MURA 31

KRS(MURA)-4

An Alternative Derivation of the Formulas for the Smooth Approximation

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The smooth approximation formulas are derived in a simpler form, and as the first of a sequence of successive approximations. Formulas are also worked out for the two dimensional case.

1. The One-Dimensional Case

Let the alternating gradient equations of motion in one dimension be written in the form

$$\begin{aligned} x' &= p, \\ p' &= f(x,s), \end{aligned} \tag{1}$$

where primes denote differentiation with respect to s, and f(x,s) is periodic in s with period S. It will be shown in Appendix A that if we assume the truth of Powell's conjecture, that under the transformation

$$\begin{aligned} x(s) &\rightarrow x(s + S) \\ p(s) &\rightarrow p(s + S) \end{aligned} \tag{2}$$

the stable part of the x,p-plane is covered by a family of closed invariant curves, then the solution of Eqs. (1) can be written in the form

$$\begin{aligned} x &= X(s) + \xi\,(X,P,s), \\ p &= P(s) + \eta\,(X,P,s), \end{aligned} \tag{3}$$

where X,P satisfy the equations

$$X' = \frac{\partial H}{\partial P},$$
$$P' = \frac{\partial H}{\partial X}, \tag{4}$$

where H(X,P) is independent of s, and where ξ , η are periodic functions of s with period S (for fixed X,P) which vanish at an arbitrary reference point in the sector. We will call X,P the smooth motion, and ξ , η the ripple.

Let us substitute Eqs. (3) in Eqs. (1):

$$X'(1+\xi_X) + P'\xi_P + \xi_s = P + \eta \tag{5}$$

$$P'(1+\eta_P) + X'\eta_X + \eta_s = f(X+\xi, s)$$

If P,X are held fixed (X',P' are then also fixed by Eqs. (4)), all terms in Eqs. (5) are periodic in s. Let us average these equations over a sector length S. We denote such an average by a bar. Then, since the derivative of a periodic function has zero mean value, we obtain

$$X'(1+\bar{\xi}_X) + P'\bar{\xi}_P = P + \bar{\eta} \tag{6}$$

$$X'\bar{\eta}_X + P'(1+\bar{\eta}_P) = \overline{f(X+\xi, s)}$$

We subtract these equations from Eqs. (5), and denote the periodic part of a periodic function by a double bar, that is, for example,

$$\bar{\bar{\xi}} = \xi - \bar{\xi}. \tag{7}$$

We then obtain

$$\xi_s = \bar{\bar{\eta}} - X'\bar{\bar{\xi}}_X - P'\bar{\bar{\xi}}_P$$
$$\eta_s = \overline{\overline{f(X+\xi, s)}} - X'\bar{\bar{\eta}}_X - P'\bar{\bar{\eta}}_P \tag{8}$$

Eqs. (6) can be solved for X', P':

$$X' = \frac{(1+\bar{\eta}_P)(P+\bar{\eta}) - \bar{\xi}_P\,\overline{f(X+\xi, s)}}{(1+\bar{\eta}_P)(1+\bar{\xi}_X) - \bar{\xi}_P\,\bar{\eta}_X} \tag{9}$$

$$P' = \frac{(1+\bar{\xi}_X)\,\overline{f(X+\xi, s)} - \bar{\eta}_X(P+\bar{\eta})}{(1+\bar{\eta}_P)(1+\bar{\xi}_X) - \bar{\xi}_P\,\bar{\eta}_X}$$

Equations (8) are subject to the initial condition that ξ, η vanish at the reference point in the sector.

We will solve these equations by successive approximations based on the assumption that ξ, η, X' and P' are small. If we neglect terms in X', P' in Eqs. (8) and neglect ξ, η in comparison with X,P, we obtain in zero order approximation:

$$\xi_{0s} = \bar{\bar{\eta}}_0$$
$$\eta_{0s} = \overline{\overline{f(X, s)}} \tag{10}$$

$$X' = P$$
$$P' = \overline{f(X, s)} \tag{11}$$

If we take s = 0 at the reference point, Eqs. (10) can be solved:

$$\eta_0 = \int_0^s \overline{f(X, s)}\, ds \qquad \xi_0 = \int_0^s \bar{\bar{\eta}}_0\, ds \tag{12}$$

We substitute these expressions in the neglected terms in Eqs. (8) and (9), and obtain in first approximation

$$\xi_{1s} = \bar{\bar{\eta}}_1 - P\bar{\bar{\xi}}_{0X} \qquad \eta_{1s} = \overline{\overline{f(X,s)}} + \overline{\overline{\xi_0 f_X(X,s)}} - P\bar{\bar{\eta}}_{0X} \tag{13}$$

$$X' = \frac{P + \bar{\eta}_0}{1 + \bar{\xi}_{0X}} \qquad P' = \overline{f(X,s)} + \overline{\xi_0 f_X(X,s)} - \frac{\bar{\eta}_{0X}(P + \bar{\eta}_0)}{1 + \bar{\xi}_{0X}} \tag{14}$$

Equations (14) are not in Hamiltonian form, since

$$\frac{\partial X'}{\partial X} + \frac{\partial P'}{\partial P} = - \frac{\bar{\xi}_{0XX}(P + \bar{\eta}_0)}{(1 + \bar{\xi}_{0X})^2} \neq 0. \tag{15}$$

We can bring Eqs. (14) into Hamiltonian form, however, by adding a term to the second equation which is of the order of the terms which have been neglected:

$$X' = \frac{P + \bar{\eta}_0}{1 + \bar{\xi}_{0X}} \tag{16}$$

$$P' = \overline{f(X,s)} + \overline{\xi_0 f_X(X,s)} - \frac{\bar{\eta}_{0X}(P + \bar{\eta}_0)}{1 + \bar{\xi}_{0X}} + \frac{\bar{\xi}_{0XX}(P + \bar{\eta}_0)^2}{2(1 + \bar{\xi}_{0X})^2}$$

The Hamiltonian is then

$$H = \frac{(P + \bar{\eta}_0)^2}{2(1 + \bar{\xi}_{0X})} - \int \overline{f(X,s)}\, dX - \int \overline{\xi_0 f_X(X,s)}\, dX \tag{17}$$

Equations (16) have still the disadvantage that the betatron period will depend on the choice of reference point through the term $(1 + \bar{\xi}_{0X})$. This is presumably a symptom of the inaccuracy in the approximation when the ripple is large. We will assume the reference point is so chosen that

$$\bar{\xi}_0 = \bar{\xi}_{0X} = \bar{\xi}_{0XX} = 0. \tag{18}$$

With the simplification, Eqs. (16) can be combined to give the following equation for X:

$$X'' = \overline{f(X,s)} + \overline{\xi_0 f_X(X,s)}\ . \tag{19}$$

This equation agrees with the approximation equation obtained in KRS(MURA)-1.

2. The Two-Dimensional Case

The equations of motion in two dimensions have the form

$$
\begin{aligned}
x' &= P_x \\
y' &= P_y \\
P_x' &= f(x, y, s) \\
P_y' &= h(x, y, s)
\end{aligned}
\qquad (20)
$$

We substitute

$$
\begin{aligned}
x &= X + \xi(X, Y, s) \\
y &= Y + \zeta(X, Y, s)
\end{aligned}
\qquad (21)
$$

and proceed in precise analogy with the development in Section 1. The theorem of Appendix A does not apply to the two-dimensional case, but it turns out that to a first approximation the substitution (21) can be made to fit Eqs. (20). The resulting equations corresponding to (19) are

$$
\begin{aligned}
X'' &= \overline{f(X, Y, s)} + \overline{\xi_0 f_x} + \overline{\zeta_0 f_y} \\
Y'' &= \overline{h(X, Y, s)} + \overline{\xi_0 h_x} + \overline{\zeta_0 h_y}
\end{aligned}
\qquad (22)
$$

where

$$
\begin{aligned}
\xi_0 &= \iint \overline{\overline{f(X, Y, s)}}\, ds\, ds \\
\zeta_0 &= \iint \overline{\overline{h(X, Y, s)}}\, ds\, ds
\end{aligned}
\qquad (23)
$$

The constants of integration in Eqs. (23) are to be so chosen that ξ_0 , ζ_0 are periodic functions with zero mean value. It follows that only the periodic parts of the functions f and h need be used in computing the second and third terms in Eqs. (22).

We consider now the two-dimensional equations in the form

$$\begin{aligned} f(x,y,s) &= F(x,y)\,g(s) \\ h(x,y,s) &= h(x,y)\,g(s) \end{aligned} \tag{24}$$

where g(s) is a unit square wave.

$$g(s) = \begin{cases} 1 & 0 < s < \frac{1}{2}S \\ -1 & \frac{1}{2}S < s < S \end{cases} \tag{25}$$

We have taken f and h with zero mean values (diagonal of necktie), but the resulting equations are easily modified for the off diagonal case, simply by adding the mean values of $\bar{f}$ and $\bar{h}$ to the smoothed forces.

We calculate ξ_0 and ζ_0 :

$$\begin{aligned} \xi_0 &= F(X,Y)\,\mu(s)\,g(s), \\ \zeta_0 &= h(X,Y)\,\mu(s)\,g(s), \end{aligned} \tag{26}$$

where μ (s) is a periodic function with period $\frac{1}{2}$S defined by

$$\begin{aligned} \mu(s) &= \frac{1}{2}\left(s - \frac{1}{4}S\right)^2 - \frac{S^2}{32}, \quad 0 \le s \le \frac{1}{2}S \\ \mu\left(s + \frac{1}{2}S\right) &= \mu(s) \end{aligned} \tag{27}$$

We have now to calculate

$$\overline{\zeta_0 f_x} = F(X,Y)F_x(X,Y)\overline{\mu(s)g^2(s)}$$

$$= F F_x \overline{\mu(s)}$$

$$= -\frac{S^2}{48} F(X,Y) F_x(X,Y) \tag{28}$$

$$\overline{\zeta_0 f_y} = -\frac{S^2}{48} h(X,Y)F_y(X,Y)$$

$$\overline{\zeta_0 h_x} = -\frac{S^2}{48} F(X,Y)h_x(X,Y)$$

$$\overline{\zeta_0 h_y} = -\frac{S^2}{48} h(X,Y)\, h_Y(X,Y).$$

The smooth equations of motion are then

$$X'' = -\frac{S^2}{48}\left(FF_X + hF_Y\right)$$

$$Y'' = -\frac{S^2}{48}\left(hh_Y + Fh_X\right) \tag{29}$$

Inhomogeneities, bumps, displaced sectors, etc., can be treated in either of two ways. The smooth equations (22) may be written down for the distorted sector (as if it were periodic) and used to carry the smooth solution through this sector. Alternatively, the difference between f(X,Y,s) for the distorted sector and the undistorted periodic f(X,Y,s) can be included on the right side of Eqs. (22) as an added inhomogeneous term.

Appendix A. Proof of the Existence of Smooth Equations of Motion

We will assume the truth of the conjecture of J. L. Powell, based on numerical solutions carried out on the Illiac, that every point in the stable region of the phase plane lies on an invariant curve which is transformed into itself under the transformation (2) generated by Eqs. (1). It follows that there is a constant of the motion $\alpha(x,p,s)$ which is periodic in s with the sector period S, such that the invariant curves are given by

$$\alpha(x,p,s) = \text{a constant}, \tag{30}$$

$$\alpha(x,p,s+S) = \alpha(x,p,s). \tag{31}$$

We now choose any reference point, say s = 0, in the sector, and we define the function $\alpha_0(X,P)$ of the "smooth variables" X,P by the equation

$$\alpha_0(XP) = \alpha(X,P,0). \tag{32}$$

Following a suggestion of Powell, let $H(\alpha_0)$ be a function of α_0 to be specified later, and consider the equations

$$X' = \frac{\partial H}{\partial P} = H_\alpha(\alpha_0)\frac{\partial \alpha_0}{\partial P},$$

$$P' = -\frac{\partial H}{\partial X} = -H_\alpha(\alpha_0)\frac{\partial \alpha_0}{\partial X}. \tag{33}$$

These equations define a motion X(s), P(s) for which H, and hence also α_0, is a constant of the motion:

$$\alpha_0 (X, P) = a \text{ constant}. \tag{34}$$

Equation (34) defines the orbits of X,P.

Comparing Eqs. (30), (32), and (34), we see that the orbits for the smooth motion X(s), P(s) are the same as the invariant curves (30) at the reference point in each sector.

We now show that if $H(\alpha_0)$ is properly chosen, the point X,P traces out the invariant curve at such a rate that at the reference points s = nS it coincides with the phase x,p of the actual motion. We first note that changes in the factor $H_\alpha(\alpha_0)$ in Eqs. (33) correspond to changes in the time scale with which the orbit is traversed. Consider now any particular value of α, and assume that the invariant curve C for this value of α is a simply connected closed curve (Fig. 1).

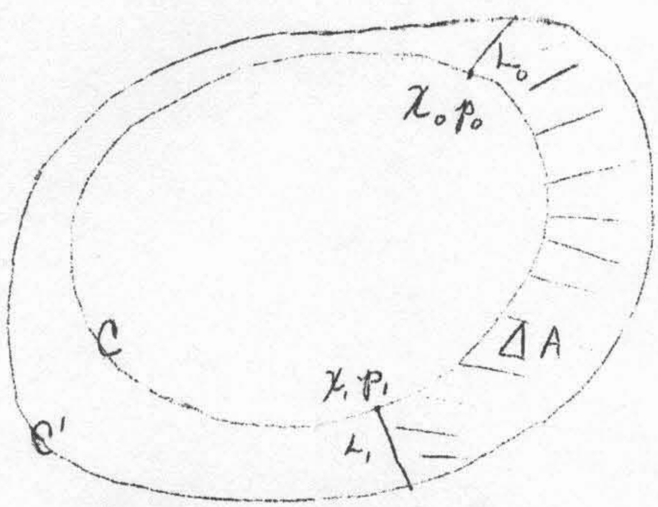

Fig. 1

Let C' be a nearby invariant curve. Take any point x_0, p_0 of C and connect it by a short line L_0 to the curve C'. Let the line L_0 move according to Eqs. (1) during the period

At s = S the curves C,C' will return to their original positions and the line L_o will have transformed into a line L_1 connecting the transform x_1, p_1 of x_o, p_o with C'. Let the total area between the curves C,C' be A, and let the portion of A between the lines L_o, L_1 be ΔA. (The ambiguity as to which of the two parts of A is between L_o and L_1 is to be settled in any consistent and continuous way.) We then define the betatron wavelength on C as follows:

$$\Upsilon = S \lim_{C' \to C} \frac{A}{\Delta A} . \tag{35}$$

By Liouville's theorem, the areas ΔA swept out by L_o on successive transformations through a sector are all equal. If for some integers n, N, $nS \doteq N\Upsilon$ and if C' is sufficiently close to C, then after n sectors, the line L_o will have swept out a total area $n \Delta A = NA$, and the final transform L_n will nearly coincide with L_o, so that x_n, p_n nearly coincides with $x_o p_o$. By continuity, if $x_o p_o$ is moved around the curve, the same pair of integers n,N will give $nS = N\Upsilon$, and hence the definition (35) of Υ is independent of the starting point x_o, p_o on C.

We now choose $H_\alpha(\alpha_o)$ for the value $\alpha_o = \alpha$ corresponding to the curve C so that the period of the smooth motion of X,P around C according to Eqs. (33) is Υ. (The sign of $H_\alpha(\alpha_o)$ is chosen so that the smooth motion traces the boundary of the shaded region ΔA in Fig. 1) This determines the function $H(\alpha_o)$ to within an arbitrary additive constant. Now by letting the line L_o in Fig. 1 move according to the smooth equations (33), and using the fact that the smooth motion is also area preserving by (33), we can show that

$$T = S \lim_{C' \to C} \frac{A}{\Delta A'} , \tag{36}$$

where $\Delta A'$ is the area between L_0 and L_1', L_1' being the transform of L_0 during one sector for the smooth motion. As C' approaches C, $\Delta A'/\Delta A$ approaches 1, and L_1' must therefore coincide with L_1 at least at the point $x_1, p_1 = X_1, P_1$. This shows that the smooth variables X,P coincide periodically with the actual variables x,p at the reference point in each sector (if they coincide initially).

The case where the invariant curve is not simply connected can also be accommodated. We consider as an example the case where the invariant curve C consists of two separate closed curves which transform into one another through a sector. In this case, we take the period 2S in which each piece of S transforms into itself and proceed as in the preceding two paragraphs. The result will be that the point X,P moving around one branch of C will coincide at the reference point in every other sector with the actual phase point x,p. At the reference points in the alternate sectors, the point x,p will coincide with a point X,P moving around the other branch of C.

We now define the "ripple variables" ξ, η :

$$\begin{aligned} \xi &= x(s) - X(s) , \\ \eta &= p(s) - P(s) , \end{aligned} \tag{37}$$

where $x(s)$, $p(s)$ is any particular solution of Eqs. (1), and $X(s)$, $P(s)$ is the corresponding solution of Eqs. (33), i.e. the solution which agrees at the reference point in each sector. The ripple variables ξ, η then vanish at the reference points. We define ξ, η as functions of X,P,s in the following way. For a given X,P,s, solve Eqs. (33) backward (or forward) to the next previous reference point $s_0 = nS$, and let X_0, P_0 be the values of X,P at $s = s_0$. Then, starting at $s = s_0$ with $x_0, p_0 = X_0, P_0$, solve Eqs. (1) forward (or backward) to s. The functions $\xi(X,P,s)$, $\eta(X,P,s)$ are then defined by Eqs. (37):

$$
\begin{aligned}
x(s) &= X(s) + \xi(X,P,s) \;, \\
p(s) &= P(s) + \eta(X,P,s) \;.
\end{aligned}
\qquad (38)
$$

It is clear from the periodicity of Eqs. (1)(and (3)) and the fact that x,p and X,P agree at the reference points that the functions ξ, η are periodic in s with period S, for fixed X,P.

In the case where the invariant curve C is composed of two separated closed curves, the ripple functions should be defined as above, but taking the sector length as 2S. Two sets of ripple functions ξ, η are then obtained, depending on whether we begin the double sectors on odd or even multiples of S. The ripple functions are then periodic in s with period 2S, vanishing at every other reference point, and, at the alternate reference points, taking on values which are the differences of the coordinates X,P of the two smooth phase points, tracing out the two parts of C in synchronism with the actual motion. More complicated connectedness properties of the invariant curves C can be treated similarly.

Corrigendum:

The last sentence in Section 1 is incorrect. Equation (19) agrees with the formulas of KRS(MURA)-1 only when $\overline{f(x,s)} = 0$ (on the diagonal). Off the diagonal, the formulas of KRS(MURA)-1 are slightly more accurate when $\overline{f(x,s)} \ll \overline{\overline{f(x,s)}}$, but become inaccurate when $\overline{f(x,s)}$ is large enough to produce oscillations of period comparable with the sector length. The formulas of the present paper are simpler and only slightly less accurate slightly off the diagonal, and are furthermore accurate for $\overline{f(x,s)} \gg \overline{\overline{f(x,s)}}$. The dotted curve for the predicted edge of the necktie in Fig. 2 of KRS(MURA)-1, for example, is displaced about 2 to 5% closer to the correct curve if we use the formulas of the present paper. The formulas given here give presumably a good approximation as long as the ripple is small in comparison with the smooth part of the motion.