CHARACTERISTICS OF NON-LINEAR LOCK-IN CAUSED BY FIELD INHOMOGENEITY

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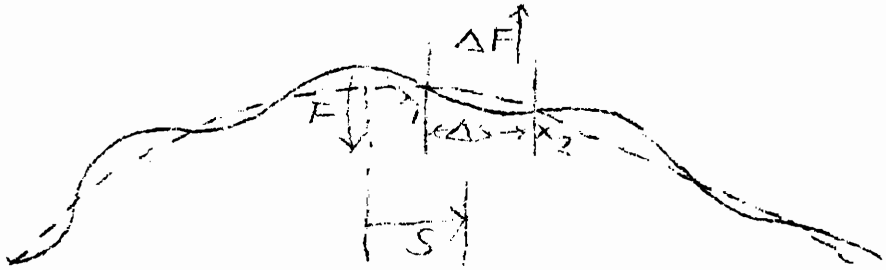
Crane's early suggestions[1] about locked-in motions pertained to the hunting about the locked-in stable orbit being driven by a field inhomogeneity. The study of these has not yet been carried out on the Illiac, but they have been seen by Courant using a 5% deviation in n in one out of 50 sectors. The special case at the tip of the necktie ($\sigma = \pi$) has been under investigation with the Illiac because without inhomogeneities of field (i.e. with the homogeneous differential equation) it was easy to find the characteristic throbbing which goes with lock-in. In this case the sectors themselves act as inhomogeneities which put energy into and which take energy out of the oscillation depending on the location of the peak of the oscillation in the focussing sector. An approximate calculation of throbbing frequency and other characteristics was made for this case.[2]

A similar primitive estimate using the method of MAC-DWK 4 follows for the case in which $\sigma \lesssim \pi$ and the energy addition or substraction is caused by an inhomogeneity, ΔF, occurring in some sector. We can be operating on an integral resonance or with a closed orbit fixed in the machine with a kink at the inhomogeneity.

1. MAC 8 and Memo H.R. Crane "Locked-In Motion".
2. MAC-DWK 4 "Approx. Calculation of Non-Linear Lock-In at $\sigma = \pi$".



The energy added to the peak of the oscillation by the action of the bump is:

(1) $-F_p \, dX_p = \Delta F \frac{ds}{dx} |\Delta s|$

where F_p . . . . occurs in a focussing sector and dX_p is the change in amplitude as a result of one passage over the bump. $|\Delta s|$ is the circumferential length of the inhomogeneity, in the sketch this is a sector length.

$\frac{dx}{ds}|\Delta s|$ is the net displacement of the particle in the direction of or opposed to the driving inhomogeneity, ΔF. $\frac{dx}{ds}$ is not the slope of the actual trajectory, but it is $\frac{X_2 - X_1}{|\Delta s|}$ where X_2 is orbit position at exit of inhomogeneity X_1 is orbit position at entrance of inhomogeneity; it is the slope of the dotted and approximately sinusoidal curve about which the scalloped motion due to alternating gradients occurs. This sinusoidal motion is: $X = X_p \cos \omega_0 S$

and it is a closed orbit in the machine; i.e., it fits integrally or is an integral resonance except for the slight kink at the inhomogeneity.

(2) $\frac{dx}{dS} = -\omega_0 X_p \sin \omega_0 S \cong -\omega_0^2 X_p S$ for small hunting.

If η is the number of passages over the bump, (1) gives:

(3) $\frac{dX_p}{d\eta} = -\frac{\Delta F \Delta S}{F_p} \frac{dx}{dS}$

or: (4) $\frac{dx_p}{d\eta} = \frac{\Delta F \Delta S}{F_p} \omega_0^2 x_p \varsigma$

ς is the displacement of the bump from the peak of the sine curve. (This is the negative of the ς used in the MAC-DWK 4)

but $\frac{d\varsigma}{\ell} = \frac{d\omega}{\omega_0}$ (as in (2) MAC-DWK 4) gives the change of ς per passage over the bump due to a deviation from the equilibrium frequency of oscillation ω_0.

ℓ is the distance travelled between passages over the bump.

So

(5) $\frac{d\varsigma}{d\eta} = \frac{\ell}{\omega_0} \frac{d\omega}{dx_p} \delta x_p$

where δx_p is the stray from the equilibrium x_p and $\frac{d\omega}{dx_p} \neq 0$ because the restoring forces are non-linear.

Differentiating (5) and remembering $\frac{d}{d\eta}\delta x_p = \frac{dx_p}{d\eta}$ of (3):

(6) $\frac{d^2\varsigma}{d\eta^2} = \ell\,\omega_0 \frac{d\omega}{dx_p} \frac{\Delta F \Delta S}{F_p} x_p \varsigma$

which is a second order differential equation with an oscillatory solution if $\frac{\Delta F}{F_p}\frac{d\omega}{dx_p} < 0$ remember F_p

If $\delta x_p \ll x_p$ then $\frac{d\omega}{dx_p} \sim$ constant and

(7) $\varsigma = \varsigma_0 \sin \Omega\eta$

where

(8) $\Omega = \omega_0 \ell \sqrt{-\frac{1}{\omega_0}\frac{d\omega}{dx_p}\frac{\Delta F \Delta S}{F_p}\frac{x_p}{\ell}}$

but $\frac{d\omega}{dx_p} \doteq \frac{4}{3}\frac{\omega_0}{x_p} f$ is known to be roughly true from digital computor observation.

$f \equiv \frac{e\,x_p^2}{3m}$ is the fractional cubic force at peak displacement giving regard to sign of e and m and where our differential equation is:

$x'' = \pm m x \mp \frac{e}{3} x^3$ (Powell MAC-JLP - 1) then:

$$(9) \quad \Omega = \omega_0 \, \ell \sqrt{-\frac{4}{3} f + \frac{\Delta F}{F_p}\frac{\Delta S}{\ell}}$$

If n_w is the number of bump passages per throb or wow and λ is the wavelength of the oscillation,

$$(10) \quad n_w = \frac{\sqrt{3}\,\lambda}{2\ell\sqrt{-f + \frac{\Delta S \Delta F}{\ell F_p}}}$$

or the number of wavelengths per wow is:

$$(11) \quad n_\lambda = \frac{\ell}{\lambda} n_w = \frac{\sqrt{3}}{2\sqrt{-f + \frac{\Delta S}{\ell}\frac{\Delta F}{F_p}}}$$

A hunting orbit in the machine can be found whether $f < 0$ or $f > 0$ by just inverting the oscillation so $\frac{\Delta F}{F_p}$ has the proper sign. Thus there is no question of the existence of a stable oscillation. An oscillation which fits half integrally around the machine will lock in with an erroneous n in one sector as Courant has found; for ΔF and F_p reverse sign together at an n bump. Oscillations which take more than two revolutions to repeat were also found locked into an n bump by Courant. This is understandable because the bump action goes to zero for trajectories crossing the ideal closed orbit in the sector with the erroneous n. The main lock-in action is still greatest for the lobe of the orbit with its peak hunting around the bump.

The two dimensional case with coupling and the case of random bumps must be investigated for the actual accelerator. Also the limits of ΔF and δx_p and δ_s before lock-in is broken should be sought by closer examination of the problem. For example, large negative δx_p will cause $\frac{d\omega}{dx_p}$ to vary and then (6) has an added term which weakens the restoring force for the δ oscillation.

This craniac motion is an example of a special type of oscillation about the kinked closed orbit in an accelerator with an inhomogeneity. The stability of this behavior in the presence of secular changes in n and larger inhomogeneities which would produce pronounced kinks are additional problems.