MURA 30

KRS(MURA)-3

An Adiabatic Theorem for Motions Which Exhibit Invariant Phase Curves

Keith R. Symon, Wayne University, August 4, 1954

We consider a particle with one degree of freedom moving according to a Hamiltonian function $H(p,q,\lambda,t)$, depending on a parameter λ, and which may also depend on the time. We assume that for each fixed value of λ of interest, the motion is stable and exhibits invariant phase curves, that is, curves which are either continuously or periodically transformed into themselves by the transformation generated by the Hamiltonian equations of motion. This implies that there is a constant of the motion $\alpha(p,q,\lambda,t)$, which is either independent of t or periodic in t, the invariant curves being given by the equation

$$\alpha(p,q,\lambda,t) = \alpha = \text{a constant.} \tag{1}$$

When the Hamiltonian is independent of t, the invariant curves are simply the closed orbits on the phase plane. When the Hamiltonian is periodic in t, there is evidence, at least in many cases, for the existence of invariants of the form (1) with the same period in t. We will further assume that there is only one set of invariant curves, that is, that all constants of the motion are functions of α. This seems to be true except in the case of the linear alternating gradient equations when the betatron period is a rational multiple of the sector period; in the latter case it is probable that the adiabatic theorem can fail.

The theorem to be proved is that under the above assumptions, if the parameter λ is changed sufficiently slowly (in comparison with the period of oscillation or the betatron period), a set of particles which lie initially on an invariant curve will continue to lie on some invariant curve. Liouville's theorem states that a set of particles which lie on any closed curve (not necessarily invariant) will after any change in parameters lie on a closed curve having the same area. An immediate corollary of these two theorems is that for sufficiently slow changes in the parameter λ, the area of the invariant curve on which a particle is located will remain constant. When the Hamiltonian is independent of t, this reduces to the usual statement of the adiabatic invariance of the area of the orbit in the phase plane.

In order to prove the theorem, we show that the change in α, when λ is varied slowly, is a function only of the initial value of α, and hence that particles having initially the same value of α will continue to have the same values of α. The time rate of change in α is

$$\dot{\alpha} = \frac{\partial \alpha}{\partial p}\dot{p} + \frac{\partial \alpha}{\partial q}\dot{q} + \frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial \lambda}\dot{\lambda} . \tag{2}$$

The sum of the first three terms is zero, since α is a constant of the motion for constant λ. Suppose now that at $t = t_1$, $\lambda = \lambda_1$, and during the time $t_2 - t_1$ λ changes to $\lambda_2 = \lambda_1 + \Delta\lambda$. We will take $\Delta\lambda$ sufficiently small so that the character of the motion is not appreciably different for values of λ between λ_1 and λ_2, and we will set

$$\lambda = \lambda_1 + \dot{\lambda}\,(t - t_1)\,, \tag{3}$$

where $\dot{\lambda}$ for convenience is to be held constant during this time $t_2 - t_1$. The change in α is

$$\Delta\alpha = \int_{t_1}^{t_2} \frac{\partial\alpha}{\partial\lambda}\,\dot{\lambda}\,dt\,. \tag{4}$$

Since we are taking $\dot{\lambda}$ to be constant, we can write this in the form

$$\frac{\Delta\alpha}{\Delta\lambda} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{\partial\alpha}{\partial\lambda}\,dt\,. \tag{5}$$

The right side is the time average of $\partial\alpha/\partial\lambda$ during the time $t_2 - t_1$. If we assume that we can calculate this average for a fixed λ between λ_1 and λ_2 ($\Delta\lambda$ sufficiently small), then if the time $t_2 - t_1$ is made sufficiently long ($\dot{\lambda}$ small), the ergodic theorem states that the right side of (5) may be made to approach a constant of the motion, and hence is a function of α :

$$\frac{\Delta\alpha}{\Delta\lambda} = f(\alpha, \lambda)\,. \tag{6}$$

For adiabatic variations in λ , the change in α is therefore governed by the differential equation

$$\frac{d\alpha}{d\lambda} = f(\alpha, \lambda), \tag{7}$$

whose solution will depend only on the initial value of α .

The essential point in the application of the ergodic theorem is that if λ changes sufficiently slowly, the particle will have time to pass many times through all parts of the phase space corresponding to its value of α , and hence the time average of $\partial\alpha / \partial\lambda$ will not depend on where the particle is at any particular time. It is therefore necessary that the time required for a significant change in λ must be many oscillation periods. How many will depend upon the details of the motion and the nature of the function $\partial\alpha / \partial\lambda$. If a point on the invariant curve is almost fixed, or almost periodic with low periodicity, it may be necessary to go through very many oscillations in order to sample the entire invariant curve in the time average of $\partial\alpha / \partial\lambda$. It is clear that since the oscillation period becomes infinite on a separatrix separating two different types of motion in the phase plane, the transition from one type of motion to another can never be adiabatic.

As a simple application of the above theorem (suggested by F. T. Cole), we consider a simple harmonic oscillation with a harmonic driving term:

$$\ddot{x} + \omega_0^2 x = A \cos \omega t . \tag{8}$$

The Hamiltonian is

$$H = \frac{p^2}{2} + \frac{\omega_0^2 x^2}{2} - Ax \cos \omega t . \tag{9}$$

The solution, if the parameter A is constant, is

$$x = \frac{A \cos \omega t}{\omega_0^2 - \omega^2} + C \cos (\omega_0 t + \theta) , \tag{10}$$

$$p = \dot{x} = \frac{-\omega A \sin \omega t}{\omega_0^2 - \omega^2} - \omega_0 C \sin (\omega_0 t + \theta), \tag{11}$$

where C, θ are arbitrary constants. The first terms have the same periodicity as the Hamiltonian. The transient terms represent a motion around an ellipse with frequency ω_0; the center of the ellipse, represented by the steady state terms, moves with frequency ω, and hence the ellipse returns periodically to the same position. The transient ellipse:

$$\frac{\left(x - \frac{A \cos \omega t}{\omega_0^2 - \omega^2}\right)^2}{C^2} + \frac{\left(p + \frac{\omega A \sin \omega t}{\omega_0^2 - \omega^2}\right)^2}{\omega_0^2 C^2} = 1 \tag{12}$$

is an invariant curve. According to the adiabatic theorem proved above, the area of this curve remains constant for slow changes in the parameter A, that is, the transient amplitude remains constant. We can verify that the conclusion is correct in this case by noting that if we set

$$A = at, \tag{13}$$

then Eq. (8) has the exact solution

$$x = \frac{A\cos\omega t}{\omega_0^2 - \omega^2} + C\cos(\omega_0 t + \theta) + \frac{2\omega a \sin\omega t}{(\omega_0^2 - \omega^2)^2}, \tag{14}$$

$$p = \dot{x} = \frac{-\omega A\sin\omega t}{\omega_0^2 - \omega^2} - \omega_0 C\cos(\omega_0 t + \theta) + \frac{a(\omega_0^2 + \omega^2)\cos\omega t}{(\omega_0^2 - \omega^2)^2}, \tag{15}$$

where C, θ are arbitrary comstants. If <u>a</u> is vanishingly small, the last terms are negligible, and the motion consists of the steady state plus a transient of constant amplitude, in agreement with the adiabatic theorem.