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An Estimate of Effects of Non-Linear Restoring Forces for Avoiding Resonances.

D. W. Kerst - Midwest Accelerator Conference
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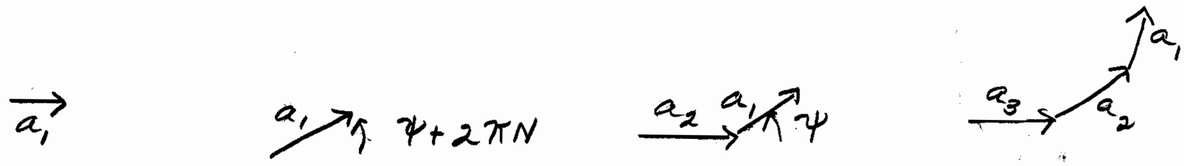
Computations on the effects of non-linear restoring forces have been made by digital computors on the CERN, Princeton, and Brookhaven projects. One of the recent results from Brookhaven is that in the equation $x'' + n(\theta)\, x + \alpha(\theta)\, x^3 = 0$ the motion is stable until $f = \alpha x_n^2 / n = 40\%$ at the displacement, x_n. This represents a rather large possible non-linear restoring force which may be used. Courant has emphasized a crucial question in the use of such non-linearities -- namely if there is a slow time variation of n due to saturation of the iron in the magnet or remnance or eddy currents, will the resulting displacement of the resonance frequency require the limiting amplitude of the oscillations to grow beyond allowable bounds in the effort to detune the oscillation? A crude estimate is made here of the size of f required to track a secular variation, $\Delta n/n$, without much growth of amplitude.

The approximation used considers the orbit in an alternating gradient machine to be a periodic motion roughly like a sine wave and with a representative phase which progresses in equal steps on every revolution about the machine. We will use the sine wave even though non-linear restoring forces will distort the wave shape somewhat since we are mainly interested in just the phase change after each revolution. The exact wave shape will influence the way successive revolutions connect; but for a start, neglect this.

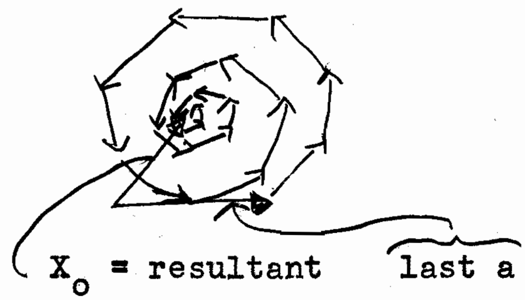
We will start with a perfectly circular orbit in a perfect machine and convert it to the orbit in a real machine with one field bump by adiabatically increasing the strength of the bump to its full size. The result is still an orbit which repeats itself exactly revolution by revolution. This is the so-called equilibrium orbit which finds its way through a bumpy field and repeats itself - the existance of which was pointed out by Courant and Snyder. Let the oscillation of the particle be represented by the projection on the vertical of a vector, a, which rotates with the betatron frequency. For one passage over a narrow magnetic bump of length W and strength ΔH, $a = (W \Delta H)/\sqrt{n}\, H$ for the Z motion of a conventional betatron, or $a = 2(W\Delta H)/H \sqrt{n}/2$ for an A.G. Field where $\sqrt{n}/2$ replaces the $\sqrt{n}$ and the factor of 2 in front is Courant's safety factor representing the fact that A.G. orbits stray about twice as far as pure sine wave orbits of the same wave length.

If the orbit contains N wavelengths plus an additional phase, ψ, in one revolution, then the a rotates N times plus ψ radians before another $\vec{a}$ is generated by the next passage over the bump.

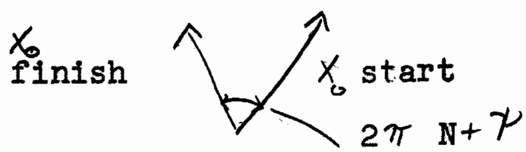


first passage | once around | once around and over the bump again | third passage over bump

In between pasages over the bump the oscillation is described by the N rotations plus ψ of the resultant of all the $\bar{a}_1$ vectors. If we let Δ H grow from zero, a_1 will be infinitesimal and successive a's will be larger so the vector diagram becomes a spiral somewhat like Cornu's spiral.

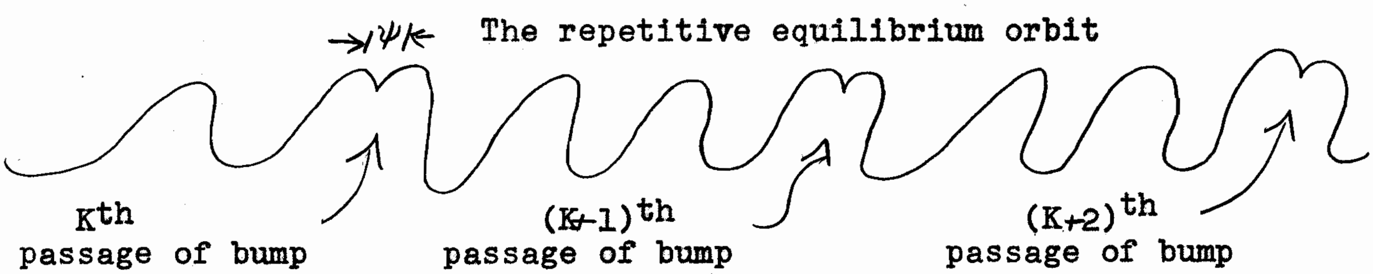


If now Δ H is held constant, the resultant rotates $2\pi N + \psi$ each revolution,



and then a is added to it to make the new resultant which has the same angle it had at the start

thus the motion repeats exactly on each revolution in a machine with a bump. The orbits connect like this:



N = 3 wavelengths around + ψ radians.

We now want to calculate the magnitude of X_o, the amplitude of oscillation.

(1) $X_o \doteq a/\psi = a/\omega_x T - 2\pi N_1 = a/\ T(\omega_x - N_1 \omega_\theta)$

T = time of revolution

ω_x= angular frequency of X oscillation

ω_θ= angular velocity of particle going around orbit.

We see the integral resonances resulting from zeros of the denominator.

We want to separate ω_x into 3 parts, a constant ω_1 which may be taken to be a resonant ω for the case of linear restoring forces, plus a $\delta(t)$ which describes the secular change in ω_x, and plus the function of X which gives the non-linear restoring forces. First calculate the term describing non-linear forces. If we have an oscillating orbit for linear restoring forces, the true orbit with non-linear forces drops away from it by a distance S which can be estimated as follows:

In $\ddot{X} + \omega_x^2 X + F(X) = 0$ break the acceleration into two parts $\ddot{X}_k + \ddot{X}_\ell$. Where $\ddot{X}_k + \omega_x^2 X = 0$ is the part of the equation giving the pure sine solution. The remnant,

$$S = \int_0^{t_1}\int_0^{t'} \ddot{X}_\ell \, dt\, dt' = -\int_0^{t_1}\int_0^{t'} F(X)\, dt\, dt'$$

gives approximately the over shoot, S, caused by the non-linear term in one period, t_1.

If $F(X) = \alpha x^3$ then

$$S = -\alpha X_0^3 \int_0^{t_1}\int_0^{t'} \sin^3\omega t \, dt\, dt' = -\frac{4\pi}{3}\frac{\alpha X_0^3}{\omega_x^2}$$

and $X_o \sin\Delta\psi = S$ where $\Delta\psi$ is the discrepancy of phase per oscillation caused by turning on the non-linear restoring force.

(2) $\Delta\psi \doteq S/X_o = -\frac{4\pi}{3} \quad \frac{\alpha X_0^2}{\omega_x^2}$ and in terms of change of ω,

(3) $\Delta\omega = \frac{2}{3} \frac{\alpha X_o^2}{\omega_x}$ this factor 2/3 (3/8 in next approximation)

is crucial in estimating the value of f which is needed to detune a secular change in n or ω. Thus $\omega_x = \omega_1 + \delta(t) + \frac{2}{3}\frac{\alpha X_0^2}{\omega_1}$ and

(1) becomes IF $\omega_1 = N_1 \ \omega_\theta$

(4) $X_o = a/T(\delta\,(t) + \frac{2}{3}\,\frac{\alpha X_o^2}{\omega_1} = \dfrac{a\omega_\theta\ /\omega_1}{2\pi(\frac{\delta(t)}{\omega_1} + \frac{2}{3}\ \frac{\alpha\ X_o^2}{\omega_1^2})}$

But $\omega_1 / \omega_\theta \doteq \sqrt{n}/2$

And $f = \alpha X_n^2\ /\omega_1^2$ where X_n is the maximum X_o or the point for which f is evaluated.

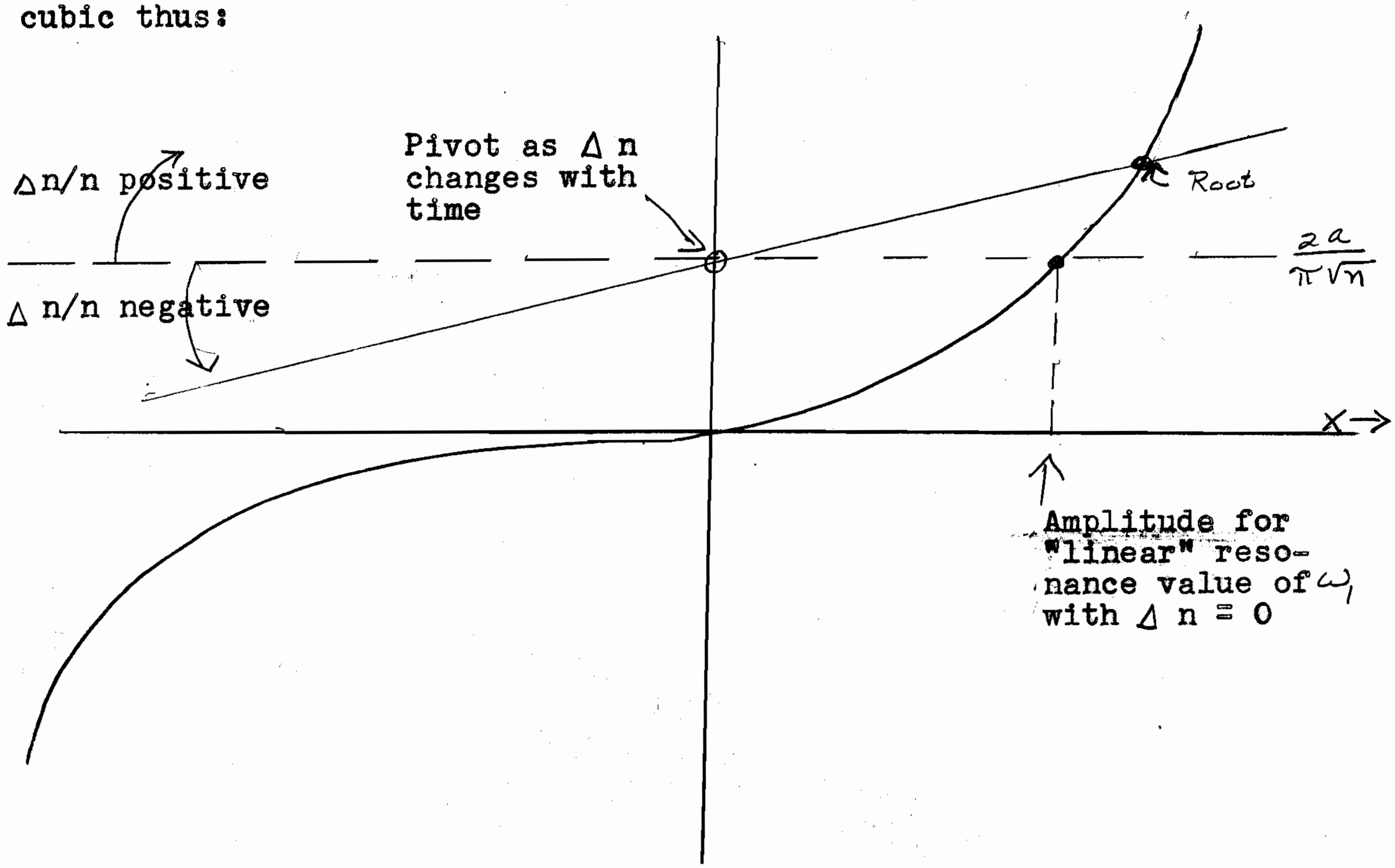
(5) Then $$X_o = \frac{2a}{\pi\sqrt{n}\left[\frac{\Delta\ n(t)}{n} + \frac{4}{3}\ f\frac{X_o^2}{X_n^2}\right]}$$

If $\Delta\ n > 0$, then one moves away from the resonance; but if $\Delta\ n < 0$, the resonance is approached for the case of $\Psi > 0$ in (1)

We want to see roots of the cubic equation

$$\frac{4}{3}\ \frac{f}{X_n^2}\ X_o^3 + \frac{\Delta\ n(t)}{n}\ X_o = \frac{2a}{\pi\sqrt{n}}$$

The roots are at the intersection of the straight line and the cubic thus:



and if at X_n we want a very modest amplitude, $X_o = 2a = X_n$ or $\psi \sim 30°$, then

$$f = + \frac{8}{6} \quad \frac{|\Delta n|}{n} + \frac{8}{6\,\pi\sqrt{n}}$$

So a 12% change of n would require only a 17% non-linear term.

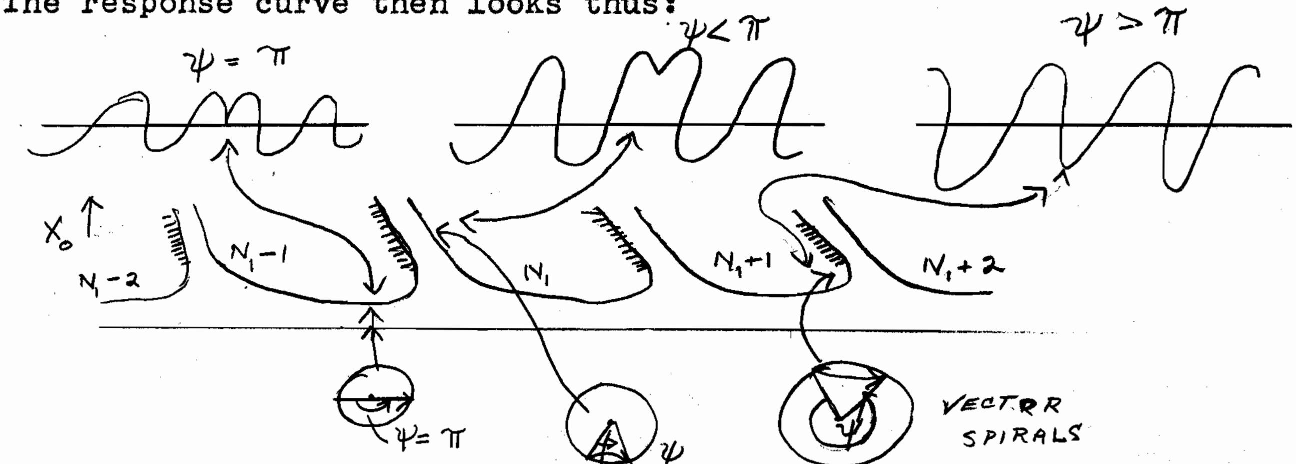
(See Joos "Theoretical Physics" p. 93 Non-Harmonic Vibrations.)

If F(X) is odd, we get either a lengthening or a shortening of the wavelength of the betatron oscillation; but if F(X) is an even function, the non-linear terms to a first approximation do not change ω_x.

As Δ n increases we reach points where ψ approaches 2π and where the number of wavelengths suddenly becomes $N_1 + 1$ or successively $N_1 + K$, K = 0, 1, 2, 3 ... and jumps back to a small angle. The whole multiple resonance response curve is given if we remenber that actually $X_o = \frac{a}{2 \sin \psi/2}$ instead of a/ψ then (5) becomes

$$(6) \qquad X_k = \frac{a}{2\sin\frac{\pi}{4}\sqrt{n_1}\left(\frac{\Delta n}{n_1} + \frac{4}{3}\frac{f X_o^2}{X_n^2} - \frac{4k}{\sqrt{n_1}}\right)}$$

and the slope of the linear line suddenly jumps back by $4/\sqrt{n_1}$ at each new resonance and the graph line continues rocking as Δ n progresses until the next jump back. There may be a phase oscillation (added free oscillation) generated at each jump since the energy taken from or given to the driving force source may not be exactly right during the jump. If Δ n is negative, no jumps are necessary. The response curve then looks thus:



The tendency will be for all oscillations to get trapped with the largest number of wavelengths because of synchrotron phase oscillations which carry the beam back and forth over regions of different n. Some of the features of this type of motion in non-linear fields were pointed out by H.R. Crane who stimulated the development of this subject.