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Twistor strings with flavour

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Abstract: We explore the tree-level description of a class of $\mathcal{N} = 2$ UV-finite SYM theories with fundamental flavour within a topological B-model twistor string framework. In particular, we identify the twistor dual of the Sp(N) gauge theory with one antisymmetric and four fundamental hypermultiplets, as well as that of the SU(N) theory with $2N$ hypermultiplets. This is achieved by suitably orientifolding/orbifolding the original $\mathcal{N} = 4$ setup of Witten and adding a certain number of new topological 'flavour'-branes at the orientifold/orbifold fixed planes to provide the fundamental matter. We further comment on the appearance of these objects in the B-model on $\mathbb{C}\mathrm{P}^{3|4}$. An interesting aspect of our construction is that, unlike the IIB description of these theories in terms of D3 and D7-branes, on the twistor side part of the global flavour symmetry is realised geometrically. We provide evidence for this correspondence by calculating and matching amplitudes on both sides.

Keywords: Chern-Simons Theories, Topological Strings, D-branes.

Contents

1. Introduction

Four-dimensional conformal field theories are relatively rare, and their existence depends crucially on the presence of a large amount of symmetry. The most celebrated example is $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory, which, especially via its strong-weak duality with IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ [1], has provided a very useful testbed for understanding the physics of strongly-coupled gauge theory. In this duality, the exact quantum conformal invariance of the theory is reflected in the AdS_5 factor of the string background, which encodes the unbroken four-dimensional conformal group $\mathrm{SO}(2,4)$ of the gauge theory.

A very different duality involving $\mathcal{N} = 4$ SYM was proposed by Witten in 2003 [2]. The idea stems from the fact that certain scattering amplitudes in Yang-Mills theory, when expressed in appropriate (spinor helicity) variables, turn out to take an unexpectedly simple form. This indicates that there might exist some reformulation in which this simplicity is evident, and in this context Witten proposed that it is useful to consider the open-string topological B-model on supertwistor space $\mathbb{CP}^{3|4}$. The isometries of $\mathbb{CP}^{3|4}$ capture the superconformal group $\mathrm{PSU}(2,2|4)$ of the gauge theory, and the spectrum of the string theory can be mapped to the field content of $\mathcal{N} = 4$ SYM via the Penrose transform [3].

In this framework, gluon scattering amplitudes can be calculated by noting that they are supported on certain simple algebraic curves in twistor space, the degree of which is linked to the number of external negative helicity gluons. For instance, Maximally Helicity Violating (MHV) amplitudes, which have two negative and any number of positive helicity gluons, are supported on degree one curves in $\mathbb{CP}^{3|4}$. In [2] it was proposed that these curves are wrapped by D1-instantons in the B-model, and, adapting a method originally due to Nair [4], it was shown that appropriately integrating over the moduli space of these D1-instantons leads to the correct expressions for tree-level amplitudes in $\mathcal{N} = 4$ SYM.

Beyond tree level, however, the situation is very different. Apart from difficulties in understanding the appropriate measure for higher-genus curves in supertwistor space, at one loop it seems that one cannot avoid unwanted contributions from the closed B-model sector which would correspond to conformal supergravity states in spacetime [5]. As the action for conformal supergravity is the square of the Weyl tensor whose kinetic term is fourth order in derivatives, it is generally believed to be non-unitary and thus a highly undesirable feature. Nonetheless, loop amplitudes in such a theory have been investigated [6] using an alternative twistor string theory due to Berkovits [7] and it is hoped that one might still be able to learn something about loop amplitudes in Yang-Mills this way.

Despite the above shortcoming, the application of twistor-inspired techniques to gauge theory has resulted in great progress in the understanding of perturbative field theory. At tree-level, the realisation that amplitudes localising on degree d curves can be equivalently calculated by integrating over the moduli space of d disconnected degree 1 curves [8 − 11], underlies the so-called MHV (or CSW) rules proposed by Cachazo, Svrček and Witten [11]. The CSW rules elevate tree-level MHV amplitudes to effective vertices, which are then glued together using simple scalar propagators to form tree amplitudes with successively greater numbers of negative helicity particles. Of particular interest is the fact that these techniques are applicable to a far larger class of theories than $\mathcal{N} = 4$ SYM, and include gauge theories with reduced or no supersymmetry and Einstein (super-)gravity - see [11 – 15] and references therein.

Even more remarkable is the fact that, despite the apparent failure of the twistor string duality at loop-level, the MHV rules *can* be straightforwardly applied at one loop in $\mathcal{N} = 4$ SYM [16], $\mathcal{N} = 2, 1$ SYM [17, 18], pure YM [19], a certain effective Higgs-YM action [20] and $\mathcal{N} = 8$ supergravity [21]. These results would seem to indicate that it is possible to overcome the current difficulties at one-loop and eventually extend Witten's prescription to the quantum level not only for $\mathcal{N} = 4$ SYM, but also for the other theories above. It is possible that such a dual string theory would have to be an appropriate (non-topological?) extension of the B-model, perhaps combined with a modification of the bosonic part of the target space geometry away from $\mathbb{CP}^3$ to reflect the fact that conformal invariance is typically lost at the quantum level. Finding such a quantum completion of the twistor string framework would certainly deepen our understanding of perturbative gauge theory.

As an intermediate step towards this goal, it is important to map out the range of four-dimensional theories that can potentially admit a twistor string description. If, to restrict the question somewhat, we insist that the full quantum theory have a perturbative string dual containing twistor space as part of the target manifold, we should clearly look among the known quantum conformally invariant theories, and, if we require that the conformal symmetry holds order-by-order in the coupling, we should focus in particular on the subset of the above which are finite. The hope is that, by explicitly constructing the twistor string duals of a wide range of such theories, which are expected to retain $\mathbb{CP}^3$ as part of the geometry at loop level, and by understanding why this construction might not work for other theories which look similar classically but which lack conformal invariance at the quantum level, one may learn something about the properties of the elusive quantum twistor string. In the process, one might also hope to gain further insight into the B-model twistor string description (or any of the several alternatives [7, 22]) even at tree-level.

Following this programme, it was shown in [23] (see also [24]) that the $\mathcal{N} = 1$ exactly marginal deformations of $\mathcal{N} = 4$ SYM can be incorporated into the B-model description by turning on a particular closed string mode, which (via a certain open/closed correlation function) effectively introduces non-anticommutativity between some of the fermionic co-ordinates of $\mathbb{CP}^{3|4}$. Another class of known finite 4d gauge theories are the quiver theories that arise as $\mathcal{N} = 1$ and $\mathcal{N} = 2$ orbifolds of $\mathcal{N} = 4$ SYM and in [25, 26] it was shown that these theories also admit a very natural twistor string description.[1]

In the present work we extend this investigation to other types of 4d gauge theories by including matter transforming in the fundamental representation. These are the $\mathcal{N} = 2$ SYM theories with gauge groups $\mathrm{Sp}(N)$ and $\mathrm{SU}(N)$, which are UV-finite when the number of flavours is $N_f = 4$ and $N_f = 2N$ respectively, and where the $\mathrm{Sp}(N)$ theory also contains

[1]Twistor string duals have also been constructed for truncations of self-dual $\mathcal{N} = 4$ SYM [27], lower dimensions [28 – 32], chiral mass terms [33] as well as for a number of gravity theories including $\mathcal{N} = 1, 2$ conformal supergravity [34, 35] and Einstein supergravity [36].

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a hypermultiplet in the antisymmetric representation. For brevity we will refer to these simply as the $N_f = 4$ and $N_f = 2N$ theories.

In direct analogy with the stringy description of the $N_f = 4$ gauge theory, in order to obtain a symplectic gauge group it will be necessary to perform an orientifold of the B-model on $\mathbb{CP}^{3|4}$. Similarly, for the $N_f = 2N$ theory we will perform an orbifold projection. Given the similarities of these techniques with previous orbifold constructions of [25, 26], the above steps are relatively straightforward. The main novelty, compared to the previous twistor string literature, is the presence of the fundamental flavours. We propose a natural mechanism to incorporate this sector of the theory, leading to an additional term in the B-model action, and show that the tree-level twistor string amplitudes precisely match those calculated on the gauge theory side.

A parallel promising development in the twistor string programme has been the introduction of effective actions on twistor space [37−39], which extend Witten's holomorphic Chern-Simons (hCS) action and, after appropriate gauge fixing, reproduce the 4d MHV-rules prescription for Yang-Mills theory.[2] By construction, this approach does not suffer from the conformal supergravity problem. It is not yet known whether such actions can be derived from a more fundamental (B-model or alternative) string description (in particular, they do not seem to arise from simple summation over the effects of D-instantons). Such an effective action for either of the $\mathcal{N} = 2$ theories that we will consider in this work, constructed by inserting the relevant matter multiplets (as described in [37]), and choosing the gauge group appropriate for each case, would provide an alternative way to reproduce the MHV amplitudes we will calculate. However, we do not follow that path since via such an approach we would not expect to gain insight into the novel features that arise when introducing fundamental flavours from a *topological string* point of view. Nevertheless, as we will point out, some aspects of our construction will turn out to be similar to those in [37].

The rest of this paper is organised as follows: In Section 2 we discuss some preliminary details related to formulating the spacetime action for the $N_f = 4$ theory. We then review Witten's construction of the twistor string for $\mathcal{N} = 4$ SYM and proceed to give the equivalent description for the theory under present study in section 3. In section 4 we elaborate on the comparison between amplitudes calculated from the spacetime and twistor points of view and demonstrate the agreement between the two pictures with a number of specific examples. Section 5 extends the above to the case with $N_f = 2N$. We describe the construction of the spacetime action, obtain the dual twistor string description and finally match the two by comparing amplitude ratios. We conclude in section 6 with a discussion of our results and directions for future research.

2. Preliminaries for the $N_f = 4$ theory

The aim of this section is to collect known facts on the $N_f = 4$ theory and it symmetries, before moving on to considering its spacetime action. It is easy to check that the matter

[2] Recently, some aspects of this formalism were extended to self-dual $\mathcal{N} = 8$ supergravity [40].

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content of this $\mathcal{N}=2$, Sp(N) theory (one hypermultiplet in the antisymmetric representation of Sp(N) and four hypermultiplets in the fundamental) is such that the one-loop β-function vanishes [41, 42]. Since $\mathcal{N}=2$ supersymmetry implies one-loop exactness of the β-function [43], perturbative finiteness is guaranteed. In the rank one case, where the gauge group reduces to SU(2) and there is no antisymmetric hypermultiplet, this theory was considered by Seiberg and Witten [44, 45], who found (for arbitrary hypermultiplet masses) the curve describing its low energy dynamics. In the massless case, these results can be used to argue that the gauge coupling does not run even at the nonperturbative level.[3] One of the intriguing outcomes of [45] was the conjecture that the $N_f=4$ theory enjoys an analogue of the Montonen-Olive (electric-magnetic) duality of $\mathcal{N}=4$ SYM, in which SL$(2,\mathbb{Z})$ mixes in a nontrivial way with SO(8) triality to produce a duality-invariant spectrum.

2.1 Review of the IIB/F-theory embedding

The $N_f=4$ theory has a very useful realisation in terms of a physical string theory description, which first arose in Sen's explorations of F-theory [47] on $K3$ [48]. In particular, Sen considered a special elliptically fibred $K3$, the orbifold $T^4/\mathbb{Z}_2$, realised as a T^2 fibration over the base $T^2/\mathbb{Z}_2$. Requiring that the axion-dilaton modulus have no dependence on the internal torus, this configuration reduces to an orientifold [49] of type IIB[4] on T^2, and thus produces four orientifold fixed planes, each carrying -4 units of D7-brane charge. Constancy of the axion-dilaton requires that four D7-branes (along with their mirrors) be placed on each orientifold plane, resulting in an SO$(8)^4$ non-abelian gauge symmetry. This type IIB setup can also be obtained from the type I string by a T-duality on both coordinates of the base [48].

Sen then argued that the F-theory moduli space close to one of the orbifold fixed points, where T^2 locally reduces to $\mathbb{R}^2$, can be accurately described by the physics of the 4d $\mathcal{N}=2$, SU(2) Seiberg-Witten theory with four fundamental hypermultiplets. Moreover, Banks et al. [52] showed that this gauge theory can be naturally realised as the low energy effective theory on the worldvolume of a probe D3-brane in the limit where the rest of the orientifold singularities are taken to be very far away, and the moduli space of the theory is captured by the dynamics of the worldvolume fields. By considering multiple coincident D3-branes as probes [53, 54] the SU(2) $\cong$ Sp(1) gauge group can be extended to higher rank to obtain an Sp(N) gauge theory, at the expense of introducing an extra hypermultiplet in the antisymmetric representation of Sp(N).[5]

Let us summarise the setup and field content of the above physical-string configuration: We will consider the low energy worldvolume action on a stack of N coincident D3-branes (and their mirrors) living in the $(x^0,\ldots,x^3)$ directions. These probe the background generated by 4 D7s (and their mirrors) and a single O7-plane lying in $(x^0,\ldots,x^7)$.

[3]Certain discrepancies in matching the results of [45] to explicit instanton calculations were resolved in [46].

[4]For related reviews on brane dynamics in the presence of orientifolds see e.g. [50, 51].

[5]The alternative extension to SU(N) will be discussed in section 5. However, it is not the natural generalisation from the F-theory point of view.

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Component	SO(1,3)	SU(2)$_a$	SU(2)$_A$	U(1)$_R$	Sp(N)	SO(8)
A, G	$(2,2)$	1	1	0	$N(2N+1)$	1
ϕ	$(1,1)$	1	1	$+2$	$N(2N+1)$	1
$\phi^\dagger$	$(1,1)$	1	1	-2	$N(2N+1)$	1
$\lambda_{\alpha,a}$	$(2,1)$	2	1	$+1$	$N(2N+1)$	1
$\bar{\lambda}_{\dot{\alpha},a}$	$(1,2)$	2	1	-1	$N(2N+1)$	1
z_{aA}	$(1,1)$	2	2	0	$N(2N-1)-1$	1
$\zeta_{\alpha,A}$	$(2,1)$	1	2	-1	$N(2N-1)-1$	1
$\bar{\zeta}_{\dot{\alpha},A}$	$(1,2)$	1	2	$+1$	$N(2N-1)-1$	1
q_a^M	$(1,1)$	2	1	0	$2N$	8
η_α^M	$(2,1)$	1	1	-1	$2N$	8
$\bar{\eta}_{\dot{\alpha}}^M$	$(1,2)$	1	1	$+1$	$2N$	8

Table 1: The on-shell field content of the $N_f = 4$ theory in component form. The representations in the first column are actually in terms of the Euclidean Lorentz group SO(4) $\sim$ SU(2)$_L \times$ SU(2)$_R$. The fundamental fields carry an SO(8) flavour index $M = 1, \ldots, 8$, while the antisymmetric fields an SU(2) 'flavour' index $A = 1, 2$. Note that $(z, \zeta, \bar{\zeta})$ transform in the irreducible second-rank antisymmetric representation of Sp(N), which in the text we call "antisymmetric" for brevity. We write SO(8) rather than the more accurate O(8) since we will not keep track of discrete groups.

The orientifold plane is added in such a way so as to preserve the same 8 supersymmetries as the D3-D7 system and the 3-3 and 7-7 strings would generate respective SU($2N$) and SU(8) gauge symmetries. However, since all the branes are sitting at the orientifold fixed plane, these project to Sp(N) and SO(8) because of the orientation reversal action on the open string Chan-Paton indices, which imposes symmetric or antisymmetric conditions on the gauge group matrices. Ramond-Ramond (RR) tadpole cancellation further restricts one to only retain antisymmetric matrices for the D7s; one is then forced to consider symmetric matrices for the D3s [49].

In the low-energy limit, the dynamical fields corresponding to 7-7 strings decouple and SO(8) becomes a global symmetry of the system. The massless spectrum of 3-3 strings fluctuating in the worldvolume $(x^0, \ldots, x^3)$ and overall transverse (x^8, x^9) directions yields the degrees of freedom corresponding to the $\mathcal{N} = 2$ vector multiplet in the adjoint (symmetric) representation of Sp(N). The fluctuations in the directions relatively transverse to the D3s $(x^4, \ldots, x^7)$ furnish a hypermultiplet transforming in the antisymmetric tensor representation of the gauge group, which captures the motion of the D3s in these directions. Therefore, the low energy D3 worldvolume action describes 4d $\mathcal{N} = 2$ SYM with gauge group Sp(N), four hypermultiplets in the fundamental and one in the antisymmetric representation, sitting at the conformal point of its moduli space.

As far as the global symmetries are concerned, the presence of the D7-branes breaks the D3 transverse group of rotations down to SO(4) $\times$ U(1)$_R \subset$ SO(6). Furthermore, we write this SO(4) as SU(2)$_a \times$SU(2)$_A$, SU(2)$_A$ being a flavour-like symmetry for the antisymmetric fields; no other field transforms nontrivially under its action. The rest of the SO(6) global symmetry subgroup accounts for the $\mathcal{N} = 2$ R-symmetry, U(2)$_R \cong$ SU(2)$_a \times$ U(1)$_R$ and we

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remind the reader that the fundamental fields transform as vectors under the global SO(8) flavour group. The precise transformation properties of all degrees of freedom under the symmetries of the system are summarised in table 1, which is adapted from [55].

By considering a large number of coincident D3-branes and taking their near-horizon limit, it is possible to obtain the supergravity dual of the $N_f = 4$ theory in terms of strings in $\mathrm{AdS}_5 \times \mathrm{S}^5/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is an orientifold action on the S^5 [56, 57] (see also [58]). Instanton effects in the AdS/CFT context have been studied in [59, 55, 60], while the plane-wave limit of the theory has been investigated in [61, 62]. Higher derivative corrections were considered in [63, 64], and the geometry of the holographic dual of the Higgs branch of the theory was described in [65]. Recently, [66] used the AdS/CFT dual to discuss the behaviour of *strongly coupled* $N_f = 4$ scattering amplitudes.

2.2 The spacetime action

We now turn to the construction of a Lagrangean for the above $\mathcal{N} = 2$ theory by taking its formulation in terms of $\mathcal{N} = 1$ superfields as a starting point.[6] This reads

$$\begin{aligned}\mathcal{L} =& \frac{1}{8\pi}\mathrm{Im}\ \mathrm{Tr}\left[\tau\left(\int d^2\theta\ W^\alpha W_\alpha + 2\int d^2\theta d^2\bar{\theta}\ e^{2V}\Phi^\dagger e^{-2V}\Phi\right)\right] + \int d^2\theta d^2\bar{\theta}\ Q^{\dagger I}e^{-2V}Q_I \\ &+ \int d^2\theta d^2\bar{\theta}\ Q'^I e^{2V} Q'^\dagger_I + \mathrm{Tr}\left(\int d^2\theta d^2\bar{\theta}\ e^{2V}Z^\dagger e^{-2V}Z + \int d^2\theta d^2\bar{\theta}\ e^{-2V}Z' e^{2V} Z'^\dagger\right) \\ &+ \sqrt{2}\left(\int d^2\theta (Q'^I \Phi Q_I + \mathrm{Tr}\left(Z'[\Phi, Z]\right)) + h.c.\right)\ .\end{aligned} \tag{2.1}$$

The $\mathcal{N} = 2$ vector multiplet consists of the $\mathcal{N} = 1$ vector and chiral superfields (V, Φ), the antisymmetric hypermultiplet of the chiral and antichiral $(Z, Z'^\dagger)$ and the four fundamental hypermultiplets of the four chiral and four antichiral superfields $(Q^I, Q'^{\dagger I})$ respectively. $(Q^{\dagger I}, Q'^I)$ are four antichiral and chiral superfields transforming in the conjugate fundamental representation and the SU(4) flavour index I runs from 1 to 4. The fundamental representation of Sp(N) is pseudoreal, which means that it is related to its conjugate simply by raising and lowering indices. The flavour symmetry is thus enhanced to SO(8). However in this $\mathcal{N} = 1$ notation this SO(8) flavour symmetry is not explicit. It is instead implicitly realised via the subgroup $\mathrm{SU}(4) \times \mathrm{U}(1) \subset \mathrm{SO}(8)$ and the decomposition $\mathbf{8}_\mathrm{s} = \mathbf{4_1} + \bar{\mathbf{4}}_{\mathbf{-1}}$, which reflects the fact that we are considering four kinds of 3-7 and 7-3 strings. Also hidden in (2.1) is the $\mathrm{SU}(2)_A$ symmetry, which we will restore in due course together with explicit SO(8) invariance. Lastly, the $\mathrm{SU}(2)_a$ part of the $\mathcal{N} = 2$ R-symmetry is also not manifest at this stage. The complexified coupling is $\tau = \frac{\Theta_{YM}}{2\pi} + \frac{4\pi i}{g^2}$ but, since we are only interested in the perturbative behaviour of the theory, we can safely set the total derivative terms to zero by requiring that $\Theta_{YM} = 0$. We will also ignore any total derivative terms coming from integration by parts.

In component form we have $(A^\mu, \lambda, \mathsf{D})$ for V, $(\phi, \chi, \mathsf{F}_\phi)$ for Φ and (q, η, F_q) for Q, with similar superfield expansions for $Q'^\dagger$, Z and $Z'^\dagger$. Since we are constructing this $\mathcal{N} = 2$

[6] Our notation and conventions are summarised in appendix A. General reviews of superspace techniques and $\mathcal{N} = 2$ supersymmetric gauge theories can be found, for instance, in [67 – 71].

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action out of $\mathcal{N}=1$ quantities, the coupling appearing in front of the superpotential terms can, in principle, be different to the coupling of the kinetic terms for the $\mathcal{N}=2$ vector multiplet. However, $\mathcal{N}=2$ supersymmetry requires that they all be equal [68]. After expanding the superfields and performing the Grassmann integration one obtains the expression

$$
\begin{aligned}
\mathcal{L} =& \frac{1}{g^2}\mathrm{Tr}\left(-\frac{1}{4}F^2 + (D^\mu\phi)^\dagger(D_\mu\phi) - i\lambda\ \not{D}\bar{\lambda} - i\bar{\chi}\not{D}\chi - i\sqrt{2}\,[\lambda,\chi]\phi^\dagger - i\sqrt{2}\,[\bar{\lambda},\bar{\chi}]\phi\right)\\
&+ (D^\mu q)^{\dagger I}(D_\mu q)_I + (D^\mu q')^I(D_\mu q')^\dagger_I - i\bar{\eta}^I\not{D}\eta_I - i\eta'^I\not{D}\bar{\eta}'_I - i\sqrt{2}\,q^{\dagger I}\lambda\eta_I\\
&+ i\sqrt{2}\,\bar{\eta}^I\bar{\lambda}q_I - i\sqrt{2}\,q'^I\bar{\lambda}\bar{\eta}'_I + i\sqrt{2}\,\eta'^I\lambda q'^\dagger_I + \mathrm{Tr}\Big((D^\mu z)^\dagger(D_\mu z) + (D^\mu z')(D_\mu z')^\dagger\\
&- i\bar{\zeta}\not{D}\zeta - i\zeta'\not{D}\bar{\zeta}' - i\sqrt{2}\,[\lambda,\zeta]z^\dagger - i\sqrt{2}\,[\bar{\lambda},\bar{\zeta}]z - i\sqrt{2}\,[\bar{\lambda},\bar{\zeta}']z' - i\sqrt{2}\,[\lambda,\zeta']z'^\dagger\Big)\\
&- \sqrt{2}\left[\left(\eta'^I\chi q_I + \eta'^I\phi\eta_I + q'^I\chi\eta_I\right) + \mathrm{Tr}\left(-[\chi,\zeta']z + \zeta'[\phi,\zeta] + [\chi,\zeta]z'\right) + h.c.\right] - V_S\,,
\end{aligned}
\tag{2.2}
$$

where our convention for the covariant derivative is $D_\mu = \partial_\mu - iA^a_\mu T^a_R$ and $\not{D}^{\dot{\alpha}\alpha} = (\bar{\sigma}^{\dot{\alpha}\alpha})^\mu D_\mu$. V_S is the scalar potential obtained by integrating out the F- and D-terms. It is given by

$$
V_S = \mathsf{F}^\dagger_q\mathsf{F}_q + \mathsf{F}_{q'}\mathsf{F}^\dagger_{q'} + \mathrm{Tr}\left(\mathsf{F}^\dagger_z\mathsf{F}_z + \mathsf{F}_{z'}\mathsf{F}^\dagger_{z'} + \frac{1}{g^2}\mathsf{F}^\dagger_\phi\mathsf{F}_\phi\right) + \frac{1}{2g^2}\mathsf{D}^2\,, \tag{2.3}
$$

where the individual terms with their index structure made explicit are

$$
\begin{aligned}
(\mathsf{F}_q)^i_I &= -\sqrt{2}\,(\phi^\dagger)^i{}_j q'^{\dagger j}_I\,, \quad (\mathsf{F}_{q'})^I_i = -\sqrt{2}\,q^{\dagger I}_j(\phi^\dagger)^j{}_i\\
(\mathsf{F}_z)^i{}_j &= -\sqrt{2}\left[\phi^\dagger, z'^\dagger\right]^i{}_j\,, \quad (\mathsf{F}_{z'})^i{}_j = -\sqrt{2}\left[z^\dagger,\phi^\dagger\right]^i{}_j
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
(\mathsf{F}_\phi)^i{}_j &= -g^2\sqrt{2}\left[z'^\dagger, z^\dagger\right]^i{}_j - \frac{g^2}{\sqrt{2}}\left(q'^{\dagger i}_I q^{\dagger I}_j + q'^\dagger_{Ij}q^{\dagger Ii}\right)\\
\mathsf{D}^a &= -\mathrm{Tr}\left(T^a[\phi^\dagger,\phi] + g^2T^a[z^\dagger,z] - g^2T^a[z',z'^\dagger]\right) + g^2\left(q^{\dagger I}T^a q_I - q'^I T^a q'^\dagger_I\right)\,.
\end{aligned}
\tag{2.5}
$$

The $(T^a)^i{}_j$'s are the generators of the fundamental representation of $\mathrm{Sp}(N)$ and in obtaining the full scalar potential one also needs to make use of the following identity

$$
(T^a)^i{}_j(T_a)^k{}_l = \frac{1}{2}(\delta^i_l\delta^k_j - \Omega^{ik}\Omega_{jl})\,. \tag{2.6}
$$

To further reorganise the action (2.2), we recall that the twistor approach to gauge theory amplitudes breaks the symmetry between positive and negative helicity states [2]. Here we implement this by splitting the action into a piece independent of the gauge coupling and another piece which is of order g^2. This is done by performing a series of rescalings which read as follows: For the adjoint fields we have

$$
(\phi,\phi^\dagger) \to \left(ig\sqrt{2}\phi, -\frac{ig}{\sqrt{2}}\phi^\dagger\right) \quad , \quad (\lambda,\bar{\lambda},) \to (g^{1/2}\lambda, g^{3/2}\bar{\lambda},) \quad , \quad (\chi,\bar{\chi}) \to (g^{1/2}\chi, g^{3/2}\bar{\chi})\,, \tag{2.7}
$$

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for the antisymmetric ones

$$\begin{aligned}
&(z, z^\dagger) \to (z, z^\dagger), && (z', z'^\dagger) \to (iz', -iz'^\dagger) \\
&(\zeta, \bar{\zeta}) \to \Big(-\frac{i\zeta}{g^{1/2}\sqrt{2}}, ig^{1/2}\sqrt{2}\bar{\zeta} \Big), && (\zeta', \bar{\zeta}') \to \Big(\frac{\zeta'}{g^{1/2}\sqrt{2}}, g^{1/2}\sqrt{2}\bar{\zeta}' \Big),
\end{aligned} \tag{2.8}$$

while for the fundamentals

$$\begin{aligned}
&(q_I, q^{\dagger I}) \to (q_I, q^{\dagger I}), && (q'^I, q'^\dagger_I) \to (iq'^I, -iq'^\dagger_I) \\
&(\eta_I, \bar{\eta}^I) \to \Big(-\frac{i\eta_I}{g^{1/2}\sqrt{2}}, ig^{1/2}\sqrt{2}\bar{\eta}^I \Big), && (\eta'^I, \bar{\eta}'_I) \to \Big(\frac{\eta'^I}{g^{1/2}\sqrt{2}}, g^{1/2}\sqrt{2}\bar{\eta}'_I \Big).
\end{aligned} \tag{2.9}$$

We will also make the symmetries of table 1 explicit by appropriately arranging the antisymmetric fields into $\mathrm{SU}(2)_A$ doublets and collecting the fundamentals into SO(8) spinors (which can be exchanged for vectors by SO(8) triality). We finally collect the hypermultiplet scalars and the adjoint fermions into doublets of $\mathrm{SU}(2)_a$. The above statements are summarised by the definitions

$$\begin{aligned}
\lambda^a &= \begin{pmatrix} \lambda \\ -\chi \end{pmatrix}, \qquad \bar{\lambda}_a = \big(\bar{\lambda}, -\bar{\chi}\big), \quad \bar{\lambda}^a = \begin{pmatrix} -\bar{\chi} \\ -\bar{\lambda} \end{pmatrix}, \quad \lambda_a = (\chi, \lambda) \\
\eta^M &= \begin{pmatrix} \eta_I \\ \eta'^I \end{pmatrix}, \qquad \bar{\eta}_M = \big(\bar{\eta}^I, \bar{\eta}'_I\big), \quad \bar{\eta}^M = \begin{pmatrix} \bar{\eta}'_I \\ \bar{\eta}^I \end{pmatrix}, \quad \eta_M = \big(\eta'^I, \eta_I\big) \\
\bar{\zeta}^A &= \begin{pmatrix} \bar{\zeta} \\ \bar{\zeta}' \end{pmatrix}, \qquad \zeta_A = \big(\zeta, \zeta'\big), \quad \zeta^A = \begin{pmatrix} \zeta' \\ -\zeta \end{pmatrix}, \quad \bar{\zeta}_A = \big(-\bar{\zeta}', \bar{\zeta}\big) \\
z^a{}_A &= \begin{pmatrix} z & z' \\ -z'^\dagger & z^\dagger \end{pmatrix}, \quad z^A{}_a = \begin{pmatrix} z^\dagger & -z' \\ z'^\dagger & z \end{pmatrix} \\
q^a{}_M &= \begin{pmatrix} -q'^I & q'^\dagger_I \\ -q^{\dagger I} & q_I \end{pmatrix}, \quad q^M{}_a = \begin{pmatrix} -q'^\dagger_I & -q_I \\ -q^{\dagger I} & -q'^I \end{pmatrix}.
\end{aligned} \tag{2.10}$$

Having made the SO(8) flavour symmetry of the fundamental fields manifest in terms of components, we can also collect them into 8 $\mathcal{N} = 2$ 'half-hypermultiplets' $Q_M = (\bar{\eta}_M, q^a_M, \eta_M)$, each of which contains two bosonic and two fermionic fields. This type of multiplet arises only for pseudoreal representations, allowing a description in terms of half the usual field content of $\mathcal{N} = 2$ supersymmetry [72, 73, 68]. Note that it is not possible to have a description in terms of full $\mathcal{N} = 2$ hypermultiplets that manifestly preserves the SO(8).

For the gauge field we introduce an anti-selfdual two-form $G^{\mu\nu}$ as a Lagrange multiplier, via which (up to a topological term which will not play a role in our perturbative study) we can rewrite the Yang-Mills action in first order form [74]

$$-\frac{1}{4g^2}\mathrm{Tr}F^{\mu\nu}F_{\mu\nu} \to -\frac{1}{2}\mathrm{Tr}\left(G_{\mu\nu}F^{\mu\nu} - \frac{1}{2}g^2 G_{\mu\nu}G^{\mu\nu} \right). \tag{2.11}$$

The final expression for the action, including the full quartic contributions arising from the scalar potential, takes the form

$$
\begin{aligned}
\mathcal{L} =& \mathrm{Tr}\left[-\frac{1}{2}GF+\frac{1}{4}g^2G^2+D\phi^\dagger D\phi+i\bar\lambda^a\,\not{D}\lambda_a-\lambda^a\lambda_a\phi^\dagger+2g^2\bar\lambda^a\bar\lambda_a\phi\right]+\mathrm{Tr}\left[\frac{1}{2}Dz^a{}_ADz^A{}_a\right.\\
&\left.+i\bar\zeta^A\,\not{D}\zeta_A-z^a{}_A[\lambda_a,\zeta^A]-2g^2z^A{}_a[\bar\zeta_A,\bar\lambda^a]+\zeta^A\zeta_A\phi-2g^2\bar\zeta^A\bar\zeta_A\phi^\dagger\right]+\frac{1}{2}Dq^a{}_MDq^M{}_a\\
&-i\bar\eta_M\,\not{D}\eta^M+q^a{}_M\lambda_a\eta^M-\frac{1}{2}\eta_M\phi\eta^M-2g^2\left(\bar\eta_M\bar\lambda^aq^M{}_a+\frac{1}{2}\bar\eta_M\phi^\dagger\bar\eta^M\right)\\
&+g^2\left(-\frac{1}{2}q^a{}_M\{\phi^\dagger,\phi\}q^M{}_a+\frac{1}{4}q^a{}_M[z^b{}_A,z^A{}_a]q^M{}_b\right)-\frac{g^2}{8}\Big((q^a{}_Mq^N{}_a)(q^b{}_Nq^M{}_b)\\
&+(q^a{}_Mq^b{}_N)(q^N{}_aq^M{}_b)\Big)-g^2\,\mathrm{Tr}\left(\frac{1}{2}[\phi^\dagger,\phi]^2+\frac{1}{4}[z^a{}_A,z^A{}_b][z^b{}_B,z^B{}_a]+[z^a{}_A,\phi][\phi^\dagger,z^A{}_a]\right)\,.
\end{aligned}
\tag{2.12}
$$

By taking the $g\to 0$ limit one obtains the 'selfdual' truncation of the Lagrangean, which has the same field content but only a subset of the interactions of the full theory. The $\mathcal{O}(g^2)$ terms can be thought of as perturbations around the selfdual theory.

In anticipation of the twistor approach, we will perhaps surprise the reader by once again hiding the global SO(8) symmetry that we just made manifest. This is done by decomposing the flavour index $M\to A'\otimes X$ according to the special maximal embedding $\mathrm{SO}(8)\supset\mathrm{SU}(2)_{A'}\times\mathrm{Sp}(2)$ where the indices run over $A'=1,2$ and $X=1,\dots,4$. One motivation for this is that each doublet indexed by A' has the field content of a full $\mathcal{N}=2$ hypermultiplet, but the main reasoning behind it will become clear in the next section. In the interim — and to facilitate comparison with the twistor analysis — we include the action for this selfdual truncation, which takes the simple form

$$
\begin{aligned}
\mathcal{L} =& \mathrm{Tr}\left[-\frac{1}{2}GF+D\phi^\dagger D\phi+i\bar\lambda^a\,\not{D}\lambda_a-\lambda^a\lambda_a\phi^\dagger\right]\\
&-\mathrm{Tr}\left[\frac{1}{2}Dz^{aA}Dz_{aA}+i\bar\zeta^A\,\not{D}\zeta_A+z^{aA}[\lambda_a,\zeta_A]+\zeta^A\zeta_A\phi\right]\\
&-\left(\frac{1}{2}Dq_{aA'X}Dq^{aA'X}+i\bar\eta_{A'X}\,\not{D}\eta^{A'X}+q_{aA'X}\lambda^a\eta^{A'X}+\frac{1}{2}\eta_{A'X}\phi\eta^{A'X}\right)\,.
\end{aligned}
\tag{2.13}
$$

This action is the $N_f=4$ analogue of the selfdual truncation of $\mathcal{N}=4$ SYM introduced by Siegel [75].

3. Twistor strings

We now turn our attention to constructing a twistor string dual to the gauge theory we have just described. To begin with, we will briefly review the most relevant parts of the twistor string dual for $\mathcal{N}=4$, SU(N) gauge theory in four dimensions [2],[7] subsequently modifying it appropriately for the $N_f=4$ case.

[7] To be precise, Witten studied the U(N) theory but since that only involved gluon amplitudes it is essentially the same to consider SU(N); the gluons don't couple to the U(1) 'photon'. Moreover, when considering colour-stripped amplitudes the extra U(1) piece will not affect the results, even for external scalars or fermions.

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3.1 The open B-model

In this section we collect a few well-known facts on the open topological B-model which will be useful in what follows. This is not intended to be a thorough review, for which we refer the reader to e.g. [76 – 79]. The B-model [80] arises as an axial-type twisting of the $\mathcal{N} = (2,2)$ supersymmetric 2d nonlinear σ-model, which turns out to only be consistent when the target space is Calabi-Yau. For a bosonic target space, the worldsheet field content of the theory consists of bosonic scalars $\phi^m, \phi^{\bar{m}}$ providing the map to the target manifold, and ghost-number one fermions $\eta^{\bar{m}}$ and θ_m (plus a worldsheet one-form $\rho^m_{\bar{z}}$ which will not play a role in our analysis). The action of the BRST charge Q_B on these fields is such that it can be precisely mapped to the Dolbeault operator $\bar{\partial}$ on the target space Calabi-Yau, and this identification leads to the following well-known relations between worldsheet fields and the geometry of the target space

$$\phi^m \sim Z^m \ , \ \phi^{\bar{m}} \sim \bar{Z}^{\bar{m}} \ , \ \eta^{\bar{m}} \sim d\bar{Z}^{\bar{m}} \ , \ \theta_m \sim \frac{\partial}{\partial Z^m} \ . \tag{3.1}$$

We will only be interested in the BRST transformations of these fields in the presence of a boundary, which are given by [81]

$$\delta_B \phi^m = 0 \ , \quad \delta_B \phi^{\bar{m}} = i\alpha\eta^{\bar{m}} \ , \quad \delta_B \eta^{\bar{m}} = 0 \ , \quad \delta_B \theta_m = 0 \ . \tag{3.2}$$

This implies (see e.g. [82, 83]) that imposing Neumann boundary conditions along a particular holomorphic direction (say m) requires that $\theta_m = 0$, while imposing Dirichlet directions along an antiholomorphic direction $\bar{m}$ leads to $\eta^{\bar{m}} = 0$.

A generic open string vertex operator, giving rise to a local observable, can be written as

$$\mathcal{V} = \theta_{m_1} \cdots \theta_{m_p} \eta^{\bar{n}_1} \cdots \eta^{\bar{n}_q} V(\phi, \bar{\phi})^i{}_j{}^{m_1 \cdots m_p}{}_{\bar{n}_1 \cdots \bar{n}_q} \tag{3.3}$$

where i,j denote the Chan-Paton indices. BRST invariance of this operator requires that V be a $(0,q)$-form with values in $\wedge^p T^{(1,0)}$ (times the Chan-Paton group). Since *physical* open string vertex operators arise at ghost number one, in practice one needs to consider two types of vertex operators

$$\text{(a)} \ \ \mathcal{V} = \eta^{\bar{m}} V^i{}_{j\bar{m}} \qquad \text{and} \qquad \text{(b)} \ \ \mathcal{V}' = \theta_m V'^i{}_j{}^m \ . \tag{3.4}$$

Recalling the identifications in (3.1), we see that these states correspond to either matrix-valued $(0,1)$-forms or tangent vectors on the target manifold. Therefore, when considering space-filling ('D5') branes on the Calabi-Yau [81], by imposing Neumann-Neumann (NN) boundary conditions on all open strings, the physical open string spectrum is just given by a $(0,1)$ form $\mathcal{A} = \mathrm{d}\bar{Z}^{\bar{m}} \mathcal{A}_{\bar{m}}$. The target space interactions can be encoded in the cubic holomorphic Chern-Simons theory

$$S = \frac{1}{2} \int_{\mathrm{CY}} \mathbf{\Omega} \wedge \mathrm{Tr}\Big(\mathcal{A} \cdot \bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big) \tag{3.5}$$

which is written with the help of the $(3,0)$ holomorphic volume form of the Calabi-Yau.

In the following we will assume the straightforward generalisation of the above statements to the super-Calabi-Yau case.

3.2 Review of the dual for $\mathcal{N}=4$ SYM

In [2], Witten showed that the tree level n-gluon MHV amplitudes, that is the amplitudes with $n-2$ positive and 2 negative helicity gluons (when all external particles are taken to be outgoing) can be reconstructed from an open string theory in supertwistor space. Essential to this was the observation that these amplitudes localise on holomorphically embedded, degree-one curves of genus zero in $\mathbb{CP}^{3|4}$, and the string theory in question is the open string sector of the topological B-model with $\mathbb{CP}^{3|4}$ target space, which is well defined since the latter is a super-Calabi-Yau. The isometries of $\mathbb{CP}^{3|4}$ encode the PSU$(2,2|4)$ superconformal symmetry of the $\mathcal{N}=4$ theory in a linear way, while the open string sector is realised by introducing Euclidean 'D5'-branes wrapping the bosonic directions of $\mathbb{CP}^{3|4}$ $(Z,\bar{Z})$ but only the holomorphic part of the fermionic directions ψ^I $(I=1,\ldots,4)$. This can be interpreted as a localisation of the D5s in the transverse fermionic coordinates and in [2] this locus was taken to be at $\bar{\psi}^{\bar{I}}=0$. Since this imposes Dirichlet boundary conditions only on the antiholomorphic fermionic directions $\bar{\psi}^{\bar{I}}$ (which would not have been possible had they been bosonic), it follows that $\theta_m,\theta_I=0$ and from (3.4) we see that the only physical field is a nonabelian $(0,1)$-form $\mathcal{A}=d\bar{Z}^{\bar{m}}\mathcal{A}(Z,\bar{Z},\psi)_{\bar{m}}$, which in addition is independent of the antiholomorphic fermionic coordinates. Therefore the superfield expansion of $\mathcal{A}$ is

$$\mathcal{A}=A+\psi^I\lambda_I+\frac{1}{2!}\psi^I\psi^J\phi_{IJ}+\frac{1}{3!}\epsilon_{IJKL}\psi^I\psi^J\psi^K\tilde{\lambda}^L+\frac{1}{4!}\epsilon_{IJKL}\psi^I\psi^J\psi^K\psi^L G\,, \tag{3.6}$$

where we will from now on suppress the gauge indices and form structure.

As mentioned above, the open string field theory of the B-model reduces to a holomorphic version of Chern-Simons theory [81], which can be straightforwardly extended to super-Calabi-Yau manifolds, yielding the following action [2]

$$S=\frac{1}{2}\int_{\mathrm{D5}}\mathbf{\Omega}\wedge\mathrm{Tr}\left(\mathcal{A}\cdot\bar{\partial}\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)\,, \tag{3.7}$$

where in this case

$$\mathbf{\Omega}=\frac{1}{4!}\mathbf{\Omega}'\epsilon_{IJKL}d\psi^I d\psi^J d\psi^K d\psi^L \qquad \left(\text{with}\quad \mathbf{\Omega}'=\frac{1}{4!}\epsilon_{IJKL}Z^I dZ^J dZ^K dZ^L\right)\,, \tag{3.8}$$

is the globally defined holomorphic volume form.[8] The classical equations of motion following from (3.7) are

$$\bar{\partial}\mathcal{A}+\mathcal{A}\wedge\mathcal{A}=0\,, \tag{3.9}$$

while an infinitesimal gauge transformation takes the form[9]

$$\delta\mathcal{A}=\bar{\partial}\epsilon+[\mathcal{A},\epsilon]\,. \tag{3.10}$$

[8]Note that, as mentioned in [2], $\mathbf{\Omega}$ does not actually define a top form in the fermionic directions and ideally should be promoted to a so-called *integral* form, which does. A thorough discussion of integration on supermanifolds in similar contexts appears in [84], where more references can be found. However, as in [2], the choice of $\mathbf{\Omega}$ in (3.8) appears to be sufficient for our purposes, and we will content ourselves with this naïve choice in the following.

[9]Here and in the following we use the standard commutator of forms $[\alpha_p,\beta_q]=\alpha_p\wedge\beta_q-(-1)^{pq}\beta_q\wedge\alpha_p$.

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By linearising around the trivial solution $\mathcal{A} = 0$, the above reduce to $\bar{\partial}\mathcal{A} = 0$ and $\mathcal{A}' - \mathcal{A} = \bar{\partial}\epsilon$ respectively, which show that $\mathcal{A}$ is in the $\bar{\partial}$ cohomology class H^1 and thus a good physical state of the open B-model. As is further explained in [2], the ψ^Is carry an additional $\mathrm{U}(1)_S$ $(+1)$ charge, under which the B-model is anomalous and the superfield $\mathcal{A}$ is neutral. The component fields $(A, \lambda_I, \phi_{IJ}, \tilde{\lambda}^I, G)$ carry charge $(0, -1, -2, -3, -4)$ under this symmetry and are then $(0,1)$-forms with values in the line bundle $\mathcal{O}(-k)$, where $-k$ is the appropriate S-charge. Each component field in (3.6) is then an element of the sheaf cohomology class $\mathrm{H}^1(\mathbb{CP}^3, \mathcal{O}(-k))$[10] and the Penrose transform [3] maps these fields to the space of solutions of massless free wave equations for fields of helicity $1 - k/2$ in Minkowski space. In this fashion one recovers the spectrum of $\mathcal{N} = 4$ SYM.

The action (3.7) thus contains all the fields of $\mathcal{N} = 4$ SYM and at least some of the interactions. It does not contain *all* the interactions, however. Rather, it describes the subset corresponding to the so-called selfdual sector of $\mathcal{N} = 4$ SYM, as can be seen via a nonlinear form of the Penrose transform, which precisely maps the hCS action to this selfdual truncation [85]. In order to recover the full set of interactions it is necessary to introduce nonperturbative objects, called 'D1-instantons' in [2], which are Euclidean 2-branes wrapping the curves on which the desired amplitudes are supported. We will postpone a review of these aspects to section 4.1, and concentrate for the moment on obtaining the analogue of the above construction for the $N_f = 4$ theory.

3.3 Orientifolding the twistor string

Having reviewed how the spectrum and selfdual interactions of $\mathcal{N} = 4$ SYM can be recovered from the B-model on $\mathbb{CP}^{3|4}$, we now begin the analogous construction for the $N_f = 4$ theory. It is clear from the above that the problem can be split into two steps: First, we will need to recover a B-model target space action corresponding to the selfdual sector of the gauge theory, and then, introducing D1-instantons wrapping appropriate curves, we can proceed to reproduce the non-selfdual amplitudes of the theory. In the following we will focus on the former part, while the second step will be considered in section 4.

Following the intuition gained from the IIB description of the $N_f = 4$ theory, reviewed in section 2.1, and the twistor description of quiver gauge theories in [25, 26], it is clear that some sort of fermionic orientifold projection will be necessary in our approach.[11] We begin by considering the $\mathcal{N} = 4$ setup of the previous section, choosing the number of 'D5' branes to be $2N$. This produces an $\mathrm{SU}(2N)$ gauge group, and accordingly the indices of $\mathcal{A}^i{}_j$ run over $i, j = 1, \ldots, 2N$. Conformal invariance of the dual gauge theory requires us to choose the orientifold action such as to leave the bosonic part of $\mathbb{CP}^{3|4}$ fixed.[12] However, we would like to reduce the amount of supersymmetry, which implies that the orientifold should act

[10] Actually, as noted in [2], the class is really $\mathrm{H}^1(\mathbb{CP}^{3'}, \mathcal{O}(-k))$ where $\mathbb{CP}^{3'}$ is a suitable open set of supertwistor space. However, we will ignore such subtleties here.

[11] Orientifolds in a topological string context were first considered (for the A-model) in [86].

[12] Although we should emphasise that, in order to discuss a specific spacetime signature, we will eventually need to pick a contour (e.g. $\mathbb{RP}^3$) within $\mathbb{CP}^3$, which can be imposed via a bosonic orientifold-type operation (albeit a trivial one from our perspective, being already present for $\mathcal{N} = 4$ [2]). We thank Dave Skinner for a relevant discussion.

on the fermions ψ^I asymmetrically, in order to break the $\mathrm{SU}(4)_R$ symmetry. Therefore, we begin by splitting the four fermionic coordinates ψ^I of $\mathbb{CP}^{3|4}$ into $I = \{a, A\}$, with $a = 1, 2$ and $A = 3, 4$. The appropriate orientifold action is the combination of a $\mathbb{Z}_2$ orbifold (acting trivially on the Chan-Paton indices), the worldsheet parity transformation $\hat{\omega}$ and an action on the Chan-Paton indices brought about by acting with an antisymmetric hermitian matrix $\tilde{\gamma} = i\Omega$, where $\Omega_{2N\times 2N}$ is the $\mathrm{Sp}(N)$ invariant tensor (see appendix A)

$$\begin{array}{ll} (a) & \psi^a \to \psi^a \quad , \quad \psi^A \to -\psi^A \\ (b) & \mathcal{A}^i{}_j \to \Omega^{ik}(\mathcal{A}^T)_k{}^l\Omega_{lj} = (\mathcal{A}^T)^i{}_j \equiv \mathcal{A}_j{}^i \ , \end{array} \tag{3.11}$$

which is a superorientifold operation in $\mathbb{CP}^{3|4}$.[13] Note that the orbifold action (a) breaks the fermionic coordinate symmetry $\mathrm{SU}(4)_R \to \mathrm{SU}(2)_a \times \mathrm{SU}(2)_A$.[14] Also note that it leaves the holomorphic volume form (3.8) invariant, indicating that the target space is still super-CY and that we can legitimately define a proper B-model action. In (b) we have used Ω to raise and lower indices.

Requiring $\mathcal{A}$ to be invariant under this operation (which, on lowering indices, translates to $\mathcal{A}_{ij} = \mathcal{A}_{ji}$), and considering its action on the various component fields in the expansion (3.6), it is easy to see that one obtains the following decomposition

$$\begin{aligned} \hat{\mathcal{A}} &= (A + \psi^a\lambda_a + \psi^1\psi^2\phi + \psi^3\psi^4\phi^\dagger + \epsilon_{cd}\psi^3\psi^4\psi^c\tilde{\lambda}^d + \psi^1\psi^2\psi^3\psi^4 G) \\ &\quad + \psi^A(\zeta_A + \psi^a z_{Aa} + \epsilon_{AB}\psi^1\psi^2\tilde{\zeta}^B) \\ &= \mathcal{V} + \psi^A Z_A \\ &= \mathcal{V} + \mathcal{Z} \ , \end{aligned} \tag{3.12}$$

where in the first line we have collected the terms ($\mathcal{V}$) which are symmetric (when both indices are either up or down) under the orientifold operation of (3.11). Since these have $N(2N + 1)$ gauge degrees of freedom, we immediately conclude that they transform in the adjoint representation of $\mathrm{Sp}(N)$. Similarly in the second line we have displayed the terms ($\mathcal{Z}$) which are antisymmetric under said operation and therefore have $N(2N - 1) - 1$ degrees of freedom and transform in the (second-rank) antisymmetric tensor representation of $\mathrm{Sp}(N)$.

By repeating the analysis performed for the $\mathcal{N} = 4$ theory and studying the linearised classical equations of motion around the trivial solution $\hat{\mathcal{A}} = 0$, one obtains the superorientifold-invariant elements of the (Dolbeault) cohomology, which via the Penrose transform map to part of the spectrum of the $N_f = 4$ theory [2]. In a helicity basis this is

$$\begin{array}{llll} \mathcal{V} = & \underbrace{(A, \lambda_a, \{\phi, \phi^\dagger\}, \tilde{\lambda}^a, G)}_{\text{1-forms of S-charge } (-k) \text{ in twistor space}} & \overset{\text{Penrose}}{\longleftrightarrow} & \underbrace{(A, \lambda_a, \{\phi, \phi^\dagger\}, \bar{\lambda}^a, G)}_{\text{fields of helicity } (1-k/2) \text{ in Mink. space}} \\ \mathcal{Z} = & \overbrace{(0, \zeta_A, z_{aB}, \bar{\zeta}^A, 0)} & \overset{\text{Penrose}}{\longleftrightarrow} & \overbrace{(0, \zeta_A, z_{aB}, \bar{\zeta}^A, 0)} \end{array} \tag{3.13}$$

[13] In writing (b) we have assumed that, as in the physical string case [49], $\hat{\omega}$ has eigenvalue -1 on the $(0, 1)$-form vertex operator $\mathcal{A}$. This minus combines with the i^2 from $\tilde{\gamma} = i\Omega$ to give an overall plus in (b).

[14] We choose the subscripts having in mind the eventual identification of these symmetries with their spacetime counterparts.

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Direction	D_c–D_c	D_c–D_f	D_f–D_f
Z,$\bar{Z}$	NN	NN	NN
ψ^a	NN	NN	NN
ψ^A	NN	ND	DD
$\bar{\psi}^{\bar{a}}$,$\bar{\psi}^{\bar{A}}$	DD	DD	DD

Table 2: Boundary conditions for open strings in the B-model setup.

and we have, therefore, obtained the adjoint and antisymmetric sector of the $N_f = 4$ theory. However, to complete the derivation of the spectrum on the twistor side, we still need to recover the fundamental degrees of freedom, to which we now turn our attention.

3.4 Flavour-branes and the fundamental sector

By analogy with the IIB string description, it should be clear that incorporating the fundamental fields of the $N_f = 4$ theory will require the introduction of a new object in twistor space. We will implement this by adding a new kind of brane to our configuration, which we will call a 'flavour'-brane, as it roughly corresponds to a D7-brane in the physical string setup, in the sense that strings stretching between the 'D5's and the flavour-branes will lead to the fundamental hypermultiplets.

Recall from section 2.1 that in the IIB picture the D7-branes were located on the orientifold plane defined by $(x^8, x^9) \to -(x^8, x^9)$. We will similarly take the flavour-branes to lie on the fixed point set of our orientifold action ($\psi^A \to -\psi^A$), by imposing Dirichlet conditions in the ψ^3, ψ^4 directions. We will also keep the Dirichlet condition on the antiholomorphic $\bar{\psi}^{\bar{A}}$ directions. Since these new branes still extend along the bosonic directions of $\mathbb{CP}^{3|4}$ (as well as the fermionic ψ^a directions), from now on we will drop the possibly misleading 'D5' terminology and label the branes discussed in the last section (which led to the gauge group Sp(N)) as 'D_c' (for colour) and the new branes as 'D_f' (for flavour). We summarise the boundary conditions satisfied by open strings stretching between the branes in our setup in table 2.

Having chosen the boundary conditions defining a D_f brane, we will now need to decide on a) how many of them to introduce and b) how the orientifold and orbifold groups act on the Chan-Paton indices associated with these branes. For the first question, it turns out that (as will become clear shortly) introducing two D_f branes, which along with their mirrors lead to a 4×4 Chan-Paton group, is what is necessary to reproduce the $N_f = 4$ theory. We will call the corresponding indices $X, Y, \ldots = 1, \ldots, 4$. As for the second question, recall that for the D_c branes we chose the orientifold action $\tilde{\gamma}_c = i\Omega_{2N\times 2N}$, but the action of the orbifold was trivial: $\gamma_c = \mathbb{I}_{2N\times 2N}$. With an eye to the results we want to obtain, we will again choose the orientifold action antisymmetric ($\tilde{\gamma}_f = i\Omega_{4\times 4}$), but this time we take $\gamma_f = -\mathbb{I}_{4\times 4}$. Thus, the full specification of our orientifold action (extending (3.11)) is given by:

$$\begin{aligned} &(a) \quad \psi^a \to \psi^a \quad , \quad \psi^A \to -\psi^A \\ &(b) \quad \mathcal{J} \to \Omega_{(r)} \mathcal{J}^{(T)} \Omega_{(r)} \\ &(c) \quad \mathcal{J} \to \gamma_{(r)} \mathcal{J} \gamma_{(r)}^{(-1)} , \end{aligned} \tag{3.14}$$

where the generic B-model state $\mathcal{J}$ can take any of the four possible choices of Chan-Paton indices (i.e. $\mathcal{J}^i{}_j, \mathcal{J}^i{}_X, \mathcal{J}^X{}_i, \mathcal{J}^X{}_Y$), $\gamma_{(r)}$ is either γ_c or γ_f depending on the index it is acting upon, and similarly $\Omega_{(r)}$ corresponds to either $\mathrm{Sp}(N)$ or $\mathrm{Sp}(2)$.

This completes the definition of our proposal for the twistor dual of the $N_f = 4$ theory. Now let us check whether we can recover the expected spectrum on the spacetime side. Of course the discussion in section 3.3 remains unchanged, so we already know that the $c-c$ strings of our construction reproduce the correct vector and antisymmetric hypermultiplet spectrum.

First, we will look at the $f-f$ strings, which will provide us with information on the Chan-Paton group corresponding to the four D_f branes. We thus need to confront the problem of interpreting the Dirichlet boundary conditions in the holomorphic ψ^A directions. Unlike what happens for the antiholomorphic fermions, simply interpreting these as imposing $\psi^A = 0$ (so that observables do not depend on ψ^A) does not seem to provide the correct degrees of freedom. The resolution comes through realising that one has to apply a fermionic analogue of dimensional reduction, which is part of a more general question of properly defining sub-supermanifolds of supermanifolds. Some aspects of this, which turn out to be sufficient for our purposes, have been discussed in [87], whose approach we will follow (and where further references can be found). In brief, the results of [87] indicate that a reasonable definition of fermionic dimensional reduction is to restrict the fermionic dependence of the original supermanifold so that fields on the sub-supermanifold can only depend on them in certain combinations. For example, one of the cases considered in [87] was the reduction $\mathbb{CP}^{3|4} \to \mathbb{CP}^{3\oplus 1|0}$, where the notation [88] means that all four ψ^I have been combined into a single nilpotent bosonic coordinate $y = \psi^1\psi^2\psi^3\psi^4$.[15]

A simple way to impose such constraints on the fermionic dependence is in terms of a suitable set of integral constraints, and indeed the particular reduction above was first performed in [89] using such an approach. However, with this choice (as well as another case considered in [87]) one is led to a completely bosonic truncation of the $\mathcal{N} = 4$ spectrum, while our D_f branes are still expected to preserve $\mathcal{N} = 2$ supersymmetry, so we will need to slightly adapt those embeddings to our setting. Given the symmetries of our system, we propose that the supermanifold reduction defining the D_f branes is $\mathbb{CP}^{3|4} \to \mathbb{CP}^{3\oplus 1|2}$, where the nilpotent coordinate is $\psi^3\psi^4$ and the ψ^1,ψ^2 coordinates are unrestricted.[16]

As discussed above, the NN directions will provide a (0,1)-form living on the D_f branes, which we denote by $\mathcal{K}^X{}_Y$. The above definition of dimensional reduction can be implemented by imposing the following eight equations (which are a subset of the truncation

[15] $\mathbb{CP}^{3\oplus 1|0}$ is an example of a *thickening* of $\mathbb{CP}^3$ [87].

[16] Such maps of supermanifolds, where one exchanges pairs of odd coordinates for even nilpotent coordinates, have also appeared, in a slightly different (superspace) context, in [90].

conditions considered in [89])

$$\int \mathrm{d}^4\psi\psi^1\psi^2\psi^A\mathcal{K} = \int \mathrm{d}^4\psi\psi^a\psi^A\mathcal{K} = \int \mathrm{d}^4\psi\psi^A\mathcal{K} = 0\,. \tag{3.15}$$

These conditions restrict the ψ dependence of $\mathcal{K}$ to take the following form

$$\mathcal{K}^X_{\ Y} = \mathrm{d}\bar{Z}^{\bar{m}}\left(K(Z,\bar{Z},\psi^a)_{\bar{m}}{}^X{}_Y + \psi^3\psi^4L(Z,\bar{Z},\psi^a)_{\bar{m}}{}^X{}_Y\right)\,. \tag{3.16}$$

It is easy to check that requiring invariance under the orientifold action results in a symmetric truncation of the Chan-Paton matrix defined by the X, Y indices and thus $\mathcal{K}$ is a 4×4 matrix transforming in the adjoint of an Sp(2) group. Thus we have specified the (0,1)-form part of the $f-f$ spectrum.

However, as can be seen in (3.4), the existence of holomorphic DD directions implies that the $(0,1)$-forms do not exhaust the possible vertex operators that can be written down at ghost number one. One can now also have states of the form

$$\mathcal{B}^A\theta_A \;\sim\; \mathcal{B}^A(Z,\bar{Z},\psi^a,\psi^A)\frac{\partial}{\partial\psi^A}\,. \tag{3.17}$$

Motivated by dimensional reduction in the physical string case, and in particular by the desire to have the same counting of states before and after the reduction, we will assume that the fermionic dependence of these DD $f-f$ states arises by considering the *complement* of the eight equations in (3.15).[17] This will restrict the general expansion for $\mathcal{B}$ to

$$\mathcal{B}^A(Z,\bar{Z},\psi)^X{}_Y\frac{\partial}{\partial\psi^A} = \psi^B B^A_B(Z,\bar{Z},\psi^a)^X{}_Y\frac{\partial}{\partial\psi^A}\,. \tag{3.18}$$

Requiring invariance under the orientifold action (under which we also have $\partial/\partial\psi^A \to -\partial/\partial\psi^A$) once again restricts the Chan-Paton indices to be those of Sp(2). It is straightforward to check that $\psi^B B_B(Z,\bar{Z},\psi^a)$ provides 4 fermionic and 4 bosonic degrees of freedom, which, together with $\mathcal{K}$, give the expected counting of states for the 8d $\mathcal{N}=1$ theory on the D7-brane (note that in this counting we suppress the index corresponding to the expansion of B in a basis of $T^{(1,0)}$, in the same way that we have been suppressing the form index $\bar{z}$ for the (0,1)-form states). These states, not being (0,1)-forms, are clearly unsuitable for a straightforward application of the Penrose transform to four dimensions. This is not unexpected, since their natural dual interpretation would be as states of the eight-dimensional D7-brane theory. We will further comment on such a potential interpretation at the end of this section.

It should also be pointed out that, again because they are not (0,1)-forms, there seems to be no obvious way to include the $\mathcal{B}$ states in a holomorphic Chern-Simons-type action (which would still need to be integrated over a (3,3)-cycle), and in particular we cannot write down the action on the D_f worldvolume including these terms by dimensional

[17] This becomes clearer if one chooses to reduce along all four ψ^I directions, as in [89]. In that case one imposes 14 equations in the NN sector, so the (0,1)-strings provide just two degrees of freedom. The remaining states should then arise from the DD sector, therefore we would want to impose just two equations on that sector.

reduction (unlike the case for bosonic DD directions, see e.g. [91]). Perhaps a suitable generalisation of the hCS action, along with a more rigorous definition of our integration measure, would be able to accommodate this more general case, but since for the purposes of this paper we will only need to know the D_c brane action, which is what is expected to have a relation to the 4d theory that we are interested in, we will not pursue this question further here.

Clearly the choice of the above geometric embedding of the D_f branes within $\mathbb{CP}^{3|4}$ has been based on rather heuristic arguments, and, although it certainly seems to provide a consistent picture, we cannot claim that it is the unique possibility. It would certainly be desirable to obtain a more fundamental understanding of this embedding starting from the basic definition of Dirichlet boundary conditions on the B-model worldsheet. Leaving this for future work, we will now turn to the last aspect of our construction, i.e. the strings stretching between the D_c and D_f branes.

Therefore, we finally consider the $c-f$ and $f-c$ strings. Recall that these are the real reason to introduce the D_f branes, since they will provide the desired fundamental matter. Looking at table 2, and recalling that (topological) DN strings do not have zero modes and thus do not provide B-model states, the only contributions arise from the NN sector. Suppressing the $(0,1)$-form index, these can be usefully written as an expansion in ψ^A

$$\mathcal{Q}^i{}_X = P(Z,\bar{Z},\psi^a)^i{}_X + \psi^A Q_A(Z,\bar{Z},\psi^a)^i{}_X + \psi^3\psi^4 R(Z,\bar{Z},\psi^a)^i{}_X \tag{3.19}$$

and similarly for the $f-c$ field $\mathcal{Q}^X_{\ i}$. Note that, due to the orientifold action (3.14.b), the $c-f$ and $f-c$ states are related by the condition

$$\mathcal{Q}^X_{\ i} = \Omega_{ij}\mathcal{Q}^j{}_Y\Omega^{YX}\ . \tag{3.20}$$

It is easy to check that the other components of (3.14) impose $P^i{}_X = R^i{}_X = 0$ and thus dictate that the $c-f$ and $f-c$ states are given by

$$\mathcal{Q}^i{}_X = \psi^A Q^i_{AX}\,, \qquad\qquad \mathcal{Q}^X{}_i = \psi^A Q^X_{A\,i}\,, \tag{3.21}$$

where we can expand

$$Q^i_{AX} = \eta^i_{AX} + \psi^a q^i_{aAX} + \psi^1\psi^2\tilde{\eta}^i_{AX} \tag{3.22}$$

and similarly for Q^X_{Ai}. Recall that here i is an $\mathrm{Sp}(N)$ gauge group index, A is an index of $\mathrm{SU}(2)_A$ and (as we previously derived) X is an index of $\mathrm{Sp}(2)$. The particular form of Q is not new: As shown in [92, 37], this is the precise twistor field content (for each value of X) corresponding to an $\mathcal{N}=2$ hypermultiplet![18] We conclude (and will make more precise shortly) that our orientifolding procedure has produced a hypermultiplet Q^i_{AX} in the fundamental representation of $\mathrm{Sp}(N)$.

Let us now investigate its transformation properties under the two global groups, given by the indices A and X. As we reviewed in section 2.2, the fundamental hypermultiplets

[18] To be more precise, these references describe a hypermultiplet as consisting of two fermionic half-hypermultiplets, while in our case they naturally appear in $\mathrm{SU}(2)_A$ doublets, at the cost of losing manifest $\mathrm{SO}(8)$ invariance.

should also transform in the fundamental representation of the global SO(8) flavour group. However at the end of that section we explicitly decomposed the SO(8) into its SU(2)×Sp(2) subgroup. The reason for that should now be evident: In the twistor string model we have constructed, the SU(2) arises geometrically as the symmetry under which the ψ^A coordinates transform as doublets, while the remaining Sp(2) arises as the Chan-Paton group of the flavour-branes. We will explore some of the implications of this decomposition of the flavour group shortly, but it is clearly an unavoidable consequence of the fundamental fields in (3.21) being linear in ψ^A. However, we can immediately comment on another consequence of this linear behaviour: It provides a very natural explanation for the fermionic nature of Q^i_{AX} which had to be assumed in the constructions of [92, 37].

We conclude that, by defining our flavour-branes to lie at the orientifold fixed point, and extending the orientifold action to act nontrivially on their Chan-Paton indices, we have reproduced the fundamental part of the spectrum of the $N_f = 4$ theory. This description has several peculiarities relative to the physical string description, not least of which is the fact that the relative sizes of the D3 and D7 branes in the IIB setup seem to be interchanged: Our D_f branes extend (have NN b.c.'s) along a subspace of that of the D_c branes and could perhaps be thought of as defects in the worldvolume theory of the latter. On the other hand, what is perhaps more relevant in comparing to the spacetime picture is the *super-dimension* of our branes, defined as the difference between the number of bosonic and fermionic NN directions.[19] Although this deserves further study, we note that it also seems to be consistent with an observation in [95] that (for non-topological strings on supermanifolds) the number of fermionic NN directions contributes to the brane tension inversely to that of bosonic NN directions, and thus a brane extending along fewer fermionic directions can be thought of as having larger mass. Although these results do not apply directly in our setting, we take them as an indication that the geometric embedding of the D_f branes is the correct one.

Another perhaps surprising feature of our model is the fact that both the D_c and D_f branes were chosen to satisfy symplectic projection conditions on their Chan-Paton indices, leading to Sp(N) and Sp(2) worldvolume gauge groups respectively. This seems to conflict with the arguments of [49] which (applied to the orientifolded D3-D7 system) would require opposite projections for the two types of branes, leading to Sp(N) and SO(8) gauge groups. However, that analysis was based on subtle properties of the $3-7$ string DN sector, which is absent in this case. Therefore it would seem that the B-model is too simple to accommodate such an effect, but confirmation of this will have to wait for a better worldsheet understanding of our orientifold prescription.[20]

Given that, in the physical string setup, our D_f branes correspond to IIB D7-branes, with an associated eight-dimensional worldvolume SYM theory, it is fascinating to speculate that our twistor string model might, via a suitable higher-dimensional generalisation of the

[19] For instance, [93] argues for the equivalence of the A-model on certain $(m|n)$-dimensional supermanifolds to that on bosonic $(m-n)$-dimensional manifolds. See also [94] for similar observations in the context of mirror symmetry.

[20] It is likely that the notions of B-parity and B-orientifolds, developed for (untwisted) (2,2) models in [96] (see also [97]), properly extended to the supermanifold case, will be of help in this regard.

Penrose transform, also have another dual description in terms of an *eight-dimensional* spacetime theory. Under this duality, the worldvolume theory of the twistor D_f brane would presumably map to some integrable subsector of 8d Yang-Mills. A preliminary remark in this direction is that a natural definition of selfduality for 8d Yang-Mills [98] also seems to require the same breaking of (Lorentz) SO(8) to Sp(2)×Sp(1) that we observe on the twistor side. Although it would be very interesting to understand this connection better, we will from now on focus on the standard four-dimensional Penrose transform that connects the spectrum and field equations of the D_c brane worldvolume theory to those of a suitable generalisation of 4d selfdual Yang-Mills.[21]

3.5 The final twistor action

In the last two sections we defined a B-model setup with certain numbers of branes that reproduced the spectrum of the $N_f = 4$ theory. The resulting superfields can be naturally embedded into the holomorphic Chern-Simons action in the following way[22]

$$\begin{aligned} S &= \frac{1}{2}\int_{\mathrm{D}_c} \boldsymbol{\Omega}\wedge\left(\mathrm{Tr}\left[\hat{\mathcal{A}}\cdot\bar\partial\hat{\mathcal{A}}+\frac{2}{3}\hat{\mathcal{A}}\wedge\hat{\mathcal{A}}\wedge\hat{\mathcal{A}}\right]+\mathcal{Q}^X\cdot\bar\partial\mathcal{Q}_X+\mathcal{Q}^X\wedge\hat{\mathcal{A}}\wedge\mathcal{Q}_X\right)\\ &= \frac{1}{2}\int_{\mathrm{D}_c} \boldsymbol{\Omega}\wedge\left(\mathrm{Tr}\left[\mathcal{V}\cdot\bar\partial\mathcal{V}+\frac{2}{3}\mathcal{V}\wedge\mathcal{V}\wedge\mathcal{V}+\mathcal{Z}\cdot\bar\partial\mathcal{Z}+2\mathcal{Z}\wedge\mathcal{V}\wedge\mathcal{Z}\right]\right.\\ &\qquad\qquad\left.+\,\mathcal{Q}^X\cdot\bar\partial\mathcal{Q}_X+\mathcal{Q}^X\wedge\mathcal{V}\wedge\mathcal{Q}_X\right)\,. \end{aligned} \tag{3.23}$$

The classical equations of motion can then be easily found to be

$$\begin{aligned} \bar\partial\mathcal{V}+\mathcal{V}\wedge\mathcal{V}+\mathcal{Z}\wedge\mathcal{Z}+\frac{1}{2}\mathcal{Q}^X\wedge\mathcal{Q}_X &= 0\\ \bar\partial\mathcal{Z}+[\mathcal{V},\mathcal{Z}] &= 0\\ \bar\partial\mathcal{Q}_X+\mathcal{V}\wedge\mathcal{Q}_X &= 0 \end{aligned} \tag{3.24}$$

and by linearising these around the trivial solutions $\mathcal{V}=0$, $\mathcal{Z}=0$, $\mathcal{Q}=0$ one obtains

$$\bar\partial\mathcal{V}=\bar\partial\mathcal{Z}=\bar\partial\mathcal{Q}=0\,. \tag{3.25}$$

In addition, (3.23) has the following three gauge invariances, related to three different (0,0)-form gauge parameters $\epsilon^i{}_j$, $\varepsilon^i{}_j$ and $e^i{}_X$

$$(a)\ \delta\mathcal{V}=\bar\partial\epsilon+[\mathcal{V},\epsilon]\,,\quad \delta\mathcal{Z}=[\mathcal{Z},\epsilon]\,,\quad \delta\mathcal{Q}^X_i=\mathcal{Q}^X_j\epsilon^j{}_i\,,\quad \delta\mathcal{Q}^i{}_X=-\epsilon^i{}_j\mathcal{Q}^j{}_X\,, \tag{3.26}$$

$$(b)\ \delta\mathcal{Z}=\bar\partial\varepsilon+[\mathcal{V},\varepsilon]\,,\quad \delta\mathcal{V}=[\mathcal{Z},\varepsilon]\,, \tag{3.27}$$

[21]In doing this we will assume that the Penrose transform can be applied just to the D_c brane theory, comprising the $c-c$ strings plus their interactions with the $c-f$ and $f-c$ strings, ignoring interactions with the D_f worldvolume theory. In the physical string setting such interactions are frozen at low energies essentially due to the difference in spatial extent of the D3 and D7-branes. It would be interesting to identify a mechanism providing such a decoupling in our topological string setting.

[22]Here we write the fundamental part of the action by analogy with that for the antisymmetric fields. However, note the different relative coefficient of the interaction terms, which is due to their different Sp(N) transformation properties.

and

$$
(c)\ \delta\mathcal{Q}^i{}_X = \bar\partial e^i{}_X + \mathcal{V}^i{}_j e^j{}_X\ , \quad \delta\mathcal{Q}^X{}_i = \bar\partial e^X{}_i - e^X{}_j\mathcal{V}^j{}_i\ , \quad \delta\mathcal{V}^i{}_j = \frac{1}{2}(\mathcal{Q}^i{}_X e^X{}_j - e^i{}_X\mathcal{Q}^X{}_j)\ . \tag{3.28}
$$

The first of these is the ordinary gauge invariance while the other two are clearly very unusual, and are due to the fact that on the twistor side $\mathcal{Z}$ and $\mathcal{Q}$ are (0,1) forms.[23] Essentially the same transformations have been discussed in [37], where they arise as symmetries of the (non-cubic) twistor space effective action which, in the formalism there, would correspond to full (non-selfdual) $\mathcal{N}=2$ SYM with matter.

As such, the linearised equations of motion and these symmetries are enough to put the superfields $\mathcal{V}, \mathcal{Z}$ and $\mathcal{Q}$ in the appropriate cohomology classes for their component fields to map to spacetime states. In particular, the components of $\mathcal{Q}$ then map to Minkowski space fields of helicity $(\frac{1}{2}, 0, -\frac{1}{2})$ via the Penrose transform

$$
\mathcal{Q} = \underbrace{(0, \eta_{AX}, q_{aAX}, \tilde\eta^{AX}, 0)}_{\text{1-forms of S-charge } (-k) \text{ in twistor space}} \quad \overset{\text{Penrose}}{\longleftrightarrow} \quad \underbrace{(0, \eta_{AX}, q_{aAX}, \bar\eta^{AX}, 0)}_{\text{fields of helicity } (1-k/2) \text{ in Mink. space}} \ .
$$

We have thus obtained the complete spectrum of the $N_f = 4$ theory from twistor string theory. Expanding (3.23) in components and integrating out the fermionic variables gives

$$
\begin{aligned}
S_{hCS} = \int_{\mathbb{CP}^3} \mathbf{\Omega'} \wedge \Big(&\mathrm{Tr}[G\wedge F + \phi^\dagger \wedge \bar D\phi - \tilde\lambda^a \wedge \bar D\lambda_a + \lambda^a \wedge \lambda_a \wedge \phi^\dagger] \\
&+ \mathrm{Tr}\Big[-\frac{1}{2} z^{aA} \wedge \bar D z_{aA} - \tilde\zeta^A \wedge \bar D\zeta_A - z^{aA} \wedge \lambda_a \wedge \zeta_A + \zeta^A \wedge \zeta_A \wedge \phi\Big] \\
&+ \tilde\eta_{AX} \wedge \bar D\eta^{AX} - \frac{1}{2} q_{aAX} \wedge \bar D q^{aAX} - q_{aAX} \wedge \lambda^a \wedge \eta^{AX} + \frac{1}{2}\eta_{AX} \wedge \phi \wedge \eta^{AX}\Big)\ ,
\end{aligned} \tag{3.29}
$$

where the covariant derivatives are defined as $\bar D = \bar\partial + [A,\ ]$ for tensor fields and $\bar D = \bar\partial + A\wedge$ for fundamental ones. This looks very much like the selfdual truncation of the $N_f = 4$ theory that we obtained in (2.13), which we present again to facilitate the comparison

$$
\begin{aligned}
S_{4d} = \int d^4x\ \mathrm{Tr}&\left[-\frac{1}{2}GF + D\phi^\dagger D\phi + i\bar\lambda^a \not{D}\lambda_a - \lambda^a\lambda_a\phi^\dagger\right] \\
&- \mathrm{Tr}\left[\frac{1}{2}Dz^{aA}Dz_{aA} + i\bar\zeta^A \not{D}\zeta_A + z^{aA}[\lambda_a, \zeta_A] + \zeta^A\zeta_A\phi\right] \\
&- \left(\frac{1}{2}Dq_{aA'X}Dq^{aA'X} + i\bar\eta_{A'X}\not{D}\eta^{A'X} + q_{aA'X}\lambda^a\eta^{A'X} + \frac{1}{2}\eta_{A'X}\phi\eta^{A'X}\right)\ .
\end{aligned}
$$

As we have already mentioned, there should exist a nonlinear generalisation of the Penrose transform in the spirit of [85], relating these two actions exactly. Moreover, note that by comparing the two we readily observe that even though there is both an $\mathrm{SU}(2)_A$ and an

[23] In fact (a) and (b) can be straightforwardly derived from the transformation of $\hat{\mathcal{A}}$ ($\delta\hat{\mathcal{A}} = \bar\partial E + [\hat{\mathcal{A}}, E]$), by splitting $\hat{\mathcal{A}} = \mathcal{V} + \mathcal{Z}$ and $E = \epsilon + \varepsilon$ into symmetric and antisymmetric parts and considering the symmetry properties of the resulting terms.

$SU(2)_{A'}$ symmetry for the gauge theory, we only see a single $SU(2)_A$ on the B-model side. This is a hint that these two symmetries are identified in the twistor string description, a claim which we will verify during the comparison of amplitudes between the two theories.

In summary, we have introduced four D_f branes parallel to the superorientifold plane which account for the Sp(2) part of the flavour symmetry. Via the Penrose transform, this yields the right spectrum for the fundamental hypermultiplets in the $N_f = 4$ theory and mimics the behaviour of the D7-branes in the physical string setup. As we further discuss in the conclusions, it would be intriguing if there were a mechanism which exactly fixes the number of D_f branes in the B-model to four (two plus two mirrors), e.g. some analogue of the RR charge cancellation condition in string theory. The existence of such a mechanism would suggest (as expected perhaps) that our construction is only consistent at loop level for the precise case when the dual gauge theory is finite.

4. Comparison of amplitudes

Having reproduced the spectrum of the $N_f = 4$ theory, we will now establish the duality on firmer grounds by calculating amplitudes in both the gauge theory and topological string theory, and by showing precise agreement (up to a constant normalisation factor).

4.1 Review of the standard amplitude prescription

We will begin by briefly summarising the prescription of [2] for the calculation of colour-stripped partial amplitudes in $\mathcal{N} = 4$ SYM. As we indicated above, this reduces to the evaluation of particular correlators on the worldvolume of D1-instantons wrapping curves of a certain degree in $\mathbb{CP}^{3|4}$ and then integrating over the moduli space of such curves. For tree-level MHV amplitudes, the D1-instantons are localised [2] on $\mathbb{CP}^1$s in $\mathbb{CP}^{3|4}$ with the embedding given by

$$\mu_{\dot\alpha} + x_{\alpha\dot\alpha}\lambda^\alpha = 0 \quad \text{and} \quad \psi^I + \theta^I_\alpha \lambda^\alpha = 0 \, , \tag{4.1}$$

where $Z^m = (\lambda^\alpha, \mu^{\dot\alpha})$ and ψ^I are the supertwistor space coordinates, while the moduli $x_{\alpha\dot\alpha}$ and θ^I_α correspond to the coordinates of 4d Minkowski space and (on-shell) $\mathcal{N} = 4$ superspace respectively.

Following an idea due to Nair [4], the gauge theory amplitudes are reproduced by correlation functions of chiral currents on the worldvolume of these D1-instantons. Since the insertion of these objects explicitly breaks the isometries of $\mathbb{CP}^{3|4}$, one must integrate over the moduli space of instantons of the appropriate degree. The prescription for the calculation of tree-level MHV amplitudes, and therefore integration over degree one, genus zero curves, is then

$$A_{(n)} = g^2 \int d^4x \, d^8\theta \, \langle \int_{\mathbb{CP}^1} J_1 w_1 \cdots \int_{\mathbb{CP}^1} J_n w_n \rangle \, , \tag{4.2}$$

where J_i are D1 worldvolume free-fermion currents coupling to the external D5-brane fields (including both the colour and flavour-branes in our case), while the w_i's are the twistor space equivalents of wavefunctions for the external particles. The lower index $i = 1, \ldots, n$ indicates the position of the external particle in the n-point scattering process, as well as the point onto which these localise on the holomorphic curve in twistor space. The factor of g^2 is identified with the D1-instanton expansion parameter. The calculation for the product of the currents boils down to yielding a gauge group factor, which we will strip off, as well as the following denominator part of the MHV amplitudes[24]

$$\langle J_1 \cdots J_n \rangle_{\text{stripped}} = \frac{1}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle} \,. \tag{4.3}$$

The numerator of the amplitude is produced by the twistor wavefunctions w_i, which, upon integration over the positions of vertex operators for each on-shell external particle, result in a colour-stripped coefficient $v_i(\psi_i)$ equal to the one in the superfield expansion of $\mathcal{A}$ in (3.6) [99]. These contribute a number of factors of ψ, which are then integrated over the moduli space of D1-instantons via the embedding relation $\psi_i^I = \theta^{\alpha I}\lambda_{i\alpha}$. Since the fermionic part of the measure on moduli space for genus zero, degree one holomorphic curves is $d^8\theta$, the MHV amplitude is non-zero only if the Grassmann integral is saturated, that is, if the total S-charge of the external states participating in the scattering process is $\mathrm{S} = -8$. Conversely, if a process involves external states with total charge $\mathrm{S} = -8$, it is then MHV. Since in the case under study these amplitudes can include external fermions or scalars satisfying this condition in addition to gluons, it is perhaps more appropriate to refer to them as 'analytic' [100] rather than MHV, and we will mostly use the latter notation in the following. Finally we note that the integral over the bosonic moduli yields a δ-function of momentum conservation, which we omit. This prescription successfully reproduces all amplitudes localising on holomorphic, degree one, genus zero curves in $\mathcal{N} = 4$ SYM.

The above can also be extended to amplitudes which localise on higher degree, genus zero holomorphic curves. For generic scattering states this degree is given by $d = -\frac{1}{4}\sum_{i=1}^{n} \mathrm{S}_i - 1$, where the sum is over the S-symmetry charges of the n external particles. For gluon scattering these correspond to next-to-MHV (NMHV) and higher ($\mathrm{N}^{\mathrm{q}-2}$MHV) amplitudes and the appropriate degree is given by $d = q - 1$, where q is the number of negative helicity gluons. Although the original string-motivated prescription of [2] made use of one connected degree–d instanton, in practice it turned out to be more useful to consider instead a sum of d disconnected (degree one) D1-instantons, leading to the MHV-rules prescription [11]. The equivalence of these prescriptions (as well as intermediate pictures of multiple D1-instantons of degrees adding up to d) is strongly suggested by the work of [101].

4.2 Extension to the $N_f = 4$ theory

The above prescription can be straightforwardly extended to the twistor model for the $N_f = 4$ theory that we constructed in section 3. The starting point is to consider D1-instantons localised along holomorphic curves in the orientifold of $\mathbb{CP}^{3|4}$, which now includes the two types of 'D5' branes, which we have denoted D_c and D_f. Assuming that the D1 worldvolume currents couple to the external D_f fields in the same way as to the D_c's, we will take the formula (4.2) as our starting point. The difference in this case is that

[24] Here we use the widespread notation $\langle 12\rangle = \langle\lambda_1\lambda_2\rangle = \lambda_1^{\alpha}\lambda_{2\alpha}$ and $[12] = [\tilde\lambda_1\tilde\lambda_2] = -\tilde\lambda_{1\dot\alpha}\tilde\lambda_2^{\dot\alpha}$, with $2(p_i \cdot p_j) = \langle\lambda_i\lambda_j\rangle[\tilde\lambda_i\tilde\lambda_j]$. See also appendix B.

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the twistor wavefunctions w_i will now associate the appropriate term in the superfield expansion of the $\mathcal{V}, \mathcal{Z}$ of (3.12) and $\mathcal{Q}$ of (3.22) with each on-shell external particle.[25]

The fact that the gauge group is now $\mathrm{Sp}(N)$ rather that $\mathrm{SU}(N)$ does not introduce major complications, due to the fact that we consider colour stripped partial amplitudes, effectively factoring out all information about the gauge group. In the usual approach to organising amplitudes in $\mathrm{U}(N)$ gauge theories,[26] this amounts to considering definite orderings for the external scattering states and then summing over all non-cyclic permutations to obtain the full amplitude. The structure of the group theory piece leads to identities, which dramatically simplify the calculation by allowing the evaluation of a great number of partial amplitudes by simply exchanging negative helicity spinor factors. A similar procedure can be applied to the $\mathrm{Sp}(N)$ case. Naturally, from a given colour stripped result, one can recover different full amplitudes depending on the gauge group choice. Since $\mathrm{Sp}(N)$ gauge theory amplitudes seem to have no real phenomenological importance and since agreement of partial amplitudes between the gauge and twistor theory sides is enough to establish their correspondence, we will not explicitly calculate the full answer, although it is straightforward to recover it using simple group theory facts.[27] We would like to note at this point that we will not only strip the gauge group indices but also the $\mathrm{Sp}(2)$ indices X, which appear in the definitions of the fundamental fields. The motivation for this is that they are global non-geometric indices and the partial amplitude calculation is insensitive to how one chooses to contract them. In obtaining the full amplitude involving external fundamental fields, one should of course be careful to properly consider all possible contractions that lead to an $\mathrm{Sp}(2)$ scalar quantity.

In order to demonstrate that the standard twistor prescription for tree-level analytic amplitudes can be applied, essentially unmodified, to the $N_f = 4$ theory, we will now move on to explicit calculations of partial amplitudes. We will do this for a large set of amplitudes of different combinations involving external particles transforming in the adjoint, antisymmetric and fundamental representations of the $\mathrm{Sp}(N)$ gauge group.[28] The first nontrivial analytic amplitudes appear at 4-point but we will also evaluate a few 5-point amplitudes to provide further evidence for the duality. In the following subsections we will explicitly display the result on the twistor string side. In order to get the result purely from gauge theory one needs to extract the Feynman rules from the Lagrangean (2.12) and then add up the contributions from all channels for the process under consideration. In appendix B we list these Feynman rules in spinor helicity formalism, as well as various identities we have employed in order to obtain the spacetime answer. Since we do not have a precise map between the actions on the two sides of the correspondence, we cannot hope to exactly match the

[25]The reader worried about only integrating over the moduli space of $\mathbb{CP}^1$s in an orientifolded theory, which should also include $\mathbb{RP}^2$ topologies [102], should recall that these contributions are non-planar and will be absent at tree-level. They should, however, play a role in any eventual loop level calculation.

[26]See for example the reviews [103, 104].

[27]Pseudoreality of $\mathrm{Sp}(N)$ will make this step slightly more subtle compared to $\mathrm{U}(N)$, since there exist extra identities relating different orderings of the external particles.

[28]We do not need to calculate gluon scattering processes since the stripping procedure guarantees that the partial amplitudes will go through as in the $\mathrm{U}(N)$ case.

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resulting amplitudes. We therefore calculate ratios of the latter and find exact agreement up to a relative constant normalisation factor. In particular, in our conventions we find that the spacetime answer is obtained from the twistor result by multiplying by a factor of $32i$.

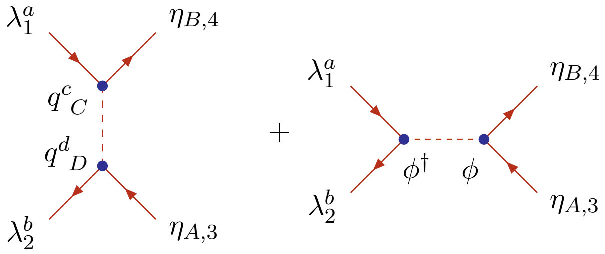
4.3 'Pre-analytic' amplitudes

Before proceeding with the analytic results, we will briefly look at the amplitudes that have a total value of S $= -4$, which we will call pre-analytic. These are $\langle\lambda^a, \lambda^b, \eta_A, \eta_B\rangle$, $\langle\lambda^a, \eta_A, \lambda^b, \eta_B\rangle$, $\langle\lambda^a, \lambda^b, \zeta_A, \zeta_B\rangle$ and $\langle\lambda^a, \zeta_A, \lambda^b, \zeta_B\rangle$ and on the twistor side they correspond to amplitudes that localise on degree zero curves in twistor space, i.e. points. This means that all particles are attached to the same point in twistor space and $\lambda_i = \lambda_j \; \forall\, i, j$. Therefore $2(p_i \cdot p_j) = \langle\lambda_i\lambda_j\rangle[\tilde{\lambda}_i\tilde{\lambda}_j] = 0$, and thus scattering amplitudes with $n \geq 4$, which depend on such nontrivial kinematic invariants, must vanish [2].

From the spacetime point of view this result is less obvious and one needs to calculate all the corresponding amplitudes explicitly. These come from interaction vertices which originate exclusively from the selfdual truncation of the $N_f = 4$ theory (2.13). In fact, this observation extends to all other theories admitting a tree-level twistor string description. Moreover, since we only focus on the colour-stripped (and Sp(2)-stripped) partial amplitudes, it suffices to calculate processes involving either fundamental or antisymmetric matter fields; the amplitude is insensitive to their gauge transformation properties. We will therefore only discuss the following examples involving the fundamental fermions η.

A. The amplitude $\langle\lambda_1^a, \lambda_2^b, \eta_{A,3}, \eta_{B,4}\rangle$

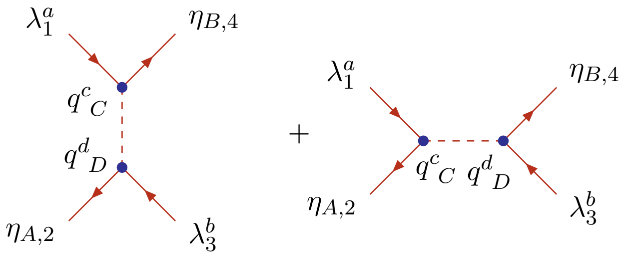
There are two channels contributing to this amplitude, namely



One can easily verify by explicit calculation, using the Feynman rules provided in appendix B, that they indeed cancel each other to give zero.

B. The amplitude $\langle\lambda_1^a, \eta_{A,2}, \lambda_3^b, \eta_{B,4}\rangle$

The contributions to this process are



and similarly we find that after summing both parts the total vanishes.

This demonstrates (at four-point level) that all pre-analytic amplitudes, which are the ones that can be constructed from the interactions in the selfdual truncation of the theory, vanish after summation over channels. The same phenomenon occurs for the selfdual truncation of $\mathcal{N} = 4$ SYM [2]. In that case, as for the selfdual truncation of pure (non-supersymmetric) Yang-Mills [105], this fact is explained by noting that the theory is classically integrable and is thus equipped with an infinite set of (nonlocal) conserved charges.[29] The corresponding Ward identities are then expected to constrain tree-level amplitudes so severely that they are forced to vanish (brief discussions on this can be found in [110, 111]). Thus, the vanishing of pre-analytic amplitudes that we observe strongly suggests that the selfdual sub-sector of the $N_f = 4$ theory (which is a very different supersymmetric extension of pure selfdual Yang-Mills from the $\mathcal{N} = 4$ case) also describes a classically integrable system. It would be interesting to check this by explicitly constructing the relevant conserved currents.

4.4 The amplitudes $\langle \phi, \phi, \phi^\dagger, \phi^\dagger \rangle$ and $\langle \phi, \phi^\dagger, \phi, \phi^\dagger \rangle$

We now turn to the analytic amplitudes of the theory. We start with two simple examples involving only external adjoint scalars. There are two possible orderings in this case and we will calculate both, to show that these indeed give rise to different partial amplitudes. On the twistor side, following the prescription (4.2) that we have discussed in some detail, we can read off and plug in the wavefunctions appropriate to the $\langle \phi, \phi, \phi^\dagger, \phi^\dagger \rangle$ amplitude from (3.12)

$$\begin{aligned} v_1(\phi) &= \psi_1^1 \psi_1^2, \quad v_2(\phi) = \psi_2^1 \psi_2^2 \\ v_3(\phi^\dagger) &= \psi_3^3 \psi_3^4, \quad v_4(\phi^\dagger) = \psi_4^3 \psi_4^4 \, . \end{aligned} \tag{4.4}$$

The result is then given by the integral

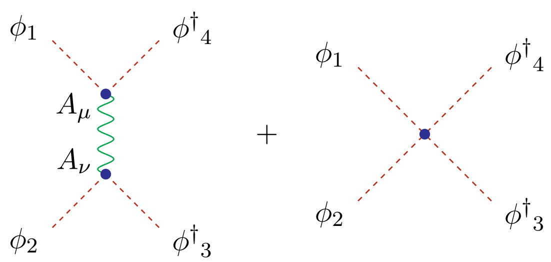
$$\langle \phi, \phi, \phi^\dagger, \phi^\dagger \rangle_{\text{Twistor}} = g^2 \int d^8\theta \frac{\psi_1^1 \psi_1^2 \psi_2^1 \psi_2^2 \psi_3^3 \psi_3^4 \psi_4^3 \psi_4^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} = \frac{g^2}{16} \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 23 \rangle \langle 41 \rangle} \, . \tag{4.5}$$

In obtaining the above we have used the anti-commutativity property of Grassmann variables and the embedding relation $\psi_i^I = \theta^{\alpha I} \lambda_{i\alpha}$ to arrive at

$$\int d^2\theta_1 \psi_i^1 \psi_j^1 = \int d^2\theta_1 \theta^{\alpha 1} \theta^{\beta 1} \lambda_{\alpha,i} \lambda_{\beta,j} = \frac{1}{2} \epsilon^{\alpha\beta} \lambda_{\alpha,i} \lambda_{\beta,j} = \frac{1}{2} \langle ji \rangle \, . \tag{4.6}$$

On the spacetime side we have contributions from two diagrams

$\langle \phi, \phi, \phi^\dagger, \phi^\dagger \rangle_{4d} =$



(4.7)

[29] For the pure selfdual YM case these can be found (for instance) via the Ward construction [106]; see [107] for a discussion and more references. For $\mathcal{N} = 4$-extended selfdual YM an associated linear system was discussed in [108] and more recently its hidden symmetries were explored in [109].

and explicit calculation shows that the final result is $\langle\phi,\phi,\phi^\dagger,\phi^\dagger\rangle_{4d} = 32i\langle\phi,\phi,\phi^\dagger,\phi^\dagger\rangle_{\text{Twistor}}$ as claimed.

For the alternative ordering $\langle\phi,\phi^\dagger,\phi,\phi^\dagger\rangle$ we have

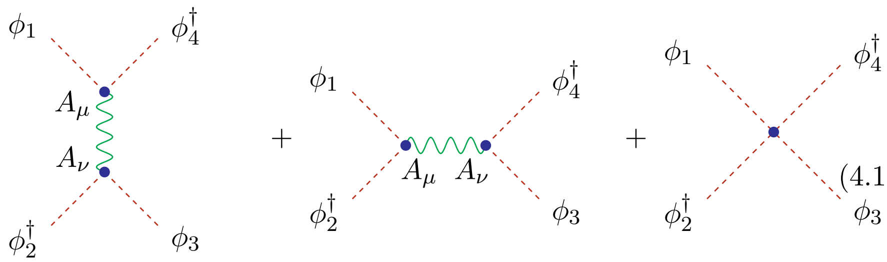
$$\begin{aligned} v_1(\phi) &= \psi_1^1\psi_1^2\ ,\ v_2(\phi^\dagger) = \psi_2^3\psi_2^4 \\ v_3(\phi) &= \psi_3^1\psi_3^2\ ,\ v_4(\phi^\dagger) = \psi_4^3\psi_4^4\ . \end{aligned} \tag{4.8}$$

On the twistor side the amplitude is

$$\langle\phi,\phi^\dagger,\phi,\phi^\dagger\rangle_{\text{Twistor}} = g^2\int d^8\theta\frac{\psi_1^1\psi_1^2\psi_2^3\psi_2^4\psi_3^1\psi_3^2\psi_4^3\psi_4^4}{\langle12\rangle\langle23\rangle\langle34\rangle\langle41\rangle} = \frac{g^2}{16}\frac{\langle13\rangle^2\langle24\rangle^2}{\langle12\rangle\langle23\rangle\langle34\rangle\langle41\rangle} \tag{4.9}$$

The spacetime side receives contributions from three Feynman diagrams

$\langle\phi,\phi^\dagger,\phi,\phi^\dagger\rangle_{4d} =$



(4.10)

By explicit evaluation we once again find that $\langle\phi,\phi^\dagger,\phi,\phi^\dagger\rangle_{4d} = 32i\langle\phi,\phi^\dagger,\phi,\phi^\dagger\rangle_{\text{Twistor}}$.

4.5 The amplitude $\langle\eta_{A'},\lambda^a,\bar\lambda^b,\bar\eta_{B'}\rangle$

Let us also examine some more detailed results concerning analytic amplitudes with non-trivial dependence on the SU(2) indices a, A, which also involve external fundamental particles. As an example we consider $\langle\eta_{A'},\lambda^a,\bar\lambda^b,\bar\eta_{B'}\rangle$. The wavefunctions can once again be read off from (3.12) and (3.22) to give

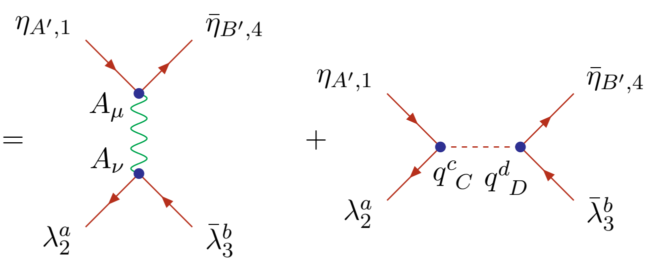
$$\begin{aligned} v_1(\eta_A) &= \psi_1^A, & v_2(\lambda^a) &= \epsilon_{da}\psi_2^d \\ v_3(\tilde\lambda^b) &= \epsilon_{cb}\psi_3^3\psi_3^4\psi_3^c, & v_4(\tilde\eta_B) &= \psi_4^B\psi_4^1\psi_4^2\ . \end{aligned} \tag{4.11}$$

The evaluation of the resulting integral is highly simplified by the use of various identities, collected in appendix A. The answer is

$$\langle\eta_A,\lambda^a,\tilde\lambda^b,\tilde\eta_B\rangle_{\text{Twistor}} = g^2\epsilon_{cb}\epsilon_{da}\int d^8\theta\frac{\psi_1^A\psi_2^d\psi_3^3\psi_3^4\psi_3^c\psi_4^B\psi_4^1\psi_4^2}{\langle12\rangle\langle23\rangle\langle34\rangle\langle41\rangle} = \frac{g^2}{16}\epsilon_{ab}\epsilon^{AB}\left(\frac{\langle34\rangle}{\langle12\rangle}+\frac{\langle34\rangle^2}{\langle23\rangle\langle14\rangle}\right). \tag{4.12}$$

On the other hand, the diagrams contributing to the gauge theory calculation are the following

$\langle\eta_{A'},\lambda^a,\bar\lambda^b,\bar\eta_{B'}\rangle_{4d} =$



(4.13)

and explicit calculation using Feynman rules leads to $\langle \eta_{A'}, \lambda^a, \bar{\lambda}^b, \bar{\eta}_{B'} \rangle_{4d} = 32i \langle \eta_A, \lambda^a, \tilde{\lambda}^b, \tilde{\eta}_B \rangle_{\text{Twistor}}$.

Here we come to a crucial point: When matching the spectra for the full $N_f = 4$ theory, we had already noticed that agreement could only be obtained if we decomposed the global flavour index in terms of its special maximal subgroups $\mathrm{SO}(8) \supset \mathrm{SU}(2)_{A'} \times \mathrm{Sp}(2)$ and then somehow related the $\mathrm{SU}(2)_{A'}$ part to the flavour group for the antisymmetric hypermultiplets $\mathrm{SU}(2)_A$. The requirement of matching amplitudes with external fundamental particles reaffirms this suggestion, since in order to get agreement the two symmetries need to be identified! This implies that the twistor string does not reproduce a gauge theory with flavour group SO(8) but a theory which has had the latter explicitly broken down to $\mathrm{SU}(2) \times \mathrm{Sp}(2)$. Moreover, this SU(2) should then be realised geometrically on the gauge theory side; recall that in the IIB description the flavour group for the antisymmetric hypermultiplet fields was related to part of the rotations of the D3 worldvolume in the transverse 6d space. The geometric realisation on the twistor side is explicit and obvious in terms of the $\mathrm{SU}(2)_A$ symmetry rotating the fermionic coordinates ψ^A. This result is quite intriguing and we will briefly return to it in the conclusions.

4.6 Further analytic amplitudes

By now, the general strategy implemented for calculating 4-point amplitudes on both sides of the correspondence should be clear to the reader. Therefore, we will simply display the twistor answer for several other analytic amplitudes which we have verified to match those arising from the gauge theory calculation, up to the same relative normalisation factor of $32i$. These amplitudes are

$$\langle \lambda^a, \phi^\dagger, \bar{\lambda}^b, \phi \rangle = \frac{g^2}{16} \epsilon_{ab} \frac{\langle 23 \rangle}{\langle 12 \rangle} \tag{4.14}$$

$$\langle z^a{}_A, z^b{}_B, z^c{}_C, z^d{}_D \rangle = \frac{g^2}{16} \bigg(-\frac{\langle 12 \rangle \langle 34 \rangle}{\langle 23 \rangle \langle 14 \rangle} \epsilon_{ad} \epsilon_{bc} \epsilon^{AD} \epsilon^{BC} - \frac{\langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 34 \rangle} \epsilon_{ab} \epsilon_{cd} \epsilon^{AB} \epsilon^{CD} \\ + \epsilon_{ab} \epsilon_{cd} \epsilon^{AD} \epsilon^{BC} + \epsilon_{ad} \epsilon_{bc} \epsilon^{AB} \epsilon^{CD} \bigg) \tag{4.15}$$

$$\langle \phi^\dagger, z^a{}_A, z^b{}_B, \phi \rangle = \frac{g^2}{16} \frac{\langle 13 \rangle \langle 24 \rangle}{\langle 23 \rangle \langle 14 \rangle} \epsilon_{ab} \epsilon^{AB} \tag{4.16}$$

$$\langle z^a{}_A, \zeta_C, \bar{\zeta}_D, z^b{}_B \rangle = -\frac{g^2}{16} \epsilon_{ab} \left(\epsilon^{AB} \epsilon^{CD} \frac{\langle 13 \rangle \langle 34 \rangle}{\langle 23 \rangle \langle 14 \rangle} + \epsilon^{AC} \epsilon^{BD} \frac{\langle 13 \rangle}{\langle 12 \rangle} \right) . \tag{4.17}$$

We recall that the partial amplitudes involving fundamental external particles can be obtained directly from the antisymmetric ones by (pair-wise) substitution of states. For example one has that $\langle q^a{}_A, q^b{}_B, q^c{}_C, q^d{}_D \rangle = \langle z^a{}_A, z^b{}_B, z^c{}_C, z^d{}_D \rangle = \langle q^a{}_A, q^b{}_B, z^c{}_C, z^d{}_D \rangle$ and so on.

These results strongly indicate that our proposed twistor duality for the $N_f = 4$ theory, as well as the assumption that (4.2) is applicable for amplitude calculations, are valid. However, the structure of 4-point analytic amplitudes is relatively trivial. A more concrete affirmation is given by examining and finding agreement for 5-point amplitudes.

This would allow us to confidently state that we are indeed considering the correct twistor string theory dual. We have indeed explicitly checked this for the following two examples

$$\langle \lambda^a, z^b{}_B, z^c{}_C, \lambda^d, \phi^\dagger \rangle = \frac{g^2}{16}\epsilon^{BC}\left(\frac{\langle 25\rangle\langle 35\rangle}{\langle 23\rangle\langle 45\rangle\langle 15\rangle}\epsilon_{ad}\epsilon_{bc} - \frac{\langle 25\rangle\langle 35\rangle\langle 14\rangle}{\langle 12\rangle\langle 34\rangle\langle 45\rangle\langle 15\rangle}\epsilon_{ab}\epsilon_{cd}\right) \quad (4.18)$$

$$\langle \phi, q^a{}_A, q^b{}_B, \eta_C, \eta_D \rangle = -\frac{g^2}{16}\epsilon_{ab}\left(\frac{\langle 13\rangle}{\langle 34\rangle\langle 15\rangle}\epsilon^{AD}\epsilon^{BC} - \frac{\langle 13\rangle\langle 25\rangle}{\langle 23\rangle\langle 45\rangle\langle 15\rangle}\epsilon^{AB}\epsilon^{CD}\right). \quad (4.19)$$

Once again, the results from the gauge theory side turn out to match those on the twistor side up to the normalisation factor of $32i$.

5. The $N_f = 2N$ theory

We now turn our attention to another class of $\mathcal{N} = 2$ UV-finite gauge theories, namely the theories with gauge group $SU(N)$ and flavour group $SU(N_f)$, where $N_f = 2N$. As discussed in the introduction, this is the alternative way of extending the $SU(2)$, $N_f = 4$ theory of Seiberg and Witten [45] beyond rank one. Here, we will identify the twistor string dual to this $N_f = 2N$ theory. Since we have done most of the work in order to describe the $N_f = 4$ case, we will omit some of the details in this case.

5.1 Physical string theory description

We will begin by reviewing the 10-dimensional string theory description which realises this gauge theory, in the same vein as for our $N_f = 4$ treatment. Unlike the previous case, this theory does not have a natural connection to F-theory, but can instead be engineered as the low energy worldvolume theory on a stack of N fractional D3-branes probing the background generated by N_f fractional D7-branes in Minkowski space with four orbifolded directions $\mathbb{R}^{1,5} \times \mathbb{R}^4/\mathbb{Z}_2$. The latter are taken to be $(x^4, \ldots, x^7)$, with $\mathbb{Z}_2$ acting on them as $(x^4, \ldots, x^7) \to (-x^4, \ldots, -x^7)$. We take the D3s to lie along $(x^0, \ldots, x^3)$, and the D7s to be in $(x^0, \ldots, x^7)$. The D3-D7 system preserves 8 supercharges and the orbifold action has been chosen such that it does not break the supersymmetry any further [112, 113]. Once more, the 3-7 (7-3) strings provide the matter hypermultiplets transforming in the fundamental (conjugate-fundamental) representation of the gauge group and in the probe limit their $SU(N_f)$ Chan-Paton index takes values in a global symmetry group. Similarly, in this limit the 'heavy' 7-7 strings decouple and one obtains a 4d $\mathcal{N} = 2$, $SU(N)$ gauge theory with N_f fundamental hypermultiplets.

We are interested in the case where the D3s and D7s are located at the same point in the transverse (x^8, x^9) directions (so there are no masses for the matter fields) and where all D3s are coincident (that is, no vevs). This is very reminiscent of the way we constructed the $N_f = 4$ theory. There are, however, some crucial differences: Firstly, there is no orientifold plane in this case and hence no gauge symmetry enhancement at any point on the moduli space; the gauge groups corresponding to the open string degrees of freedom remain $SU(N)$. Secondly, the number of flavours corresponding to the conformal point is chosen via a very different mechanism: On the supergravity side the solution exhibits a naked singularity,

Component	SO(1,3)	$SU(2)_a$	$SU(2)_A$	$U(1)_R$	$SU(N)$	$SU(2N)\times U(1)$
A, G	$(2,2)$	1	1	0	N^2-1	1
ϕ	$(1,1)$	1	1	$+2$	N^2-1	1
$\phi^\dagger$	$(1,1)$	1	1	-2	N^2-1	1
$\lambda_{\alpha,a}$	$(2,1)$	2	1	$+1$	N^2-1	1
$\bar{\lambda}_{\dot{\alpha},a}$	$(1,2)$	2	1	-1	N^2-1	1
$q_a^I, q_{aI}^\dagger$	$(1,1)$	2	1	0	$N, \overline{N}$	$\overline{2N}_{-1}, 2N_{+1}$
$\eta_\alpha^I, \bar{\eta}_\alpha^{\prime I}$	$(2,1)$	1	1	-1	N	$\overline{2N}_{-1}$
$\bar{\eta}_{\dot{\alpha}I}, \eta'_{\dot{\alpha}I}$	$(1,2)$	1	1	$+1$	$\overline{N}$	$2N_{+1}$

Table 3: The on-shell field content of the $N_f = 2N$ theory in component form. Once again, the Lorentz representations are given in terms of $SO(1,3) \to SO(4) \sim SU(2)_L \times SU(2)_R$. The fundamental fields carry an $SU(2N)$ index $I = 1, \ldots, 2N$.

a usual feature in the gravity description of non-conformal theories. For the case of non-compact orbifolds, however, the appearance of an enhançon [114] prevents the theory from being trusted all the way to the singular point since new, light degrees of freedom appear at the enhançon radius. At that point, the SQCD energy scale diverges. The excision of the region between the enhançon radius and the naked singularity corresponds to discarding energy scales where nonperturbative effects become relevant. This also prevents one from obtaining a supergravity dual to the gauge theory à la Maldacena.[30] This system therefore only describes the perturbative regime of the gauge theory, which is however precisely the one that we want to reproduce from a twistor string perspective. For $N_f = 2N$ the enhançon radius vanishes, the gauge coupling stops running, and the theory sits at the conformal point in its moduli space [118]. In the following we will focus on this conformal $N_f = 2N$ case.

Let us now take a look at the open string massless spectrum of the theory. This is very similar to the one we studied for $N_f = 4$ and is summarised in table 3. The orbifold projection discards the 3-3 open string modes responsible for the antisymmetric hypermultiplets in the $N_f = 4$ theory. This can be intuitively seen from the inability of the fractional D3s to move away from the orbifold-fixed plane and therefore the antisymmetric hypermultiplet modes, which were accounting for those degrees for freedom, are now absent.

Also note that no field transforms nontrivially under the $SU(2)_A$. The reason we include this symmetry in table 3 is to precisely highlight the similarities and differences with the massless spectrum of the $N_f = 4$ theory. The absence of the antisymmetric hypermultiplet is a sign that the discussion related to the geometric realisation of an SU(2) subgroup of the full flavour symmetry in the spacetime picture will not make an appearance in this context.

[30] The impossibility of obtaining a supergravity dual even for the conformal $N_f = 2N$ theory can also be seen by noting (e.g. [115]) that (unlike the $N_f = 4$ theory) the two coefficients a and c of the four-dimensional anomaly are not equal to leading order in $1/N$, violating a requirement of [116, 117].

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5.2 The spacetime action

We will now repeat the same steps as for the analysis of the $N_f = 4$ theory. Without further delay, let us write down the corresponding $\mathcal{N} = 2$ Lagrangean in terms of $\mathcal{N} = 1$ superfields

$$\begin{aligned}\mathcal{L} = &\frac{1}{8\pi}\text{Im Tr}\left[\tau\left(\int d^2\theta\, W^\alpha W_\alpha + 2\int d^2\theta d^2\bar{\theta}\, e^{2V}\Phi^\dagger e^{-2V}\Phi\right)\right] + \int d^2\theta d^2\bar{\theta}\, Q^{\dagger I} e^{-2V} Q_I \\ &+ \int d^2\theta d^2\bar{\theta}\, Q'^I e^{2V} Q'^\dagger_I + \sqrt{2}\int d^2\theta \left(Q'^I \Phi Q_I + h.c.\right) \,, \end{aligned} \tag{5.1}$$

where the Is are now fundamental $\text{SU}(N_f)$ indices and therefore $I = 1, \ldots, 2N$. The evaluation of the kinetic part of the action will follow directly from the previous case by setting the antisymmetric fields to zero and keeping in mind the new global flavour group. After expanding the superfields and performing the Grassmann integration the result reads

$$\begin{aligned}\mathcal{L} =& \frac{1}{g^2}\text{Tr}\left(-\frac{1}{4}F^2 + (D^\mu\phi)^\dagger(D_\mu\phi) - i\bar{\lambda}\,\not{D}\lambda - i\bar{\chi}\,\not{D}\chi - i\sqrt{2}\,[\lambda,\chi]\phi^\dagger - i\sqrt{2}\,[\bar{\lambda},\bar{\chi}]\phi\right) \\ &+ (D^\mu q)^{\dagger I}(D_\mu q)_I + (D^\mu q')^I (D_\mu q')^\dagger_I - i\bar{\eta}^I\,\not{D}\eta_I - i\eta'^I\,\not{D}\bar{\eta}'_I - i\sqrt{2}\, q^{\dagger I}\lambda\eta_I \\ &+ i\sqrt{2}\,\bar{\eta}^I\bar{\lambda}q_I - i\sqrt{2}\, q'^I\bar{\lambda}\eta'_I + i\sqrt{2}\,\eta'^I\lambda q'^\dagger_I - \sqrt{2}\left(\eta'^I\chi q_I + \eta'^I\phi\eta_I + q'^I\chi\eta_I\right) \\ &- \sqrt{2}\left(q^{\dagger I}\bar{\chi}\bar{\eta}'_I + \bar{\eta}^I\phi^\dagger\bar{\eta}'_I + \bar{\eta}^I\bar{\chi}q'^\dagger_I\right) - V_S \,.\end{aligned} \tag{5.2}$$

Once again V_S denotes the scalar potential obtained by integrating out the auxiliary F- and D-terms, whose contributing terms are now given by

$$(\mathsf{F}_q)^i_I = -\sqrt{2}\,(\phi^\dagger)^i{}_j q'^{\dagger j}_I \tag{5.3}$$

$$(\mathsf{F}_{q'})^I_i = -\sqrt{2}\, q^{\dagger I}_j (\phi^\dagger)^j{}_i \tag{5.4}$$

$$(\mathsf{F}_\phi)^j{}_i = -g^2\sqrt{2}\, q^{\dagger I}_i q'^{\dagger j}_I \tag{5.5}$$

$$\mathsf{D}^a = -\text{Tr}\left(T^a[\phi^\dagger,\phi]\right) + g^2\left(q^{\dagger I}T^a q_I - q'^I T^a q'^\dagger_I\right) \,, \tag{5.6}$$

where the $(T^a)^i{}_j$'s are the generators of the fundamental representation of $\text{SU}(N)$. In the calculation of these terms we have, in principle, the introduction of $1/N$ contributions from the coupling of the fundamental fields to the 'photon'

$$(T^a)^i{}_j (T_a)^k{}_l = \delta^i_l \delta^k_j - \frac{1}{N}\delta^i_j \delta^k_l \,. \tag{5.7}$$

These, however, will decouple along with the rest of the colour information during the stripping process. We then perform the field redefinitions (2.7) and (2.9), and once again combine fields in $\text{SU}(2)_a$ doublets. The adjoint fermions are redefined as in (2.10), while for the fundamental scalars we now have

$$\begin{aligned}(q^a)^i{}_I &= \begin{pmatrix} q^i{}_I \\ -q'^{\dagger i}{}_I \end{pmatrix} \,,\; (q^\dagger_a)^I{}_i = \left(q^{\dagger I}{}_i, -q'^I{}_i\right) \\ (q^{\dagger a})^I{}_i &= \begin{pmatrix} -q'^I{}_i \\ -q^{\dagger I}{}_i \end{pmatrix} \,,\; (q_a)^i{}_I = \left(q'^{\dagger i}{}_I, q^i{}_I\right) \,. \end{aligned} \tag{5.8}$$

The full action, including the quartic terms, now becomes

$$\begin{aligned}\mathcal{L} =&\mathrm{Tr}\left[-\frac{1}{2}GF+\frac{1}{4}g^2G^2+D\phi^\dagger D\phi+i\bar\lambda^a \not{D}\lambda_a-\lambda^a\lambda_a\phi^\dagger+2g^2\bar\lambda^a\bar\lambda_a\phi\right]-\eta'^I\phi\eta_I-2g^2\bar\eta^I\phi^\dagger\bar\eta'_I\\ &-Dq^{\dagger aI}Dq_{aI}-i\bar\eta^I\not{D}\eta_I-i\eta'^I\not{D}\bar\eta'_I+q^{\dagger aI}\lambda_a\eta_I-\eta'^I\lambda^a q_{aI}+2g^2\bar\eta^I\bar\lambda^a q_{aI}-2g^2q^{\dagger aI}\bar\lambda_a\bar\eta'_I\\ &-\frac{g^2}{2}\mathrm{Tr}[\phi^\dagger,\phi]^2+g^2q^{\dagger aI}\{\phi^\dagger,\phi\}q_{aI}-\frac{g^2}{2}\left[(q^{\dagger aI}q_{aJ})(q^{\dagger bJ}q_{bI})+(q_a^{\dagger I}q_{bJ})(q^{\dagger aJ}q_I^b)\right]\\ &+\frac{g^2}{2N}\left[(q^{\dagger aI}q_{bI})(q_a^{\dagger J}q_J^b)+(q^{\dagger aI}q_I^b)(q_b^{\dagger J}q_{aJ})\right]\ .\end{aligned} \tag{5.9}$$

In light of the twistor picture that we will discuss in a moment, it seems natural to once again choose a special maximal embedding of SU(2) into SU($2N$), namely SU($2N$) $\supset$ SU(N)$\times$SU$(2)_{A'}$ and therefore we will decompose $I\to K\otimes A'$, with $K=1,\ldots,N$. Finally, after the appropriate chiral rescalings (analogous to (2.7), (2.9)) the selfdual truncation of the above is simply

$$\begin{aligned}\mathcal{L} =\ &\mathrm{Tr}\left[-\frac{1}{2}GF+D\phi^\dagger D\phi+i\bar\lambda^a\not{D}\lambda_a-\lambda^a\lambda_a\phi^\dagger\right]-Dq^{\dagger aA'K}Dq_{aA'K}\\ &-i\bar\eta^{A'K}\not{D}\eta_{A'K}-i\eta'^{A'K}\not{D}\bar\eta'_{A'K}-\eta'^{A'K}\phi\eta_{A'K}+q^{\dagger aA'K}\lambda_a\eta_{A'K}-\eta'^{A'K}\lambda^a q_{aA'K}\ .\end{aligned} \tag{5.10}$$

5.3 The twistor action

Let us now see how we can reproduce the spectrum of this $N_f=2N$ theory on the twistor side and obtain the appropriate twistor action. We will not provide exhaustive details for this construction, since the arguments follow our previous analysis of the $N_f=4$ theory very closely. To proceed, we simply orbifold two of the fermionic directions of $\mathbb{C}\mathrm{P}^{3|4}$, namely

$$\psi^a\to\psi^a\,,\qquad \psi^A\to-\psi^A \tag{5.11}$$

and place N D_c branes spanning the bosonic and holomorphic fermionic directions, as well as N (rather than $2N$, which might seem more natural at first) D_f branes on the orbifold plane $\psi^3=\psi^4=0$ (as before, this is loose language for "branes satisfying DD boundary conditions in the ψ^3,ψ^4 directions"). The orbifold action on the Chan-Paton indices will again be given by $\gamma_c=\mathbb{I}_{N\times N}$ and $\gamma_f=-\mathbb{I}_{N\times N}$. The invariant piece of the $c-c$ superfield $\mathcal{A}$ is

$$\tilde{\mathcal{A}}=(A+\psi^a\lambda_a+\psi^1\psi^2\phi+\psi^3\psi^4\phi^\dagger+\epsilon_{cd}\psi^3\psi^4\psi^c\tilde\lambda^d+\psi^1\psi^2\psi^3\psi^4G)\,, \tag{5.12}$$

which, via the arguments of the previous sections, will be mapped to the spectrum of an $\mathcal{N}=2$ vector multiplet in the adjoint of the gauge group SU(N). Leaving aside the $f-f$ sector (the only difference from the $N_f=4$ case being that the Chan-Paton indices will be in SU(N), parametrised by $K=1\ldots N$) we will focus on the $c-f$ and $f-c$ strings. Arguing similarly to section 3.4, we find that the states surviving the orbifold projection are now the following (0,1)-forms

$$\mathcal{Q}^i{}_K=\psi^AQ^i_{AK}\,,\qquad\qquad \mathcal{Q}^{\dagger K}{}_i=\psi^AQ^{\dagger K}_{Ai}\ . \tag{5.13}$$

The † here simply denotes that these superfields transform in conjugate representations of the gauge group SU(N), namely the fundamental and conjugate fundamental respectively. We can further decompose Q_A and $Q^\dagger_A$ into their components (suppressing gauge indices from now on)

$$\begin{aligned} Q_{AK} &= \eta_{AK} + \psi^a q_{aAK} + \psi^1\psi^2\tilde{\eta}'_{AK} \ , \\ Q^{\dagger K}_A &= \eta'^K_A + \psi^a q^{\dagger K}_{aA} + \psi^1\psi^2\tilde{\eta}^K_A \ . \end{aligned} \tag{5.14}$$

The details related to identifying the BRST cohomology pertaining to the fundamental superfields $\mathcal{Q}$, presented for the $N_f = 4$ theory in section 3, will go through intact for this case as well. The above expressions therefore provide the correct field content to reproduce the spacetime spectrum for the fundamental hypermultiplets. It should now be clear that N D_f branes suffice to provide the $2N$ hypermultiplets, although in a form where the SU($2N$) global group is not manifest.

The final twistor description is given by the hCS action

$$S = \int_{\mathrm{D}_c} \mathbf{\Omega} \wedge \left(\frac{1}{2}\mathrm{Tr}\left[\tilde{\mathcal{A}} \cdot \bar{\partial}\tilde{\mathcal{A}} + \frac{2}{3}\tilde{\mathcal{A}} \wedge \tilde{\mathcal{A}} \wedge \tilde{\mathcal{A}} \right] + \mathcal{Q}^{\dagger K} \cdot \bar{\partial}\mathcal{Q}_K + \mathcal{Q}^{\dagger K} \wedge \tilde{\mathcal{A}} \wedge \mathcal{Q}_K \right) \ , \tag{5.15}$$

where $\tilde{\mathcal{A}}$ is as shown in (5.12). In component form this can be expanded into

$$\begin{aligned} S_{hCS} = \int_{\mathbb{CP}^3} \mathbf{\Omega}' \wedge \big(&\mathrm{Tr}[G \wedge F + \phi^\dagger \wedge \bar{D}\phi - \tilde{\lambda}^a \wedge \bar{D}\lambda_a + \lambda^a \wedge \lambda_a \wedge \phi^\dagger] \\ &+ \tilde{\eta}^{KA} \wedge \bar{D}\eta_{AK} + \eta'^{KA} \wedge \bar{D}\tilde{\eta}'_{AK} - q^{\dagger aKA} \wedge \bar{D}q_{aAK} \\ &+ \eta'^{KA} \wedge \phi \wedge \eta_{AK} - q^{\dagger aKA} \wedge \lambda_a \wedge \eta'_{AK} + \eta'^{KA} \wedge \lambda_a \wedge q_{aAK} \big) \ . \end{aligned} \tag{5.16}$$

The similarity with (5.10) is obvious, once one identifies A with A'. As we have already mentioned for the $N_f = 4$ theory, we expect a nonlinear form of the Penrose transform to map the above action to the selfdual truncation of the spacetime Lagrangean, given by (5.10). As expected, we cannot assign a geometric meaning to the spacetime $\mathrm{SU}(2)_{A'}$ in this case, even though the twistor string description of $\mathrm{SU}(2)_A$ is explicitly geometric. Note, however, that in the component action (5.16) (but not in (5.15)) we can trivially undo the $\mathrm{SU}(2N) \supset \mathrm{SU}(N) \times \mathrm{SU}(2)$ decomposition to exhibit the full global flavour group $\mathrm{SU}(2N) \times \mathrm{U}(1)$. On the other hand, to apply the twistor amplitude prescription (which explicitly involves the ψ^A coordinates) one is obliged to work with this symmetry non-manifest, and restore it at the end by combining the relevant sets of amplitudes.[31]

Before proceeding to compare amplitudes, we should emphasise the similarities between this construction for $N_f = 2N$ and that for the $N_f = 4$ theory which we explored in section 3: The two theories differ only by the presence of the orientifold and the number of D_f branes that are introduced. In the case of rank one (where the orientifold imposes no condition, since $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$) they reduce to the same theory — the Seiberg-Witten SU(2) SYM with four massless flavours. This simple picture is in contradistinction with

[31]Instead, we chose to compute gauge theory amplitudes in decomposed form and compare with the twistor results.

the IIB embeddings of these two theories, where (for instance) even the corresponding orbifold actions are taken in different spacetime directions, and it is difficult to see how they become equivalent for rank one. Presumably the twistor string is able to be so concise in its description of this pair of theories because (unlike their IIB string duals) it is only required to know about perturbative gauge theory physics.

5.4 Comparison of amplitudes

Finally, we move on to compare partial amplitudes on both sides of the correspondence. In fact, the similarity in field content between the $N_f = 4$ and $N_f = 2N$ theories means that the partial amplitude calculations are almost identical, since the only novelty, apart from the absence of the antisymmetric hypermultiplet, is the behaviour of the fundamental scalars and fermions due to the SU(N) gauge group. For example, it is easy to see that the partial amplitude involving adjoint external particles is exactly the same as for the $N_f = 4$ theory. Moreover, it is straightforward to replace the appropriate fundamental fields and vertices to find the same agreement between the twistor and spacetime results, including the relative normalisation factor of $32i$.

As such, we only display two amplitudes. These involve fundamental external particles and, at 4 and 5-point respectively, are

$$\begin{aligned}\langle q^{\dagger a}{}_A, q^b{}_B, q^{\dagger c}{}_C, q^d{}_D\rangle_{\text{Twistor}} &= \frac{g^2}{16}\left(-\frac{\langle 12\rangle\langle 34\rangle}{\langle 23\rangle\langle 14\rangle}\epsilon_{ad}\epsilon_{bc}\epsilon^{AD}\epsilon^{BC} - \frac{\langle 14\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle}\epsilon_{ab}\epsilon_{cd}\epsilon^{AB}\epsilon^{CD}\right. \qquad (5.17)\\ &\qquad \left. +\epsilon_{ab}\epsilon_{cd}\epsilon^{AD}\epsilon^{BC} + \epsilon_{ad}\epsilon_{bc}\epsilon^{AB}\epsilon^{CD}\right)\\ \langle \phi, q^a{}_A, q^{\dagger b}{}_B, \eta_C, \eta'_D\rangle_{\text{Twistor}} &= -\frac{g^2}{16}\epsilon_{ab}\left(\frac{\langle 13\rangle}{\langle 34\rangle\langle 15\rangle}\epsilon^{AD}\epsilon^{BC} - \frac{\langle 13\rangle\langle 25\rangle}{\langle 23\rangle\langle 45\rangle\langle 15\rangle}\epsilon^{AB}\epsilon^{CD}\right)\ . \qquad (5.18)\end{aligned}$$

A straightforward gauge theory calculation exactly reproduces these results.

6. Conclusions and outlook

In this paper we have extended the correspondence between 4d UV-finite supersymmetric gauge theories and B-model twistor string theory at tree level, by identifying the twistor string duals for theories containing fundamental matter. These theories were $\mathcal{N} = 2$, Sp(N) SYM with $N_f = 4$ and $\mathcal{N} = 2$, SU(N) SYM with $N_f = 2N$ fundamental hypermultiplets, both sitting at the superconformal point of their moduli space. We initially studied the physical string realisation of these theories and examined the open string massless spectrum, which allowed us to properly identify all the symmetries of the system. We then used this information to construct their proper spacetime Lagrangean description. On the twistor side, we performed a superorientifold and superorbifold projection respectively, which yielded the non-fundamental part of the spectrum. The fundamental degrees of freedom were introduced via new objects in the topological B-model on supertwistor space, which we baptised flavour-branes (D_f). These wrap all the bosonic but only half of the holomorphic fermionic directions spanned by Witten's Euclidean 'D5'-branes providing the colour degrees of freedom (D_c).

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We then proceeded to compare amplitudes on both sides of the proposed correspondence. We found precise agreement for a number of 4– and 5-point amplitudes, involving external particles transforming in the adjoint, fundamental, and, in the $N_f = 4$ case, antisymmetric representations of the gauge group. These results provide strong evidence for the robustness of the twistor string duals, and even though we only calculated analytic ('MHV') processes in this work, we believe that the agreement should continue to hold for tree level amplitudes supported on holomorphic curves of higher degree.

In the process of performing the identification between the two sides, the embedding of the flavour-branes into the hCS theory of the colour-branes forced us to provide a geometric realisation for an SU(2) subgroup of the flavour group, and in the $N_f = 4$ case to identify that with the flavour symmetry of the antisymmetric hypermultiplet fields. At first glance, this decomposition of the flavour group might seem slightly *ad hoc*; we could have chosen any other subgroup which contains SU(2). However, our choice is consistent with reproducing the same gauge group on the D_f branes as the one appearing on the D_c branes in the B-model, namely $\mathrm{Sp}(N)$ and $\mathrm{SU}(N)$ for the two cases. The fact that, in the $N_f = 4$ case, this decomposition leads to both kinds of branes coming with the same type of gauge group (i.e. both symplectic) is not unreasonable, if one remembers that they wrap the same number of bosonic directions in $\mathbb{CP}^3$.

For the $N_f = 4$ theory, we found that the twistor string side actually describes a gauge theory with global flavour symmetry broken down to $\mathrm{SO}(8) \to \mathrm{SU}(2)_A \times \mathrm{Sp}(2)$. This was due to two unrelated (from the gauge theory point of view) SU(2) groups being identified with the same geometric $\mathrm{SU}(2)_A$ on the twistor string side, and as such the twistor string does not seem to describe precisely the theory that we set off to recover. This could be so for a number of reasons: One possibility is clearly that we have not found the most generic twistor string description of the $N_f = 4$ theory, and that, despite the apparent rigidity of our construction, further investigation might reveal a way to disentangle these two symmetries. A second possibility is that this is indeed the correct symmetry group of the IIB setup once the effects of interactions between the fundamental and antisymmetric hypermultiplet sectors are taken into account (recall that the claim that the D3-D7 brane configuration accurately describes the $N_f = 4$ theory is based mainly on inspection of the spectrum). Checking this would entail establishing whether open string interactions involving the antisymmetric hypermultiplet in the physical string picture preserve the global SO(8) flavour group or not. A final possibility is that the twistor string actually maps to an enriched version of the original physical brane construction. For example, this could arise by taking the instantons on the D7 worldvolume theory away from the zero thickness limit, which, if localised in the relative transverse directions between the D3s and D7s, could break the global symmetry precisely in the required fashion.[32] In this case, the mechanism leading to the geometric interpretation of the $\mathrm{SU}(2)_A$ symmetry would be analogous to the usual embedding of the gauge group into the spin connection. However, one is then forced to explain why the twistor string only manages to capture the dynamics of this rather special configuration, as well as to reconcile such a solution (which would seem to move the theory

[32] We would like to thank K.S. Narain for suggesting such a possibility.

towards the Higgs branch) with the apparently unbroken conformal invariance. It would be intriguing to uncover the answer to this question, which we will, however, not address at present. We should emphasise that in the $N_f = 2N$ theory the full flavour symmetry is accurately (though not manifestly, given the decomposition $\mathrm{SU}(2N) \to \mathrm{SU}(2)_A \times \mathrm{SU}(N)$) captured by the twistor side and a spacetime geometric interpretation of the $\mathrm{SU}(2)_A$ on the IIB side is not forced, essentially due to the absence of the antisymmetric hypermultiplet.

As discussed in the introduction, the main reason for studying twistor string duals of finite theories is to potentially understand what, if anything, makes them special on the twistor side. It is clear that generic non-finite theories are not expected to have a dual with a $\mathbb{CP}^3$ component at the quantum level, while the duals of the theories we have considered in this work should have a $\mathbb{CP}^3$ description also at loop level. Unfortunately, since our understanding of twistor string theory is confined to tree level, at this stage we have not been able to identify what is the distinguishing feature of our finite theories as far as twistor strings are concerned. For example, for the theory considered in sections 2, 3 and 4, we could just as well have added one flavour-brane (and its mirror) instead of two, and the construction would have worked out in a very similar fashion, reproducing the amplitudes of an $\mathcal{N} = 2$ theory with two (rather than four) fundamental hypermultiplets, clearly not a finite theory. The challenge, therefore, is to find a condition (similar to the RR charge cancellation requirement which enforces $N_f = 4$ on the physical string side) which constrains the number of flavour-branes we can add to the B-model on $\mathbb{CP}^{3|4}$.[33] An immediate obstacle is that our D_f branes, whose number we would like to fix, have an $\mathrm{Sp}(2)$ gauge group, while in the physical string context, orientifold planes leading to symplectic (rather than orthogonal) groups on the corresponding branes have positive RR charges, and thus are not relevant in situations where the total brane charge has to cancel. However, in our topological context, this could perhaps be circumvented by recalling the arguments of Vafa [120] that topological anti-branes can be derived from branes by formally taking $N \to -N$. This, combined with the observation [121] that — as far as gauge invariant quantities are concerned — in gauge theory $\mathrm{Sp}(N)$ can be thought of as $\mathrm{SO}(-N)$, indicates that our D_f's might be best thought of as anti-branes, whose negative 'charge' could potentially cancel that of the orientifold plane. Similar comments apply to the $N_f = 2N$ theory as well, although the details will be different since in this case requiring finiteness fixes the relative number of colour and flavour-branes rather than the absolute number of D_f's. Finding a mechanism that produces the above restrictions should give considerable insight on how to properly complete the twistor string description of finite gauge theories at the quantum level.[34]

We should note that, although (as discussed above) our tree-level construction (and

[33] We note that (bosonic) topological string orientifolds in the twistor string context have been considered in [119]. However, in that context the ensuing restriction on the number of colour branes (and thus the rank of the $\mathcal{N} = 4$ SYM gauge group) was deemed an unpleasant feature, and most consideration was given to orientifolds of lower-dimensional subspaces of $\mathbb{CP}^{3|4}$. Perhaps the arguments in [119] could be revisited with our current goal of restricting the number of *flavour* branes in mind.

[34] And, applied in the other direction, might play a role in establishing the UV-finiteness of other gauge or even gravity theories admitting a twistor string description (a class which might, perhaps via a suitable extension of the self-dual results of [40], potentially include $\mathcal{N} = 8$ supergravity).

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the ensuing amplitude calculations) applies to gauge theories with different numbers of flavours than those required for finiteness, for $\mathrm{Sp}(N)$ gauge theories there seems to be a restriction to even numbers of flavours, since we required (for $N=2$) the decomposition $4N \to (2,2N)$ of the fundamental of $\mathrm{SO}(4N)$ under $\mathrm{SO}(4N) \to \mathrm{SU}(2) \times \mathrm{Sp}(N)$. At tree level (where finiteness constraints should not arise) we might expect the twistor string to also describe theories with e.g. $N_f = 3$, leading to an SO(6) flavour group, which would not fit in the above framework. Perhaps a different geometric embedding of the flavour-branes can account for such flavour groups.

Passing to other open directions suggested by our work, it is interesting to remark (extending the discussion in [45] to higher rank) that the (massless as well as massive) $N_f = 4$ theory is expected to enjoy an analogue of the $\mathrm{SL}(2,\mathbb{Z})$ Montonen-Olive symmetry of $\mathcal{N}=4$ SYM, which combines with Spin(8) triality to form the full duality group of the theory. The $\mathrm{SL}(2,\mathbb{Z})$ duality of the $\mathcal{N}=4$ theory motivated the authors of [119] to propose a strong-weak duality relating the B-model with the A-model on the same (super) Calabi-Yau. (Further discussion on the origin of this type of topological string duality can be found in [122].) It is intriguing to ask whether the duality group of the $N_f=4$ theory fits within this framework, and therefore whether there exists an A-model version of the setup we have constructed. Also, the fact that the F-theory perspective we reviewed in section 2.1 provides a natural explanation of the duality properties of the $N_f=4$ theory hints that perhaps a topological F-theory [123] point of view might provide some additional insight in this case. Furthermore, given that the standard B-model $\mathcal{N}=4$ SYM setup on $\mathbb{CP}^{3|4}$ has been conjecturally related (via the above S-duality plus mirror symmetry arguments [124]) to a B-model on the superquadric $\mathcal{L}^{5|6} \in \mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$ [125, 126], it is natural to ask whether flavour-branes could also be incorporated in the latter geometry, which should capture the dynamics of full (rather than self-dual) Yang-Mills theory without the need for D1-instantons.

From a gauge theory point of view, one of the main interesting features of the theories with fundamental matter we have considered is their richer vacuum structure as compared to $\mathcal{N}=4$ SYM, in particular the presence of Higgs branches. In the IIB embeddings we have reviewed, this moduli space acquires geometric meaning, in terms of the directions along which the various branes can be separated. Perhaps the similarities of our constructions to the physical string realisations can provide clues on how to move off the superconformal point from the twistor string perspective as well.

In conclusion, we have demonstrated that the topological B-model description of twistor strings is rich enough to accommodate finite four-dimensional theories with fundamental matter, and that the precise descriptions of these theories bear a strong resemblance to, but also intriguing differences from, the standard embeddings of these theories within physical string theory. Apart from suggesting that a thorough analysis of boundary conditions and associated D-branes for topological strings on supermanifolds (which was beyond the scope of this work) would be a worthwhile enterprise, we believe that our results reinforce the expectation that, by deciphering the (still mysterious) connection between twistor and physical strings, the current obstacles in establishing the twistor string duality at the quantum level can eventually be overcome.

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A. Notation and conventions

In this short appendix we set up the notation and conventions used throughout this paper.

Spacetime: We take the signature of spacetime to be $(+---)$ and the raising and lowering of spacetime spinor indices to be performed by

$$\begin{aligned}\psi^\alpha &= \epsilon^{\alpha\beta}\psi_\beta\ ,\ \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \\ \bar\psi^{\dot\alpha} &= \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}\ ,\ \bar\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}\ .\end{aligned} \tag{A.1}$$

We also have the following relations between the superspace variables

$$\begin{aligned}\theta^2 &= \theta^\alpha\theta_\alpha = -2\theta^1\theta^2, \quad \theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2 \\ \bar\theta^2 &= \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha} = 2\bar\theta_{\dot1}\bar\theta_{\dot2}, \quad \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta^2\ .\end{aligned} \tag{A.2}$$

The appropriate definitions for the ϵ-tensors are

$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ , \tag{A.3}$$

where the above satisfy $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^\alpha_\gamma$ and $\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \delta^{\dot\gamma}_{\dot\alpha}$. Superspace integration then obeys

$$\int d\theta\,\theta = 1\,, \qquad \int d^2\theta\,\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta} \tag{A.4}$$

and so on. During the evaluation of amplitudes in twistor space, one also encounters more complicated Grassmann integrals. The following identities dramatically simplify these superspace integrations (recall here that $\psi^I = -\theta^I_\alpha\lambda^\alpha$)

$$\int d^4\theta\ \psi^a_i\psi^1_j\psi^2_j\psi^b_k = \frac{1}{4}\epsilon^{ab}\langle ij\rangle\langle jk\rangle \tag{A.5}$$

$$\int d^4\theta\ \psi^A_i\psi^3_j\psi^4_j\psi^B_k = \frac{1}{4}\epsilon^{AB}\langle ij\rangle\langle jk\rangle \tag{A.6}$$

$$\int d^4\theta\ \psi^a_i\psi^b_j\psi^c_k\psi^d_l = \frac{1}{4}\Big(\epsilon^{ad}\epsilon^{bc}\langle ij\rangle\langle kl\rangle - \epsilon^{ab}\epsilon^{cd}\langle il\rangle\langle jk\rangle\Big) \tag{A.7}$$

$$\int d^4\theta\ \psi^A_i\psi^B_j\psi^C_k\psi^D_l = \frac{1}{4}\left(\epsilon^{AD}\epsilon^{BC}\langle ij\rangle\langle kl\rangle - \epsilon^{AB}\epsilon^{CD}\langle il\rangle\langle jk\rangle\right)\ . \tag{A.8}$$

These expressions also lead to a useful ϵ-tensor identity

$$\begin{aligned}
\int d^4\theta\ \psi_i^a\psi_j^b\psi_k^c\psi_l^d &= -\int d^4\theta\ \psi_i^a\psi_k^c\psi_j^b\psi_l^d \\
\Rightarrow \frac{1}{4}\left(\epsilon^{ad}\epsilon^{bc}\langle ij\rangle\langle kl\rangle - \epsilon^{ab}\epsilon^{cd}\langle il\rangle\langle jk\rangle\right) &= -\frac{1}{4}\left(\epsilon^{ad}\epsilon^{cb}\langle ik\rangle\langle jl\rangle - \epsilon^{ac}\epsilon^{bd}\langle il\rangle\langle kj\rangle\right) \\
\Rightarrow \epsilon^{ad}\epsilon^{bc} + \epsilon^{ab}\epsilon^{cd} &= \epsilon^{ac}\epsilon^{bd}\ .
\end{aligned} \tag{A.9}$$

We additionally make use of the following relations

$$\begin{aligned}
(\bar\sigma^\mu)^{\dot\alpha\alpha} &= \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta}\,, \quad & \theta\sigma^\mu\bar\theta\theta\sigma^\nu\bar\theta &= \frac{1}{2}\theta^2\bar\theta^2\eta^{\mu\nu} \\
\chi\sigma^\mu\bar\psi &= -\bar\psi\bar\sigma^\mu\chi\,, & (\chi\sigma^\mu\bar\psi)^\dagger &= \psi\sigma^\mu\bar\chi\ .
\end{aligned} \tag{A.10}$$

Gauge and flavour groups: The defining relation for elements of the Sp(N) algebra is that

$$M = -\Omega M^T \Omega^{-1} \tag{A.11}$$

for a hermitian matrix M, where Ω_{ij} is the invariant tensor of Sp(N). The fundamental and conjugate-fundamental indices are then raised and lowered using this tensor, which is defined via

$$\Omega_{ij} = \Omega^{ij} = -(\Omega^{-1})^{ij} = i\sigma_2 \otimes 1_{N\times N}\,,\ \text{where } \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and the indices are contracted following the 'NW-SE' rule. A useful property of matrices ${M^i}_j$ satisfying (A.11) is that they become symmetric once their upper index is lowered using Ω_{ij}. Contraction of the invariant tensor gives $\Omega^{ik}\Omega_{kj} = -\delta^i_j$, so that raising and lowering a contracted Sp(N) index in a given expression results in the appearance of an extra minus sign. In particular, in traces of products of Sp(N) generators, the raising and lowering of indices can be used to relate different permutations to each other which is of importance when relating colour-stripped sub-amplitudes to the full amplitudes. For example it is straightforward to see that

$$\mathrm{Tr}(T^aT^bT^c) = -\mathrm{Tr}(T^aT^cT^b) \text{ and } \mathrm{Tr}(T^aT^bT^cT^d) = \mathrm{Tr}(T^aT^dT^cT^b)\,,$$

where a, b, c, d here are adjoint indices. Furthermore, pseudoreality of the Sp(N) vector representation means that fundamental and conjugate fundamental fields can be related simply by raising and lowering indices. Our assignment of signs for this is that

$$\mathcal{Q}_i = -\Omega_{ij}\mathcal{Q}^j\ . \tag{A.12}$$

Finally, as noted in equation (2.6), the contraction of two Sp(N) generators gives

$${(T^a)^i}_j {(T_a)^k}_l = \frac{1}{2}(\delta^i_l\delta^k_j - \Omega^{ik}\Omega_{jl})\ . \tag{A.13}$$

More details on Sp(N) can be found, for instance, in [127].

Because of the Sp(1) $\cong$ SU(2) isomorphism, the Sp(N) conventions above for the contraction of the invariant tensor are the ones that we use for all other SU(2) symmetries (apart from the 4d Lorentz SU(2)s discussed in the previous section). In particular we take

$$\epsilon_{ab} = \epsilon^{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A.14}$$

for raising and lowering $\mathrm{SU}(2)_a$ indices, which leads to $\epsilon_{ab}\epsilon^{bc} = -\delta_a^c$. Similar remarks apply for $\mathrm{SU}(2)_A$. Note, therefore, that for the Grassmann integration in supertwistor space these conventions imply

$$\int d\psi\psi = 1 \qquad , \qquad \int d^2\psi\psi^a\psi^b = \frac{1}{2}\epsilon^{ab} \; . \tag{A.15}$$

Our conventions for the SU(N) gauge and flavour indices are the usual ones to be found in e.g. [128].

B. Feynman rules and useful identities

In this appendix we present the Feynman rules and some related identities, which we use for the calculation of amplitudes in sections 4 and 5.4.

Spinor identities. In 4d, on-shell null momenta decompose in terms of two commuting, two-component, positive and negative helicity spinors $p_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha}$. These are referred to as holomorphic and antiholomorphic spinors respectively and we define the following inner products

$$\lambda^\alpha\mu_\alpha = \langle\lambda\mu\rangle \quad \text{and} \quad -\tilde\lambda_{\dot\alpha}\tilde\mu^{\dot\alpha} = [\tilde\lambda\tilde\mu] \; . \tag{B.1}$$

These products are antisymmetric so that $\langle\lambda\mu\rangle = -\langle\mu\lambda\rangle$, $[\tilde\lambda\tilde\mu] = -[\tilde\mu\tilde\lambda]$ and $\langle\lambda\lambda\rangle = [\tilde\lambda\tilde\lambda] = 0$.

One can switch between spinor helicity and Lorentz notations using the generalised Pauli matrices $(\sigma^\mu)_{\alpha\dot\alpha} \equiv (1, \vec\sigma)$ and $(\bar\sigma^\mu)^{\dot\alpha\alpha} \equiv \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}(\sigma^\mu)_{\beta\dot\beta}$ through

$$q_{\alpha\dot\alpha} = \sigma^\mu{}_{\alpha\dot\alpha}q_\mu \, , \qquad q^\mu = \frac{1}{2}(\bar\sigma^\mu)^{\dot\alpha\alpha}q_{\alpha\dot\alpha} \; . \tag{B.2}$$

Some useful σ-matrix identities include

$$(\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma_\mu)^{\dot\beta\beta} = 2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta} \, , \qquad (\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha\alpha} = 2\eta^{\mu\nu} \; . \tag{B.3}$$

The momentum inner product can be easily shown to be given by the expression

$$p \cdot q = \frac{1}{2}\langle\lambda\mu\rangle[\tilde\lambda\tilde\mu] \, , \tag{B.4}$$

which differs by a sign from the usual QCD literature but is in-line with the majority of the twistor string literature. Momentum conservation for an n-point amplitude can be implemented in the spinor helicity formalism as[35]

$$\sum_{i=1}^{n}\langle ji\rangle[ik] = 0 \; . \tag{B.5}$$

The Schouten identity is also extremely useful

$$\langle ij\rangle\langle kl\rangle + \langle ik\rangle\langle lj\rangle + \langle il\rangle\langle jk\rangle = 0 \; . \tag{B.6}$$

[35] Here we use the common abbreviation of $\langle\lambda_i\,\lambda_j\rangle = \langle i\,j\rangle$.

Field	Helicity	Wave-function
Scalar	0	1
Fermion i	$+$	$\tilde{\lambda}_{i\,\dot{\alpha}} = -[i\|$
Fermion i	$-$	$\lambda_i^{\alpha} = \langle i\|$
Anti-fermion j	$+$	$\tilde{\lambda}_j^{\dot{\alpha}} = \|j]$
Anti-fermion j	$-$	$\lambda_{j\,\alpha} = \|j\rangle$
Vector $p = \lambda\tilde{\lambda}$	$+$	$\epsilon^{+}_{\alpha\dot{\alpha}} = \sqrt{2}\,\frac{\mu_\alpha \tilde{\lambda}_{\dot{\alpha}}}{\mu^\alpha \lambda_\alpha} = -\sqrt{2}\,\frac{\|\mu\rangle[\tilde{\lambda}\|}{\langle\mu\lambda\rangle}$
Vector $p = \lambda\tilde{\lambda}$	$-$	$\epsilon^{-}_{\alpha\dot{\alpha}} = \sqrt{2}\,\frac{\lambda_\alpha \tilde{\mu}_{\dot{\alpha}}}{\tilde{\mu}_{\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}}} = -\sqrt{2}\,\frac{\|\lambda\rangle[\tilde{\mu}\|}{[\tilde{\mu}\tilde{\lambda}]}$

Table 4: Wavefunctions corresponding to outgoing external fields of given helicity. Note that to define the vector wavefunctions we employ an arbitrary reference vector $q = \mu\tilde{\mu}$.

Field	Schematic form	Value
Adjoint scalar	ϕ $\phi^\dagger$, p	$\frac{i}{p^2}$
q, z scalars	$(q^a{}_A, z^a{}_A)$ $(q^b{}_B, z^b{}_B)$, p	$\epsilon^{ab}\epsilon_{AB}\frac{i}{p^2}$
Adjoint fermion	λ^a $\bar{\lambda}^b$, p	$\epsilon^{ab}\frac{ip_{\alpha\dot{\alpha}}}{p^2}$
Adjoint antifermion	$\bar{\lambda}^a$ λ^b, p	$-\epsilon^{ab}\frac{ip^{\alpha\dot{\alpha}}}{p^2}$
η, ζ fermions	(η_A, ζ_A) $(\bar{\eta}_B, \bar{\zeta}_B)$, p	$\epsilon_{AB}\frac{ip_{\alpha\dot{\alpha}}}{p^2}$
Vector	A_μ A_ν, p	$-ig^2\frac{\eta_{\mu\nu}}{p^2}$.

Table 5: Propagators for the various fields in our theory.

Feynman rules. Tables 4–6 list the Feynman rules for the $N_f = 4$ theory — the ones for the $N_f = 2N$ theory can be obtained straightforwardly from these. In table 4 we give the wavefunctions for external particles. Table 5 shows some examples of propagators, while table 6 includes a few sample vertices. The remaining vertices can of course be easily derived from the action. The vertices for the $N_f = 2N$ theory are almost identical to the ones listed here. The main differences are that the antisymmetric fields are absent in that case, and that the fundamental scalars are complex as opposed to real fields. In these expressions (as well as for our amplitude calculations), all external momenta are taken to be outgoing.

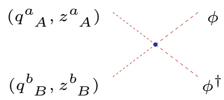
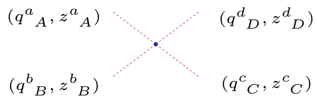
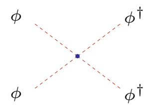
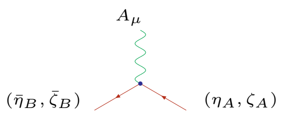
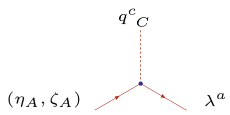
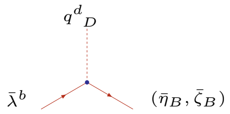
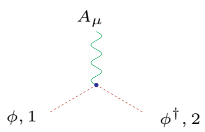
<table>
<tr><th>Schematic form</th><th>Value</th></tr>
<tr><td>
</td><td>$ig^2\epsilon_{ab}\epsilon^{AB}$</td></tr>
<tr><td>
</td><td>$i\left(2\epsilon_{ab}\epsilon_{cd}\epsilon^{AD}\epsilon^{BC}+\epsilon_{ad}\epsilon_{bc}\epsilon^{AD}\epsilon^{BC} +2\epsilon_{ad}\epsilon_{bc}\epsilon^{AB}\epsilon^{CD}+\epsilon_{ab}\epsilon_{cd}\epsilon^{AB}\epsilon^{CD}\right)$</td></tr>
<tr><td>
</td><td>ig^2</td></tr>
<tr><td>
</td><td>$-i\epsilon^{AB}\sigma^\mu$</td></tr>
<tr><td>
</td><td>$i\epsilon_{ac}\epsilon^{AC}$</td></tr>
<tr><td>
</td><td>$2ig^2\epsilon_{bd}\epsilon^{BD}$</td></tr>
<tr><td>
</td><td>$-i(p_2^\mu - p_1^\mu)$</td></tr>
</table>

Table 6: Some of the interaction vertices of the $N_f = 4$ theory.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].

[2] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [hep-th/0312171].

[3] R. Penrose, *Twistor algebra*, *J. Math. Phys.* **8** (1967) 345.

[4] V.P. Nair, *A current algebra for some gauge theory amplitudes*, *Phys. Lett.* **B 214** (1988) 215.

[5] N. Berkovits and E. Witten, *Conformal supergravity in twistor-string theory*, *JHEP* **08** (2004) 009 [hep-th/0406051].

[6] L. Dolan and P. Goddard, *Tree and loop amplitudes in open twistor string theory*, *JHEP* **06** (2007) 005 [hep-th/0703054].

[7] N. Berkovits, *An alternative string theory in twistor space for N = 4 super-Yang-Mills*, *Phys. Rev. Lett.* **93** (2004) 011601 [hep-th/0402045].

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[8] R. Roiban, M. Spradlin and A. Volovich, *A googly amplitude from the B-model in twistor space*, *JHEP* **04** (2004) 012 [hep-th/0402016].

[9] R. Roiban and A. Volovich, *All googly amplitudes from the B-model in twistor space*, *Phys. Rev. Lett.* **93** (2004) 131602 [hep-th/0402121].

[10] R. Roiban, M. Spradlin and A. Volovich, *On the tree-level S-matrix of Yang-Mills theory*, *Phys. Rev.* **D 70** (2004) 026009 [hep-th/0403190].

[11] F. Cachazo, P. Svrček and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, *JHEP* **09** (2004) 006 [hep-th/0403047].

[12] G. Georgiou and V.V. Khoze, *Tree amplitudes in gauge theory as scalar MHV diagrams*, *JHEP* **05** (2004) 070 [hep-th/0404072].

[13] G. Georgiou, E.W.N. Glover and V.V. Khoze, *Non-MHV tree amplitudes in gauge theory*, *JHEP* **07** (2004) 048 [hep-th/0407027].

[14] L.J. Dixon, E.W.N. Glover and V.V. Khoze, *MHV rules for Higgs plus multi-gluon amplitudes*, *JHEP* **12** (2004) 015 [hep-th/0411092].

[15] N.E.J. Bjerrum-Bohr, D.C. Dunbar, H. Ita, W.B. Perkins and K. Risager, *MHV-vertices for gravity amplitudes*, *JHEP* **01** (2006) 009 [hep-th/0509016].

[16] A. Brandhuber, B.J. Spence and G. Travaglini, *One-loop gauge theory amplitudes in $N = 4$ super Yang-Mills from MHV vertices*, *Nucl. Phys.* **B 706** (2005) 150 [hep-th/0407214].

[17] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *A twistor approach to one-loop amplitudes in $N = 1$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 706** (2005) 100 [hep-th/0410280].

[18] C. Quigley and M. Rozali, *One-loop MHV amplitudes in supersymmetric gauge theories*, *JHEP* **01** (2005) 053 [hep-th/0410278].

[19] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *Non-supersymmetric loop amplitudes and MHV vertices*, *Nucl. Phys.* **B 712** (2005) 59 [hep-th/0412108].

[20] S.D. Badger, E.W.N. Glover and K. Risager, *One-loop ϕ-MHV amplitudes using the unitarity bootstrap*, *JHEP* **07** (2007) 066 [arXiv:0704.3914].

[21] A. Nasti and G. Travaglini, *One-loop $N = 8$ supergravity amplitudes from MHV diagrams*, arXiv:0706.0976.

[22] W. Siegel, *Untwisting the twistor superstring*, hep-th/0404255.

[23] M. Kulaxizi and K. Zoubos, *Marginal deformations of $N = 4$ SYM from open / closed twistor strings*, *Nucl. Phys.* **B 738** (2006) 317 [hep-th/0410122].

[24] P. Gao and J.-B. Wu, *(Non)-supersymmetric marginal deformations from twistor string theory*, hep-th/0611128.

[25] J. Park and S.-J. Rey, *Supertwistor orbifolds: gauge theory amplitudes and topological strings*, *JHEP* **12** (2004) 017 [hep-th/0411123].

[26] S. Giombi et al., *Orbifolding the twistor string*, *Nucl. Phys.* **B 719** (2005) 234 [hep-th/0411171].

[27] A.D. Popov and M. Wolf, *Topological B-model on weighted projective spaces and self-dual models in four dimensions*, *JHEP* **09** (2004) 007 [hep-th/0406224].

[28] D.-W. Chiou, O.J. Ganor, Y.P. Hong, B.S. Kim and I. Mitra, *Massless and massive three dimensional super Yang-Mills theory and mini-twistor string theory*, *Phys. Rev.* **D 71** (2005) 125016 [hep-th/0502076].

[29] A.D. Popov, C. Sämann and M. Wolf, *The topological B-model on a mini-supertwistor space and supersymmetric Bogomolny monopole equations*, *JHEP* **10** (2005) 058 [hep-th/0505161].

[30] C. Sämann, *On the mini-superambitwistor space and $N = 8$ super Yang-Mills theory*, hep-th/0508137.

[31] O. Lechtenfeld and C. Sämann, *Matrix models and D-branes in twistor string theory*, *JHEP* **03** (2006) 002 [hep-th/0511130].

[32] A.D. Popov, *σ-models with $N = 8$ supersymmetries in $2+1$ and $1+1$ dimensions*, *Phys. Lett.* **B 647** (2007) 509 [hep-th/0702106].

[33] D.-W. Chiou, O.J. Ganor and B.S. Kim, *A deformation of twistor space and a chiral mass term in $N = 4$ super Yang-Mills theory*, *JHEP* **03** (2006) 027 [hep-th/0512242].

[34] C.-H. Ahn, *$N = 1$ conformal supergravity and twistor-string theory*, *JHEP* **10** (2004) 064 [hep-th/0409195].

[35] C.-h. Ahn, *$N = 2$ conformal supergravity from twistor-string theory*, *Int. J. Mod. Phys.* **A 21** (2006) 3733 [hep-th/0412202].

[36] M. Abou-Zeid, C.M. Hull and L.J. Mason, *Einstein supergravity and new twistor string theories*, hep-th/0606272.

[37] R. Boels, L. Mason and D. Skinner, *Supersymmetric gauge theories in twistor space*, *JHEP* **02** (2007) 014 [hep-th/0604040].

[38] R. Boels, L. Mason and D. Skinner, *From twistor actions to MHV diagrams*, *Phys. Lett.* **B 648** (2007) 90 [hep-th/0702035].

[39] R. Boels, *A quantization of twistor Yang-Mills theory through the background field method*, hep-th/0703080.

[40] L.J. Mason and M. Wolf, *A twistor action for $N = 8$ self-dual supergravity*, arXiv:0706.1941.

[41] P.S. Howe, K.S. Stelle and P.C. West, *A class of finite four-dimensional supersymmetric field theories*, *Phys. Lett.* **B 124** (1983) 55.

[42] A. Parkes and P.C. West, *Finiteness in rigid supersymmetric theories*, *Phys. Lett.* **B 138** (1984) 99.

[43] M.T. Grisaru and W. Siegel, *Supergraphity. 2. Manifestly covariant rules and higher loop finiteness*, *Nucl. Phys.* **B 201** (1982) 292 [*Erratum ibid.* **206** (1982) 496].

[44] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [*Erratum ibid.* **B 430** (1994) 485] [hep-th/9407087].

[45] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099].

[46] N. Dorey, V.V. Khoze and M.P. Mattis, *On $N = 2$ supersymmetric QCD with 4 flavors*, *Nucl. Phys.* **B 492** (1997) 607 [hep-th/9611016].

[47] C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B 469** (1996) 403 [hep-th/9602022].

[48] A. Sen, *F-theory and orientifolds*, *Nucl. Phys.* **B 475** (1996) 562 [hep-th/9605150].

[49] E.G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and d-manifolds*, *Phys. Rev.* **D 54** (1996) 1667 [hep-th/9601038].

[50] A. Dabholkar, *Lectures on orientifolds and duality*, hep-th/9804208.

[51] A. Giveon and D. Kutasov, *Brane dynamics and gauge theory*, *Rev. Mod. Phys.* **71** (1999) 983 [hep-th/9802067].

[52] T. Banks, M.R. Douglas and N. Seiberg, *Probing F-theory with branes*, *Phys. Lett.* **B 387** (1996) 278 [hep-th/9605199].

[53] M.R. Douglas, D.A. Lowe and J.H. Schwarz, *Probing F-theory with multiple branes*, *Phys. Lett.* **B 394** (1997) 297 [hep-th/9612062].

[54] O. Aharony, J. Sonnenschein, S. Yankielowicz and S. Theisen, *Field theory questions for string theory answers*, *Nucl. Phys.* **B 493** (1997) 177 [hep-th/9611222].

[55] E. Gava, K.S. Narain and M.H. Sarmadi, *Instantons in $N = 2$ $\mathrm{Sp}(N)$ superconformal gauge theories and the AdS/CFT correspondence*, *Nucl. Phys.* **B 569** (2000) 183 [hep-th/9908125].

[56] A. Fayyazuddin and M. Spalinski, *Large-N superconformal gauge theories and supergravity orientifolds*, *Nucl. Phys.* **B 535** (1998) 219 [hep-th/9805096].

[57] O. Aharony, A. Fayyazuddin and J.M. Maldacena, *The large-N limit of $N = 2, 1$ field theories from three-branes in F-theory*, *JHEP* **07** (1998) 013 [hep-th/9806159].

[58] I.P. Ennes, C. Lozano, S.G. Naculich and H.J. Schnitzer, *Elliptic models, type IIB orientifolds and the AdS/CFT correspondence*, *Nucl. Phys.* **B 591** (2000) 195 [hep-th/0006140].

[59] M. Gutperle, *Heterotic/type-I duality, d-instantons and a $N = 2$ AdS/CFT correspondence*, *Phys. Rev.* **D 60** (1999) 126001 [hep-th/9905173].

[60] T.J. Hollowood, *Instantons, finite $N = 2$ $\mathrm{Sp}(N)$ theories and the AdS/CFT correspondence*, *JHEP* **11** (1999) 012 [hep-th/9908201].

[61] D.E. Berenstein, E. Gava, J.M. Maldacena, K.S. Narain and H.S. Nastase, *Open strings on plane waves and their Yang-Mills duals*, hep-th/0203249.

[62] J. Gomis, S. Moriyama and J.-w. Park, *Open + closed string field theory from gauge fields*, *Nucl. Phys.* **B 678** (2004) 101 [hep-th/0305264].

[63] R. de Mello Koch and R. Tatar, *Higher derivative terms from threebranes in F-theory*, *Phys. Lett.* **B 450** (1999) 99 [hep-th/9811128].

[64] R. de Mello Koch, A. Paulin-Campbell and J.P. Rodrigues, *Non-holomorphic corrections from threebranes in F-theory*, *Phys. Rev.* **D 60** (1999) 106008 [hep-th/9903029].

[65] Z. Guralnik, S. Kovacs and B. Kulik, *Holography and the Higgs branch of $N = 2$ SYM theories*, *JHEP* **03** (2005) 063 [hep-th/0405127].

[66] Z. Komargodski and S.S. Razamat, *Planar quark scattering at strong coupling and universality*, arXiv:0707.4367.

[67] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, *Superspace, or one thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1 [hep-th/0108200].

[68] M.F. Sohnius, *Introducing supersymmetry*, *Phys. Rept.* **128** (1985) 39.

[69] L. Álvarez-Gaumé and S.F. Hassan, *Introduction to S-duality in $N = 2$ supersymmetric gauge theories: a pedagogical review of the work of Seiberg and Witten*, *Fortschr. Phys.* **45** (1997) 159 [hep-th/9701069].

[70] S. Kovacs, *$N = 4$ supersymmetric Yang-Mills theory and the AdS/sCFT correspondence*, hep-th/9908171.

[71] A. Bilal, *Introduction to supersymmetry*, hep-th/0101055.

[72] P. Breitenlohner and M.F. Sohnius, *Matter coupling and nonlinear σ-models in $N = 2$ supergravity*, *Nucl. Phys.* **B 187** (1981) 409.

[73] L. Mezincescu and Y.-P. Yao, *Nonexistence of renormalizable selfinteraction in $N = 2$ supersymmetry for scalar hypermultiplets*, *Nucl. Phys.* **B 241** (1984) 605.

[74] W. Siegel, *Fields*, hep-th/9912205.

[75] W. Siegel, *The $N = 2(4)$ string is selfdual $N = 4$ Yang-Mills*, hep-th/9205075.

[76] A. Neitzke and C. Vafa, *Topological strings and their physical applications*, hep-th/0410178.

[77] M. Mariño, *Les Houches lectures on matrix models and topological strings*, hep-th/0410165.

[78] M. Vonk, *A mini-course on topological strings*, hep-th/0504147.

[79] C. Vafa and E. Zaslow eds., *Mirror symmetry*, vol. 1, Clay Mathematics Monographs (2003).

[80] E. Witten, *Mirror manifolds and topological field theory*, hep-th/9112056.

[81] E. Witten, *Chern-Simons gauge theory as a string theory*, *Prog. Math.* **133** (1995) 637 [hep-th/9207094].

[82] I. Brunner, M.R. Douglas, A.E. Lawrence and C. Romelsberger, *D-branes on the quintic*, *JHEP* **08** (2000) 015 [hep-th/9906200].

[83] S.H. Katz and E. Sharpe, *D-branes, open string vertex operators and ext groups*, *Adv. Theor. Math. Phys.* **6** (2003) 979 [hep-th/0208104].

[84] P.A. Grassi and G. Policastro, *Super Chern-Simons theory as superstring theory*, hep-th/0412272.

[85] A.D. Popov and C. Sämann, *On supertwistors, the Penrose-ward transform and $N = 4$ super Yang-Mills theory*, *Adv. Theor. Math. Phys.* **9** (2005) 931 [hep-th/0405123].

[86] S. Sinha and C. Vafa, *SO and Sp Chern-Simons at large-N*, hep-th/0012136.

[87] C. Sämann, *The topological b-model on fattened complex manifolds and subsectors of $N = 4$ self-dual Yang-Mills theory*, *JHEP* **01** (2005) 042 [hep-th/0410292].

[88] A. Konechny and A.S. Schwarz, *On $(k \oplus l|q)$-dimensional supermanifolds*, hep-th/9706003.

[89] O. Lechtenfeld and A.D. Popov, *Supertwistors and cubic string field theory for open $N = 2$ strings*, *Phys. Lett.* **B 598** (2004) 113 [hep-th/0406179].

[90] J.P. Harnad, J. Hurtubise and S. Shnider, *Supersymmetric Yang-Mills equations and supertwistors*, *Ann. Phys. (NY)* **193** (1989) 40.

[91] C. Hofman, *On the open-closed B-model*, *JHEP* **11** (2003) 069 [hep-th/0204157].

[92] A. Ferber, *Supertwistors and conformal supersymmetry*, *Nucl. Phys.* **B 132** (1978) 55.

[93] A.S. Schwarz, *σ-models having supermanifolds as target spaces*, *Lett. Math. Phys.* **38** (1996) 91 [hep-th/9506070].

[94] R. Ricci, *Super Calabi-Yau's and special lagrangians*, *JHEP* **03** (2007) 048 [hep-th/0511284].

[95] T. Tokunaga, *String theories on flat supermanifolds*, hep-th/0509198.

[96] I. Brunner and K. Hori, *Orientifolds and mirror symmetry*, *JHEP* **11** (2004) 005 [hep-th/0303135].

[97] K. Hori and J. Walcher, *D-brane categories for orientifolds: the Landau-Ginzburg case*, hep-th/0606179.

[98] R.S. Ward, *Completely solvable gauge field equations in dimension greater than four*, *Nucl. Phys.* **B 236** (1984) 381.

[99] F. Cachazo and P. Svrček, *Lectures on twistor strings and perturbative Yang-Mills theory*, PoS(RTN2005)004 [hep-th/0504194].

[100] V.V. Khoze, *Gauge theory amplitudes, scalar graphs and twistor space*, hep-th/0408233.

[101] S. Gukov, L. Motl and A. Neitzke, *Equivalence of twistor prescriptions for super Yang-Mills*, hep-th/0404085.

[102] G.M. Cicuta, *Topological expansion for* $\mathrm{SO}(N)$ *and* $\mathrm{Sp}(2N)$ *gauge theories*, *Lett. Nuovo Cim.* **35** (1982) 87.

[103] M.L. Mangano and S.J. Parke, *Multiparton amplitudes in gauge theories*, *Phys. Rept.* **200** (1991) 301 [hep-th/0509223].

[104] L.J. Dixon, *Calculating scattering amplitudes efficiently*, hep-ph/9601359.

[105] C.N. Yang, *Condition of selfduality for* $\mathrm{SU}(2)$ *gauge fields on euclidean four-dimensional space*, *Phys. Rev. Lett.* **38** (1977) 1377.

[106] R.S. Ward, *On self-dual gauge fields*, *Phys. Lett.* **A 61** (1977) 81.

[107] A.D. Popov, *Self-dual Yang-Mills: symmetries and moduli space*, *Rev. Math. Phys.* **11** (1999) 1091 [hep-th/9803183].

[108] I.V. Volovich, *Supersymmetric Yang-Mills equations as an inverse scattering problem*, *Lett. Math. Phys.* **7** (1983) 517.

[109] M. Wolf, *On hidden symmetries of a super gauge theory and twistor string theory*, *JHEP* **02** (2005) 018 [hep-th/0412163].

[110] H. Ooguri and C. Vafa, *$N = 2$ heterotic strings*, *Nucl. Phys.* **B 367** (1991) 83.

[111] D. Cangemi, *Self-dual Yang-Mills theory and one-loop maximally helicity violating multi-gluon amplitudes*, *Nucl. Phys.* **B 484** (1997) 521 [hep-th/9605208].

[112] J. Polchinski, *$N = 2$ gauge-gravity duals*, *Int. J. Mod. Phys.* **A 16** (2001) 707 [hep-th/0011193].

[113] M. Graña and J. Polchinski, *Gauge/gravity duals with holomorphic dilaton*, *Phys. Rev.* **D 65** (2002) 126005 [hep-th/0106014].

[114] C.V. Johnson, R.C. Myers, A.W. Peet and S.F. Ross, *The enhancon and the consistency of excision*, *Phys. Rev.* **D 64** (2001) 106001 [hep-th/0105077].

[115] T.J. Hollowood, V.V. Khoze and M.P. Mattis, *Summing the instanton series in $N = 2$ superconformal large-N QCD*, *JHEP* **10** (1999) 019 [hep-th/9905209].

[116] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [hep-th/9806087].

[117] S.S. Gubser, *Einstein manifolds and conformal field theories*, *Phys. Rev.* **D 59** (1999) 025006 [hep-th/9807164].

[118] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda and R. Marotta, *$N = 2$ gauge theories on systems of fractional $D3/D7$ branes*, *Nucl. Phys.* **B 621** (2002) 157 [hep-th/0107057].

[119] A. Neitzke and C. Vafa, *$N = 2$ strings and the twistorial Calabi-Yau*, hep-th/0402128.

[120] C. Vafa, *Brane/anti-brane systems and* $\mathrm{U}(N|M)$ *supergroup*, hep-th/0101218.

[121] R.L. Mkrtchyan, *The equivalence of* $\mathrm{Sp}(2N)$ *and* $\mathrm{SO}(-2N)$ *gauge theories*, *Phys. Lett.* **B 105** (1981) 174.

[122] N. Nekrasov, H. Ooguri and C. Vafa, *S-duality and topological strings*, *JHEP* **10** (2004) 009 [hep-th/0403167].

[123] L. Anguelova, P. de Medeiros and A. Sinkovics, *On topological F-theory*, *JHEP* **05** (2005) 021 [hep-th/0412120].

[124] M. Aganagic and C. Vafa, *Mirror symmetry and supermanifolds*, hep-th/0403192.

[125] E. Witten, *An interpretation of classical Yang-Mills theory*, *Phys. Lett.* **B 77** (1978) 394.

[126] J. Isenberg, P.B. Yasskin and P.S. Green, *Nonselfdual gauge fields*, *Phys. Lett.* **B 78** (1978) 462.

[127] H. Georgi, *Lie algebras in particle physics*, Addison-Wesley (1982).

[128] M.E. Peskin and D.V. Schroeder, *An introduction to quantum field theory*, Addison-Wesley (1995).