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DISCUSSION OF SPACE-CHARGE EFFECTS IN THE ALTERNATE-GRADIENT SYNCHROTRON

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1. Introduction:

As a sequel to the January 9 meeting of the Mid-West Technical Group, Dr. Kerst suggested that it would be desirable to record equations which have been used in discussion of space-charge effects and to exhibit some of the grave consequences suggested by use of these equations. The present report is in compliance with this suggestion, but is written with the following reservations in mind.

(i) Concentration of attention on space-charge effects, which will be most prominent at low energy, should not cause one to overlook other phenomena,[1,2] not readily analyzed, which may play important rôles at injection.

(ii) Analysis of space-charge effects on the basis of an assumed form for the charge distribution may be seriously in error if the particles of the group considered can execute oscillations which result in a distribution differing from that assumed.[3]

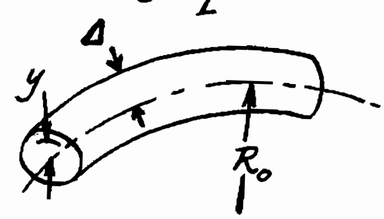
It is suggested, however, that application of the present formulas to a group of particles moving non-coherently will provide an approximate indication of dangerous values for design parameters.

2. Statement of Formulas:

A. Effective Change of "n", Rudimentary Derivation--

For a beam of constant charge density ρ throughout a cross-sectional area of constant radius Δ, the total charge q is

$$q = (\pi\Delta^2)(2\pi R_o)$$
$$= 2\pi^2 \Delta^2 R_o \rho , \qquad (1)$$



where R_o represents the orbit radius.

The net defocusing electric and magnetic force experienced, as a result of the space-charge, by a particle within the beam at a distance y from the axis, is

$$F_{s.c.} = \frac{e}{2}\left(\frac{1}{\epsilon_o} - \mu_o v^2\right) y\rho$$

$$= \frac{e}{2\epsilon_o}(1-\beta^2)\, y\rho$$

$$= \frac{qe(1-\beta^2)y}{4\pi^2 \epsilon_o \Delta^2 R_o}, \quad \text{in "rationalized" (MKS) units.} \tag{2}$$

The focusing and defocusing forces produced by the magnetic field of the accelerator are

$$F_z = n\,\frac{evB_o y}{R_o} = n\,\frac{E_T\beta^2 y}{R_o^{\,2}}, \quad \text{or} \tag{3a}$$

$$F_r = (1-n)\,\frac{E_T\beta^2 y}{R_o^{\,2}}, \tag{3b}$$

with E_T representing the <u>total</u> energy of the particle.

The force indicated in equation (2) is thus equivalent to a reduction of n in the equation of axial motion, and an increase in n in the radial equation, by

$$|\delta n| = \frac{qeR_o}{4\pi^2 \epsilon_o \Delta^2}\,\frac{1-\beta^2}{\beta^2 E_T}; \tag{4}$$

$$q = \frac{4\pi^2 \epsilon_o}{e}\,\frac{\Delta^2}{R_o}\,\beta^2\,\frac{E_T^{\,3}}{E_o^{\,2}} \cdot |\delta n|$$

$$= \frac{(E_o/e)_{volts}}{60_{ohms}}\left(\frac{2\pi R_o}{c}\right)_{sec}\left(\frac{\Delta}{R_o}\right)^2 \beta^2 \left(\frac{E_T}{E_o}\right)^3 |\delta n|. \tag{5}$$ [a]

(a)
$$\epsilon_o = \left[\frac{1}{(4\pi \times 10^{-7})_{H/M}\; c_{M/sec}^{\,2}}\right]_{farad/M} = \frac{1}{(120\pi)_{ohms}\; c_{M/sec}}.$$

If this analysis is accepted, it can be seen that the result of the space-charge is equivalent to effecting a translation of the operating point at right angles to the diagonal of the "necktie" diagram.

For comparison with similar results stated elsewhere, it is also of interest to write the associated "current"

$$i \equiv q\,(\beta c/\text{circumference})$$

$$= \frac{(E_o/e)_{\text{volts}}}{60_{\text{ohms}}}\left(\frac{\Delta}{R_o}\right)^2 \beta^3 \left(\frac{E_T}{E_o}\right)^3 \cdot |\delta n| \quad \text{amperes.} \tag{6}$$

B. Comparison with Previous Results --

Equation (5) is consistent with a non-relativistic result given by Kerst[4] for a conventional betatron, if we identify $|\delta n|$ with the limiting tolerance (1-n) for radial stability. Again in application to a conventional synchrotron, Judd[5] considers unequal radial and axial focusing and derives the aperture requirements for a beam of elliptical cross-section. His results also agree with our equation (5) in the case $n = 1-n = 1/2 = |\delta n|$. Similarly, J. P. Blewett[6] has also considered an elliptical beam in a conventional betatron with $n = 3/4$ and $R_o = 0.833$ Meter. Finally, Barden[7] originated the equation (5) in the form cited here and has suggested considering its application to an alternate-gradient accelerator in terms of the permissible variation of n.

C. Estimate of a Tolerable $|\delta n|$ --

In application of equation (5) to an alternate-gradient accelerator, Barden[7] originally suggested that one require

$$|\delta n| < 0.006\, N_s^{\,2}, \tag{7}$$

where N_s represents the number of magnet sectors. This suggested limitation was possibly motivated by the observation that the overall width of the necktie diagram, projected onto the n_1 or n_2 axis, corresponds approximately to $|\delta n| = 0.03\, N_s^{\,2}$. Thus a variation of about the amount suggested by Barden would carry the operation point from the diagonal almost half-way towards the edge of the stable region.

In view, however, of the present concern about integral and half-integral resonances (as well as sum resonances), it appears more prudent to allow variations of n only within one of the small diamonds situated along the diagonal of the necktie diagram. Since the characteristic solutions for the particle trajectories involve a factor exp(+ik) for traversal of a sector pair, the separation of integral resonances corresponds to

$$|\delta k| = \frac{2\pi}{N_s/2}, \quad \text{or} \tag{8a}$$

$$|\delta(\cos k)| \simeq \frac{2\pi \sin k}{N_s/2}; \tag{8b}$$

similarly, movement from the center of a small diamond, bounded by integral and half-integral resonances, half way[(b)] towards the edge corresponds to

$$|\delta k| = \frac{\pi/4}{N_s/2}, \quad \text{or} \tag{9a}$$

$$|\delta(\cos k)| = \frac{(\pi/4) \sin k}{N_s/2}. \tag{9b}$$

With the index n alternating between n_1 and $n_2 \equiv -m$ in sectors of equal length,[8]

$$\cos k \simeq \cos \frac{2\pi n_1{}^{1/2}}{N_s} \cosh \frac{2\pi m^{1/2}}{N_s} - \frac{n_1 - m}{2n_1{}^{1/2} m^{1/2}} \sin \frac{2\pi n_1{}^{1/2}}{N_s} \times \sinh \frac{2\pi m^{1/2}}{N_s}. \tag{10}$$

For variations such that $\delta n_1 = -\delta m$, and in the neighborhood of the diagonal,

$$\delta(\cos k) = \left\{ \frac{\pi}{n^{1/2} N_s} \left[\cos \frac{2\pi n^{1/2}}{N_s} \sinh \frac{2\pi n^{1/2}}{N_s} + \sin \frac{2\pi n^{1/2}}{N_s} \cosh \frac{2\pi n^{1/2}}{N_s} \right] + \frac{1}{n} \sin \frac{2\pi n^{1/2}}{N_s} \sinh \frac{2\pi n^{1/2}}{N_s} \right\} [-\delta n]. \tag{11}$$

A conservative limit to the acceptable $|\delta n|$ thus appears to be

$$|\delta n| \leq \frac{(n^{1/2}/2) \sin k}{\cos \frac{2\pi n^{1/2}}{N_s} \sinh \frac{2\pi n^{1/2}}{N_s} + \sin \frac{2\pi n^{1/2}}{N_s} \cosh \frac{2\pi n^{1/2}}{N_s} + \frac{N_s}{\pi n^{1/2}} \times \sin \frac{2\pi n^{1/2}}{N_s} \sinh \frac{2\pi n^{1/2}}{N_s}} \simeq \frac{(n^{1/2}/2) \sin k}{\sin \frac{2\pi n^{1/2}}{N_s} \sinh \frac{2\pi n^{1/2}}{N_s}} \tag{12}$$

continuation of denominator

[(b)] To afford some latitude for other possible variations of the accelerator characteristics, as would arise for example from remanence. It may also be noted that, as J. B. Adams[14] has pointed out, particles with momentum different from the equilibrium momentum are presented with a different n value ($|\delta n| \simeq n_0 \cdot \Delta p/p_0$).

Accordingly, near the center of the necktie where $n^{1/2}/N_s \cong 0.25$, $\sin k \cong 1$ and

$$|\delta n| \leq \frac{n^{1/2}/2}{\cosh \pi/2 + (4/\pi) \sinh \pi/2} = 0.0919\ n^{1/2} = 0.0230\ N_s; \qquad (13a)$$

similarly at an operation point for which $n^{1/2}/N_s \cong 0.1778$, $\sin k \cong 0.671$ and

$$|\delta n| \leq \frac{(n^{1/2}/2)\ 0.671}{4.314} = 0.078\ n^{1/2} = 0.0138\ N_s. \qquad (13b)$$

The above criteria suggest, as a typical tolerance in an accelerator with n in the range of 400 to 500,

$$|\delta n| \simeq 1.8.$$

Livingston[9] appears to have considered a similar approach to the problem of estimating space-charge limitations.

3. Numerical Results:

In application of equation (5) to estimate the beam which can be held in an alternate-gradient synchrotron at the time of injection, two alternative view-points may be considered. If one considers that the injected beam spirals inward,[10,11] due to the rising magnetic field, equation (5) may be considered as giving the maximum charge per turn and Δ might be taken as one-half of the pitch required for the spiral to clear the inflector comfortably;[c] in this case the acceptable injection current is the limiting charge per turn divided by the period of revolution and the total charge is the charge per turn times the number of turns accepted. If, on the other hand, the details of the injection process are ignored, equation (5) might be regarded as giving the total possible charge, with Δ representing the useful semi-aperture of the accelerator, and the acceptable injection current would be this charge divided by the estimated duration of the useful injection interval. In either case, the expected useful beam from the accelerator will be no more than about one-half of that successfully injected, due to (for example) incomplete capture into the synchrotron phase.

In estimating the manner in which the acceptable injection currents will depend on injection energy, one must take account (in the non-relativistic case) of the energy dependence of the period of revolution. The bunching action of a R.F. linear accelerator has been suggested[12] as aggravating the space-charge effects, but it appears[12,13] that a slight inherent energy inhomogeneity suffices to smooth out the charge distribution within a distance less than one circumference.

[c] Supposedly this pitch would be at least twice the beam radius plus the radial thickness of the deflecting electrode.

A numerical example of space-charge limitations has been given by J. B. Adams[14] in connection with a proposed CERN accelerator design. Adams states his conclusions in terms of maximum current, which is presumably $i = q(\beta c/\text{circumference})$. With $n = 392$, we expect $|\delta n| \cong 3$ to carry the operating point to near the edge of a diamond.[d] If, following Adams, we take $\Delta = 0.4$ cm (the radius of the injected beam), $R_0 = 8600$ cm, Kinetic Energy = 50 Mev, and $\beta = 0.314$, we find from equation (6) that

$$i = 1.2 \times 10^{-3} \cdot |\delta n| \quad \text{amperes}$$

constitutes a limiting current (for one-turn injection) similar to the 3 ma cited by Adams.

We give below a table of permissible values, calculated from equation (5), for a circular accelerator of 8650 cm radius[e] and with the permissible $|\delta n|$ limited to 1.8. Kinetic energies for proton injection of 4 Mev and 50 Mev are considered. In addition, we first consider an injected beam of 0.3 cm radius, spiraling inward so that injection continues for six turns; secondly we consider a total beam of 4.0 cm radius, without regard to the details of the injection process. It is noted that the estimated acceptable injection currents for 50 Mev injection are about 45 times those for 4 Mev (proportional non-relativistically to the three-halves power of the kinetic energy).

4. Conclusions:

From the foregoing examples it is clear that space-charge may seriously limit the beam currents in certain of the accelerator designs presently under consideration. It is important, therefore, to be as certain as possible concerning the following points:

(i) Is the conventional analysis presented here valid?[3]

(ii) Are the integral and half-integral resonances so important that space-charge should not be permitted to displace the operating point across such resonances?[9]

(iii) If the present analysis is considered adequate, is it best to associate Δ with the radial width of the proposed injected beam,[9,14] with the pitch of the spiral[10,11] described by the injected beam, or with the semi-aperture of the accelerator?

The advantage of injection at high energy is apparent, if the injector supply can deliver the currents desired. It would be unfortunate to have an injector system incapable of delivering the desired currents, but it would also be frustrating to have designed an accelerator which could not accept the injection

[d] Or see diagram VI of Adams' paper.[14]

[e] Such a radius would permit, for example, attainment of 25 Gev in a field of 10,000 gauss (1 weber/M^2).

EXAMPLES OF ESTIMATED SPACE-CHARGE LIMITATIONS

$R_o = 8650$ cm, $|\delta n| = 1.8$

			$\Delta = 0.3$ cm				$\Delta = 4.0$ cm			
Kinetic Energy of Protons	β	Revolution Period μsec	Charge / Revolution (coulombs)	i (ma)	Charge / 6 Rev. (coulombs)	Particles, assuming 50% capture	Total Charge (Coulombs)	i (ma), if inject for 1 Rev	i (ma), if inject for 6 Rev	Particles, assuming 50% capture
50 Mev	0.314	5.76	7.1×10^{-9}	1.2	42×10^{-9}	13×10^{10}	1.26×10^{-6}	220	36	3.9×10^{12}
4 Mev	0.092	19.7	5.3×10^{-10}	0.027	3.2×10^{-9}	1.0×10^{10}	9.4×10^{-8}	4.8	0.80	0.3×10^{12}

The computed acceptable charge is rather considerably greater for electrons (which one could easily inject at high energy from a linear accelerator of the Stanford type) than for protons of the same energy. For injection energies which are relativistic for electrons and non-relativistic for protons, the ratio $q_{electrons}/q_{protons}$ appears to be approximately

$$\frac{(\text{Total Electron Energy})^3}{2\,(\text{Proton Kinetic Energy})\,(\text{Electron Rest Energy})^2},$$

or about 5000 for 50 Mev injection.

currents which it was planned to attain. Attention should be given, moreover, to the avoidance of R.F. voltages which would bunch the beam to an extent that space-charge would cause the beam to expand beyond the bounds of the effective aperture. The space-charge effects appear to be considerably less serious in comparable electron accelerators.

5. References:

1. D. W. Kerst, Phys. Rev. 74, 503 (1948).
2. D. L. Judd, "A Study of the Injection Process in Betatrons and Synchrotrons", California Institute of Technology Thesis (Pasadena, 1950).
3. The possible importance of the mutual interaction between the particle trajectories and the space-charge distribution was suggested by M. Hamermesh following a discussion of this topic at the January 9 MAC meeting.
4. D. W. Kerst, Phys. Rev. 60, 47 (1941). In our equation (5), $\beta^2 E_o$ may be identified in the non-relativistic limit with $2E_{Kinetic}$ and $E_T/E_o \rightarrow 1$; we obtain in this way agreement with Kerst's expression

$$(I/f)_{coulombs} = \pi\Delta^2 \text{ (Injection Voltage)}(1-n)/(15r_o \cdot c).$$

In discussion of this phenomenon, Kerst also comments on the nature of the relativistic effect.

5. D. L. Judd, op. cit.[2] At the bottom of p. 91, Judd gives for the total charge the expression $q = 2\pi^2 \epsilon_o B_o R^2 v \left(\frac{1-n}{n}\right)^{1/2} (1-v^2/c^2)^{-1} \cdot (\text{relative radial semi-aperture})^2$. When $n = 1/2$ this result assumes the form $q = 2\pi^2 \epsilon_o B_o \frac{v}{1-\beta^2} \Delta^2$; with the substitution $\frac{B_o v}{1-\beta^2} = \frac{\beta^2 E_o}{eR}\left(\frac{E_T}{E_o}\right)^3$, one has

$$q = \frac{2\pi^2 \epsilon_o}{e} \beta^2 \frac{E_T^3}{E_o^2} \frac{\Delta^2}{R},$$ in agreement with our expression leading to equation (5) if $|\delta n|$ is set equal to one-half.

Judd's expression may be compared similarly with the result of Blewett[6] [equation (18), p. 90] by writing the circulating current as

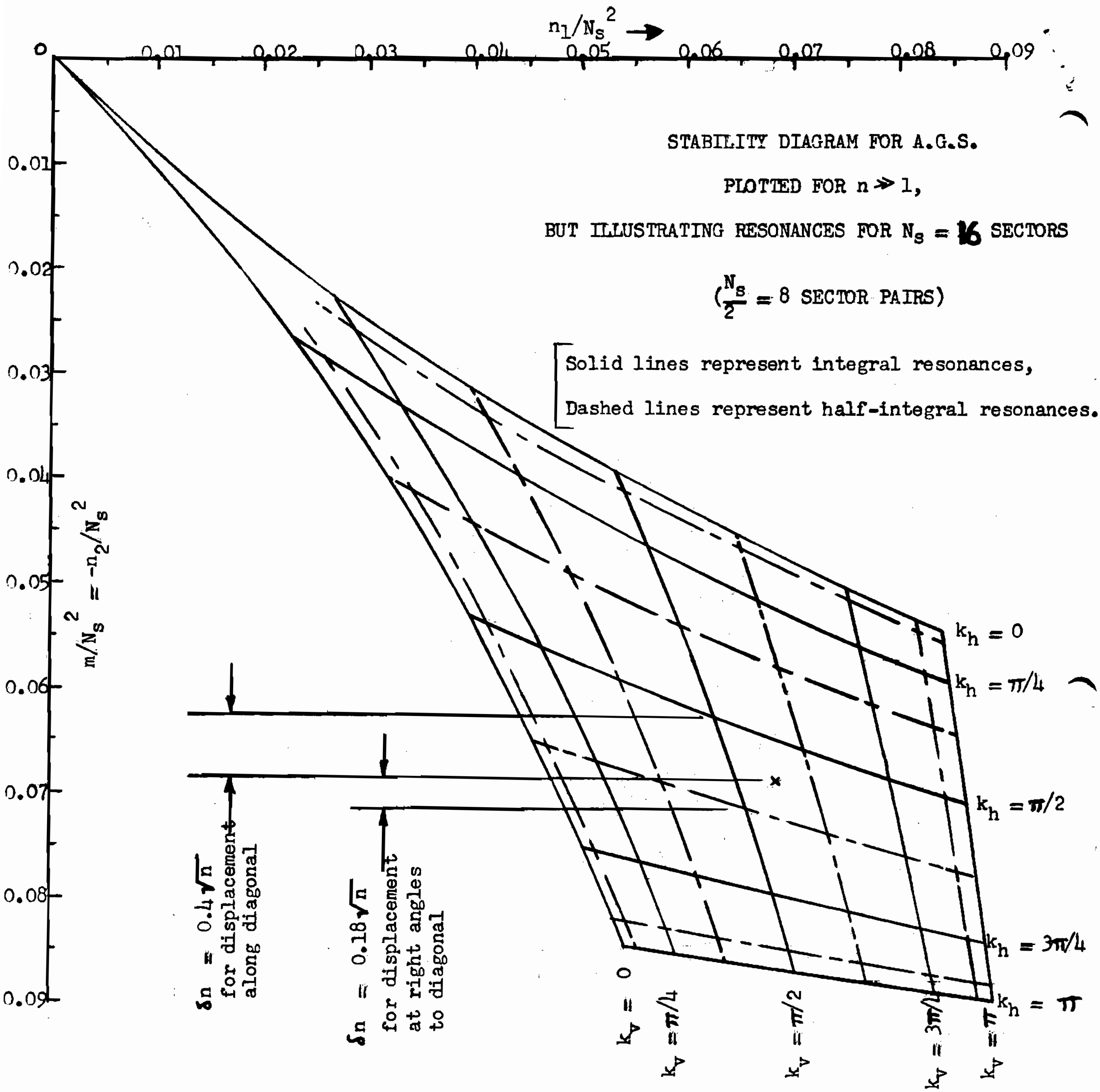
$$i = \pi \epsilon_o B_o R \frac{v^2}{1-v^2/c^2} \left(\frac{1-n}{n}\right)^{1/2} \left(\frac{w/2}{R}\right)^2 = \frac{\pi}{4} \epsilon_o \left(\frac{e}{m_o}\right)^2 B_o^3 R \left(\frac{1-n}{n}\right)^{1/2}$$

$$\times w^2 = 1.0 \times 10^{11} B_o^3 w^2,$$

after numerical substitution of Blewett's values [$n = 3/4$, $R = 0.833$ M, and $\frac{e}{m_o} = 1.76 \times 10^{-11}$ cou/Kg].

6. J. P. Blewett, Phys. Rev. 69, 87 (1946). Blewett's equation (18) appears to be consistent with our equation (6) if we take $|\delta n| = 1/2$ and replace Δ^2 by an effective value $w^2/(4\sqrt{3})$.

7. S. E. Barden, private communication to Dr. E. D. Courant (1953).
8. Courant, Livingston, and Snyder, Phys. Rev. 88, 1190 (1952). Our equation (10) results from the substitution of $n_2 = -m$ in their equation (4).
9. M. Stanley Livingston, "Design Study for a 15 Mev Accelerator", MIT Technical Report No. 60 (Cambridge, Massachusetts; June 30, 1953), §8.9. In the belief that the factor $(1-\beta^2)$ in Livingston's equations (98)-(99) has been lumped into his equation (101) for ω_p^2, his results for the defocusing effect of space-charge appear similar to the implications of our equation (5). He points out (p. 154) that for his design injection of 3 ma at 4 Mev, "the space-charge density is great enough to shift the beam through 3 or 4 π-resonances, allowing for the space-charge defocusing in the straight sections. ...We conclude that there is an upper limit to the space-charge density that can be held in stable orbits in the A.G.S. In the present design it is probable that the maximum useful current is appreciably less than 3 ma." In Livingston's estimates, a beam radius of 0.5 cm at injection was considered.
10. Livingston, Blewett, Green, and Haworth, Rev. Sci. Inst. 21, 7 (1950).
11. Kerst, Adams, Koch, and Robinson, Rev. Sci. Inst. 21, 462 (1950).
12. D. W. Kerst, private communication concerning discussion at the 26-28 October 1953 CERN conference (November 25, 1953).
13. J. H. Williams, comments at the meeting of the MAC Technical Group (January 9, 1954).
14. J. B. Adams, "The design of an alternate gradient synchrotron based on the linear theory", CERN, Geneva (October 24, 1953). Adams suggests (cf. his Fig. IX) a working diamond which "allows the n values of the different magnet sectors to have a random variation in n between the limits ± 1 per cent of n." Scaling his figure indicates a permissible consistent variation of n within the limits n±2. Adams points out explicitly that "for an injected current of 3 ma and an injection energy of 50 Mev the working point moves to the edge of the diamond. Dropping the injection energy to 25 Mev puts the working point outside the diamond. If a 4 Mev Van de Graaff generator were used the working point would be right outside the stability diagram."
15. Servo control of certain features of the beam trajectories was mentioned by L. Jones in a preliminary discussion at the meeting of the MAC Technical Group (January 9, 1954).



STABILITY DIAGRAM FOR A.G.S.
PLOTTED FOR n ≫ 1,
BUT ILLUSTRATING RESONANCES FOR N_s = 16 SECTORS
($\frac{N_s}{2}$ = 8 SECTOR PAIRS)
Solid lines represent integral resonances,
Dashed lines represent half-integral resonances.
n_1/N_s^2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
$m/N_s^2 = -n_2/N_s^2$
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
$k_h = 0$
$k_h = \pi/4$
$k_h = \pi/2$
$k_h = 3\pi/4$
$k_h = \pi$
$k_v = 0$
$k_v = \pi/4$
$k_v = \pi/2$
$k_v = 3\pi/4$
$k_v = \pi$
$\delta n = 0.4\sqrt{n}$
for displacement
along diagonal
$\delta n = 0.18\sqrt{n}$
for displacement
at right angles
to diagonal