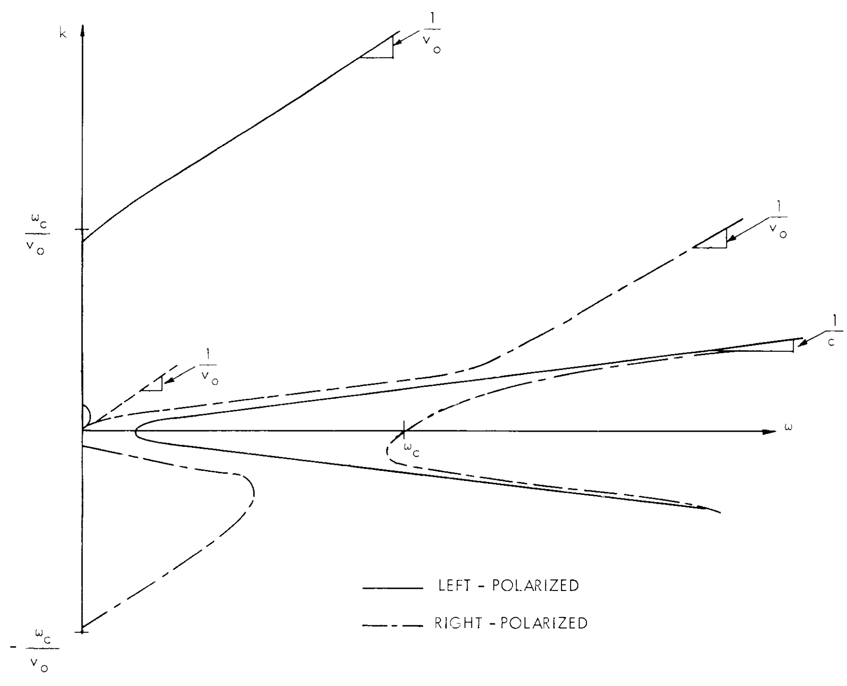

Fig. V-4. Electron-beam dispersion ($\omega_p \ll \omega_c$).

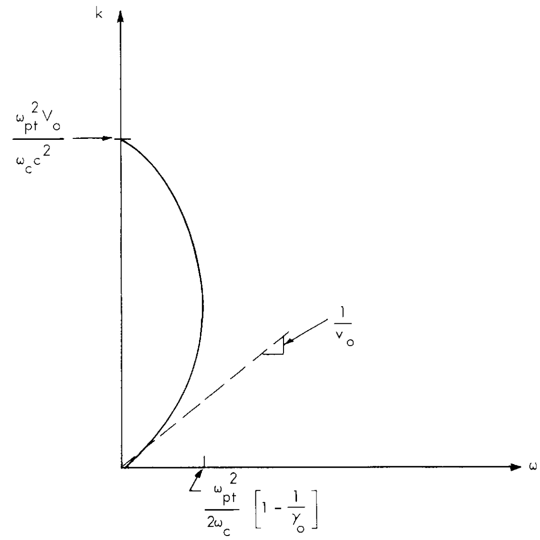

Fig. V-5. Detail of low-frequency, backward wave $\left(\text{valid if } \frac{\omega_p}{\omega_c}\frac{v_o}{c} \ll 1\right)$.

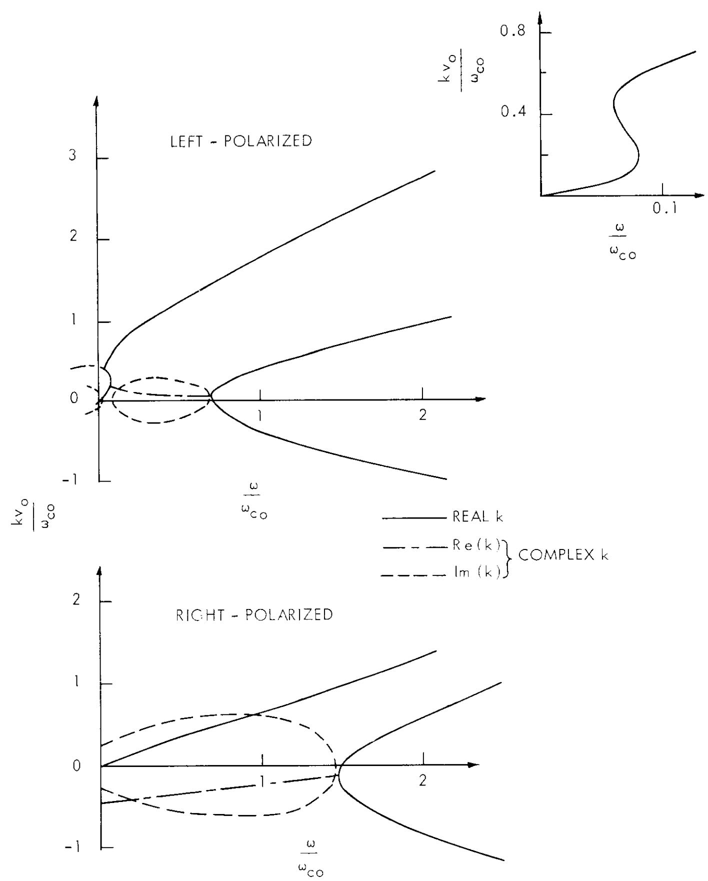

Fig. V-6. Dispersion for $\omega_p^2 = \omega_{co}^2$, $\frac{v_o}{c} = 0.5$.

these conditions. This branch is shown in detail in Fig. V-5. It will be shown in the next section that the left polarized wave carries negative energy for all phase velocities less than c. Therefore, a portion of the low-frequency branch is an active backward wave that propagates negative energy opposite to the beam velocity.

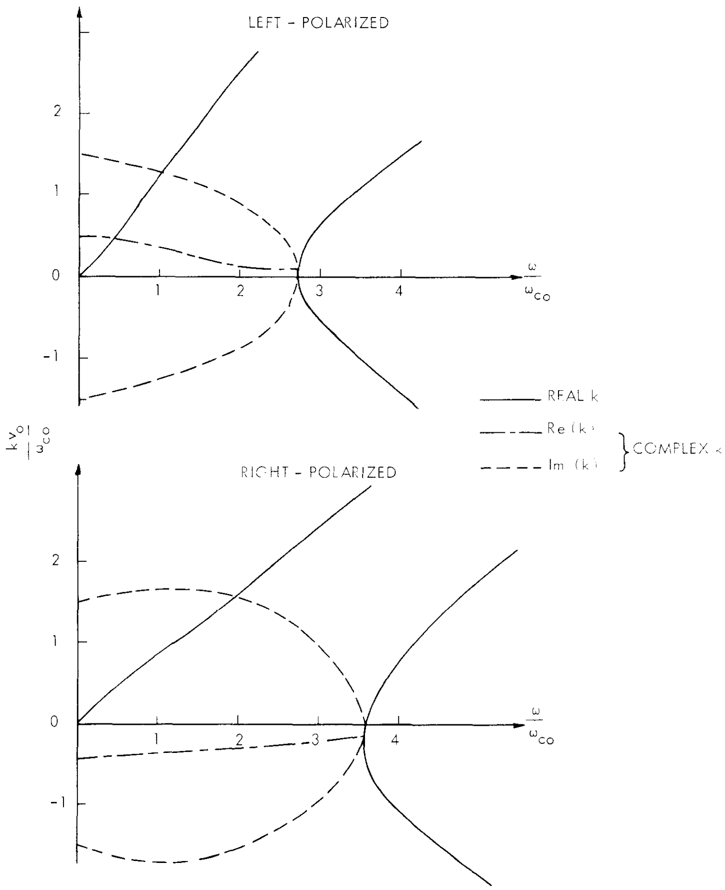

Fig. V-7. Dispersion for $\omega_p^2 = 10\omega_{co}^2$, $\frac{v_o}{c} = 0.5$.

The ω-k plots for some representative cases that are not covered by this approximate analysis are shown in Figs. V-6, V-7, and V-8. Since the transformation gives at least one real root of k for all ω, it is a simple matter to obtain the complex k values from Eqs. 6 and 7 by factoring out the real root. These are also shown in Figs. V-6, V-7, and V-8. It can be shown that the low-frequency backward wave will no longer exist when $(\omega_p/\omega_c)(v_o/c)$ exceeds a certain critical value of the order of unity. This is also evident from Figs. V-6, V-7, and V-8.

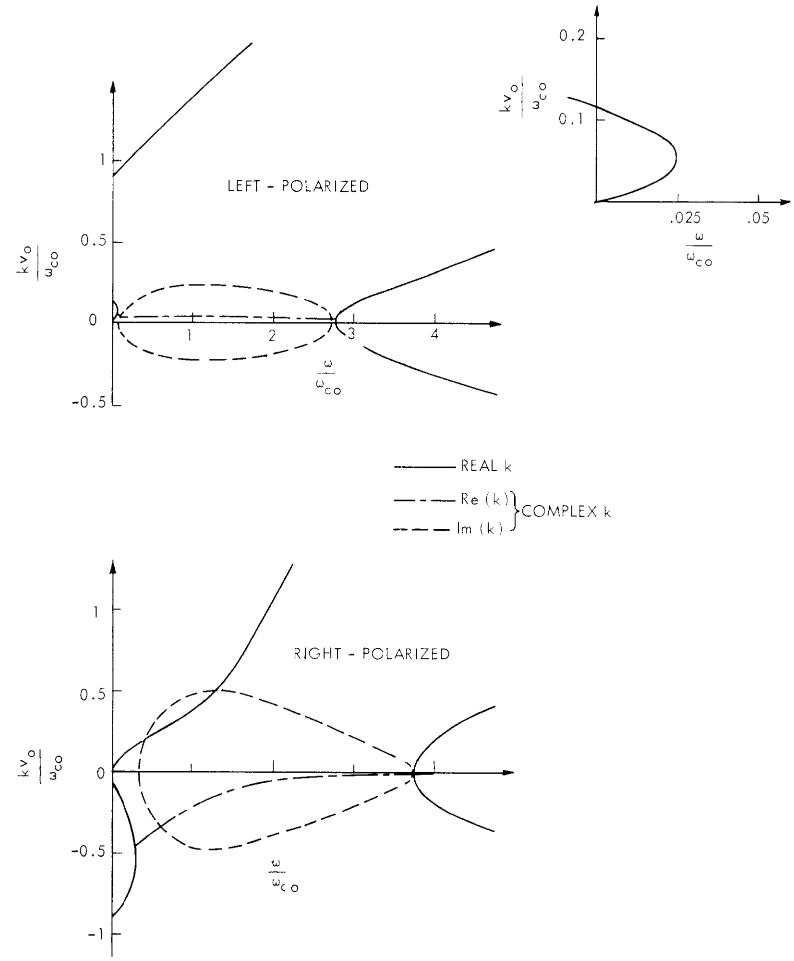

Fig. V-8. Dispersion for $\omega_p^2 = 10\omega_{co}^2$, $\frac{v_o}{c} = 0.1$.

3. Energy and Power

The application of Sturrock's criterion would show that the left polarized waves with phase velocity less than c carry negative energy, whereas the right polarized waves carry positive energy.[1] This can also be shown from an explicit evaluation of the energy density in the laboratory frame. The total energy density of a plane wave in such a medium is[2, 3]

$$W = 1/4\ \mu_o |\overline{H}|^2 + 1/4\ \epsilon_o \overline{E}^* \cdot \frac{\partial}{\partial\omega} (\omega \overline{\overline{K}}) \cdot \overline{E}. \tag{15}$$

It can be shown that the right and left polarized waves are orthogonal, and that the energy density of the left polarized wave is

$$W_\ell = 1/4\ \mu_o |\bar{H}|^2 + 1/4\ \epsilon_o |\bar{E}|^2 \frac{\partial}{\partial\omega}(\omega K_\ell)$$

$$= 1/4\ \epsilon_o |\bar{E}|^2 \left[2 - \frac{\omega_{pt}^2 \omega_c}{\omega(\omega - kv_o + \omega_o)^2} \right].$$

For a slow wave, $\omega/k < c$, it can be shown that for propagating waves the wave number must lie in the range

$$\frac{\omega}{v_o} < k < \frac{\omega + \omega_c}{v_o}.$$

By using the disperion relations, the energy density can be written as

$$W_\ell = -1/4\ \epsilon_o |\bar{E}|^2 \left[\frac{2\omega(\omega - kv_o)^2 + \omega_c(\omega - kc)^2 + 2kc\omega\left(1 - \frac{v_o}{c}\right)}{\omega(kv_o - \omega)\ (\omega + \omega_c - kv_o)} \right],$$

which is negative definite for slow waves. The energy density of the right polarized wave is

$$W_r = 1/4\ \epsilon_o |\bar{E}|^2 \left[2 + \frac{\omega_{pt}^2 \omega_c}{\omega(\omega - kv_o - \omega_c)^2} \right],$$

which is positive definite.

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References

1. P. A. Sturrock, In what sense do slow waves carry negative energy? J. Appl. Phys. 31, 2052-2056 (1960).

2. A. Bers, Properties of waves in time- and space-dispersive media, Quarterly Progress Report No. 65, Research Laboratory of Electronics, M. I. T., April 15, 1962, pp. 89-93.

3. A. Bers, Energy and power in media with temporal and spacial dispersion, Quarterly Progress Report No. 66, Research Laboratory of Electronics, M. I. T., July 15, 1962, pp. 111-116.