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A Perturbation Treatment of Non-Linear Restoring Forces

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It has been suggested by Crane and others that the variation of wavelength with amplitude afforded by non-linear restoring forces might be used to overcome the resonance difficulties in AG synchrotrons. This note reports on a preliminary investigation of the possibility of such an effect using a simple first order perturbation treatment.

Consider the equation for the uncoupled radial or vertical motion,

$$x'' + nE(\theta)\, x = \alpha \mu(\theta)\, x^3 + \alpha F(\theta) \tag{1}$$

where $E(\theta)$ and $\mu(\theta)$ have period $\frac{2\pi}{N}$ and $F(\theta)$, the driving force due to misalignments, etc., has period 2π. α is a small constant.

Note that this is not a realistic equation of motion; one cannot construct a magnetic field with a linear and a cubic term. In order to satisfy Maxwell's Equations, mixed terms in the horizontal and vertical displacements must eventually be added.

(1) satisfies the initial conditions

$$\begin{aligned} x(0) &= x_o \\ x'(0) &= x_o' \end{aligned} \tag{2}$$

We attempt to find a solution of the form

$$x = x_o + \alpha x_1 + \alpha^2 x_2 + \ldots . \tag{3}$$

where

$$\begin{aligned} x_o(0) &= x_o \\ x_o'(0) &= x_o' \\ x_i(0) &= 0 \\ x_i'(0) &= 0 \end{aligned} \quad \text{for } i \geqslant 1 \tag{4}$$

This choice of X_i and X_i' ($i \geqslant 1$) is most convenient and assures that the initial conditions (2) are independent of α.

Substituting (3) into (1), we find to first order in the two equations

$$\ddot{X}_o + nE(\theta)\, X_o = 0 \qquad (5a)$$

$$\ddot{X}_1 + nE(\theta)\, X_1 = \mu(\theta) X_o^3 + F(\theta) = \Gamma(\theta) \qquad (5b)$$

The equation (5a) is of the type of Hill's equation. For the case of stable oscillations, it has as fundamental solutions.

$$y = e^{i\beta\theta}\, p(\theta)$$

$$y^* = e^{-i\beta\theta}\, p^*(\theta) \qquad (6)$$

where $0 < \beta < \frac{N}{2}$ (β real)

and $p(\theta + \frac{2\pi}{N}) = p(\theta)$

The solution which staisfies the initial conditions (4) may be written

$$x_o = 2\,\mathrm{Re}\,(ay) \qquad (7)$$

wher "a" is a complex constant. The right hand side of Eq. (5b) is now a known function. For the initial conditions (4), it has the solution

$$X_1(\theta) = \int_o^\theta G(\theta, u)\, \Gamma(u)\, du \qquad (8)$$

where

$$G(\theta,u) = -i \left[y(\theta) y^*(u) - y^*(\theta) y(u) \right] \qquad (9)$$

It can be seen by direct substitution that (8) satisfies (5) with $G(\theta,u)$ given by (9), provided that y is so normalized that

$$\lim_{u \to \theta} \frac{d}{d\theta} G(\theta,u) = 1$$

Both, $G(\theta,u)$ and $\Gamma(u)$ are real. Eq. (8) can be written

$$x_1(\theta) = 2\,\mathrm{Im} \left\{ y(\theta) \int_o^\theta y^*(u)\, \Gamma(u)\, du \right\}$$

$$x_1(\theta) = 2\,\mathrm{Im} \left\{ y(\theta) \int_o^\theta y^*(u) \left[\mu(u)\, X_o^3(u) + F(u) \right] du \right\}$$

$$= 2\,\mathrm{Im} \left\{ y(\theta) \int_o^\theta y^*(u) \left[\mu(u) \left(a^3 y^3 + 3a|a|^2 y |y|^2 + \text{c.c.} \right) + F(u) \right] du \right\}$$

where we substituted the solution (7).

Using (6) for y and y*, the integrand may be written in the form

$$\sum_s e^{is\beta u} \quad p_s\,(u) \tag{10a}$$

with

$$
\begin{aligned}
p_o(u) &= 3\mu\, a\,|a|^2 |p|^4 \\
p_1(u) &= 0 \\
p_2(u) &= \mu\, a^3 |p|^2 \quad p^2 \\
p_3(u) &= p_4(u) = \dots = 0 \\
p_{-1}(u) &= p^* \; F(u) \\
p_{-2}(u) &= 3\mu\, a^* \,|a|^2 \; p^{*2} \,|p|^2 \\
p_{-3}(u) &= 0 \\
p_{-4}(u) &= \mu\, a^{*3} \,|p|^4 \\
p_{-5}(u) &= p_{-6}(u) = \dots = 0
\end{aligned} \tag{10b}
$$

For s even, p_s is periodic of period $2\pi/N$. We split the range of integration into an integer number of cells and the fraction of a cell left over

$$\int_o^\theta = \int_o^{2\ \ k/N} + \int_{2\pi k/N}^\theta$$

where k is the number of cells traversed. The last integral is always finite and we neglect it for our discussion of resonance effects. For the first integral we use the relation

$$\int_o^{2\pi k/N} e^{i\gamma u}\; q(u)\; du = \frac{e^{2\pi i\gamma k/N}-1}{e^{2\pi i\gamma/N}\ \ 1-1} \int_o^{2\pi/N} e^{i\gamma u}\; q(u)\; du \tag{11}$$

q(u) of period $2\pi/N$

For s odd, the only non-vanishing term is P_{-1}, which has period 2π. We assume without loss of generality that $k' = k/N$ is integral and use the same relation for the integral involving

$$\int_o^{2\pi k^1} e^{-i\beta u}\; p_{-1}\;(u)du = \frac{e^{-2\pi i\beta k^1}\quad -1}{e^{-2\pi i\beta}\quad -1} \int_o^{2\pi} e^{-i\beta u}\; p_{-1}(u)du \tag{12}$$

Using (10), (11), and (12), and keeping only the terms which might give resonance,

$$x_1^{res}(\theta) = 2\ \mathrm{Im}\left\{ y(\theta) \left[\sum_{\text{seven}} \frac{e^{2\pi i s\beta k/N} - 1}{e^{2\pi i s\beta/N} - 1} \int_0^{2\pi/N} e^{is\beta u} \cdot p_s(u)\,du + \frac{e^{-2\pi i\beta k} - 1}{e^{-2\pi i\beta} - 1} \int_0^{2\pi} e^{-i\beta u}\, p_{-1}(u)du \right.\right. \qquad (13)$$

The first term in the square bracket is due to the non-linearities, the second one due to the driving term. The latter will give resonance when $\beta = m =$ integer. Then,

$$\lim_{\beta\to m} \frac{e^{-2\pi i\beta k/N} - 1}{e^{-2\pi i\beta} - 1} = k/N$$

Thus, without the non-linear term (that is, with $\mu = 0$), the solution increases linearly with the number of revolutions. This is the well-known integral resonance of the AG machine.

On the other hand, the non-linear terms seem to give rise to resonance where $s\beta/N = m'$, an integer, since then, in an analogous way, the factor in front of the first integral has the value k. Thus, without the driving term (that is, with $F = 0$) the perturbation solution of the non-linear equation increases linearly with the number of cells traversed (and thus with the number of revolutions).

We wish to investigate the conditions under which these linear increases will cancel - to the first order in α. Probably the exact (at least for $\alpha < 0$), but the initial linear rise indicated by this first order perturbation solution is correct.

Let us enumerate the cases where $s\beta/N = m'$

1. $s = 0$, any β
2. $s = \pm 2$, $\beta = \pm \frac{m'N}{2}$
3. $s = -4$, $\beta = -\frac{m'N}{4}$

For all other s,

$$p_s = 0$$