EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH
Laboratory for Particle Physics

Departmental Report

CERN/AT 2007-17 (MEL)

THE PAIRING MATRIX IN DISCRETE ELECTROMAGNETISM
On the Geometry of Discrete de Rham Currents

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To be published in European Physical Journal - Applied Physics

CERN, Accelerator Technology Department
CH - 1211 Geneva 23
Switzerland

15 May 2007

EPJ manuscript No.
(will be inserted by the editor)

The Pairing Matrix in Discrete Electromagnetism

On the Geometry of Discrete de Rham Currents

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Received: date / Revised version: date

Abstract. We introduce pairing matrices on simplicial cell complexes in discrete electromagnetism as a means to avoid the explicit construction of a topologically dual complex. Interestingly, the Finite Element Method with first-order Whitney elements – when it is looked upon from a cell-method perspective – features pairing matrices and thus an implicitly defined dual mesh. We show that the pairing matrix can be used to construct discrete energy products. In this exercise we find that different formalisms lead to equivalent matrix representations. Discrete de Rham currents are an elegant way to subsume these geometrically equivalent but formally distinct ways of defining energy-products.

PACS. 02.40.Re Algebraic topology – 02.40.Sf Manifolds and cell complexes – 02.70.Dh Finite-element and Galerkin methods – 41.20.Cv Electrostatics; Poisson and Laplace equations, boundary-value problems – 41.20.Gz Magnetostatics; magnetic shielding, magnetic induction, boundary-value problems

1 Introduction

A discrete theory of electromagnetism (DEM) features discrete fields, discrete derivative operators, and discrete material operators. Cell methods provide all of these features based on an oriented cell complex C, representing a bounded n-dimensional domain Ω, $n = 0, 1, 2, 3$, and its topologically dual complex $\underset{\sim}{C}$. In this paper we consider simplicial cell complexes and their barycentric duals.

To the above framework we add the concept of pairing matrices, first postulated in [1]. They translate coefficient vectors of a cochain defined on the primal complex into coefficient vectors of cochains on the (barycentric) dual. Discrete fields need only be defined on the primal complex. The explicit construction of a dual complex is not required.

Section 2 briefly recalls the mathematical framework of discrete electromagnetism. The dual complex, in particular its boundary, is investigated in Section 3. Section 4 introduces the pairing matrix. Its coefficients are defined by the principle of interpolation of simplices in Section 4.1, and its properties are discussed in Section 4.2. Eventually we discover that the pairing matrix features in a finite-element method with Whitney elements in Section 5.

The second part of the paper deals with the role of the pairing matrix in discrete energy products. Two formalisms are introduced in Section 6: the discrete wedge product, which mimics the definition of continuous energy products; and the chain/cochain formalism, which discretizes electromagnetic fields as chains and cochains on one cell complex. Realizing that both formalisms lead to identical matrix equations, we introduce a further layer of abstraction in Section 7: the formalism of discrete de Rham currents, which unifies the above approaches.

Appendix A shows that the pairing operator induces a chain map between the primal and the closed dual complex. In Appendix B we discuss the equivalence of four seemingly unrelated results in this paper. Eventually, Appendix C lists the transformation properties of discrete fields and operators under a global change of orientation.

2 Mathematical Framework and Notation

We denote $\bar{F} \in \mathcal{F}^p(\Omega)$ a differential p-form, and $\langle \bar{F} \,|\, m \rangle$ the integral of a differential p-form over a p-manifold $m \subset \Omega$. Stokes' theorem for differential forms reads

$$\langle \mathrm{d}\,\bar{F} \,|\, m \rangle = \langle \bar{F} \,|\, \mathrm{b}\, m \rangle, \tag{1}$$

with the boundary operator b, and the coboundary- or exterior derivative operator d.

Discrete electromagnetism uses concepts of algebraic topology such as (co)chains and (co)boundaries. Metric is introduced in form of discrete Hodge operators. We denote $\hat{F} \in \mathcal{C}^p(C)$ (hat) the p-cochain of a field and $\{\hat{F}\} \in \mathbb{R}^{n_p}$ the coefficient vector of the cochain. $\mathcal{C}^p(C)$ or simply $\mathcal{C}^p$ denotes the space of cochains on C. n_p denotes the number of p-simplices $\check{\sigma}_p$ in the complex C. The ith coefficient of a cochain is obtained from a differential form by the de Rham map, $\{\hat{F}\}^i = \langle \bar{F} \,|\, \check{\sigma}_p^i \rangle$. Furthermore $\check{c} \in \mathcal{C}_p(C)$ (check) denotes a chain of cells and $\{\check{c}\} \in \mathbb{R}^{n_p}$ the chain's coefficient vector. $\mathcal{C}_p(C)$ denotes the space of chains. Objects that are defined on the dual complex are underlined, e.g. $\underline{\hat{F}}$ for a dual cochain. Finally we denote $\tilde{F} \in \mathcal{W}^p(C)$ (tilde) the Whitney form, that is the interpolation of a discrete field based on its coefficients $\{\hat{F}\}$. The duality product of chains and cochains is written $\langle \hat{F} \,|\, \check{c} \rangle = \{\hat{F}\}^{\mathrm{T}}\{\check{c}\}$, and operators acting on chains and cochains are written in bold font, e.g., $\mathbf{d}$ for the discrete derivative-, $\mathbf{b}$ for the discrete boundary-, and $\bigstar$ for the discrete Hodge operator.

The discrete Stokes theorem reads for $\hat{F} \in \mathcal{C}^{p-1}$

$$\langle \mathbf{d}\,\hat{F} \,|\, \check{c} \rangle = \langle \hat{F} \,|\, \mathbf{b}\,\check{c} \rangle \quad \text{or} \tag{2}$$

$$\left([\mathrm{D}^{p-1}]\{\hat{F}\}\right)^{\mathrm{T}}\{\check{c}\} = \{\hat{F}\}^{\mathrm{T}}[\mathrm{B}^p]\{\check{c}\}, \tag{3}$$

where $[\mathrm{B}^p]$ and $[\mathrm{D}^{p-1}]$ represent the discrete boundary and coboundary operators acting on p-chains and $(p-1)$-cochains on the coefficient level.

3 Some Remarks on the Dual Complex

We list a few properties of boundary- and coboundary operators on topologically dual cell complexes. The matrices of discrete boundary- and coboundary operators are related via

$$[\mathrm{B}^p] = [\mathrm{D}^{p-1}]^{\mathrm{T}}, \tag{4 a}$$

$$[\underline{\mathrm{B}}^p] = [\underline{\mathrm{D}}^{p-1}]^{\mathrm{T}}, \tag{4 b}$$

$$(-1)^p[\underline{\mathrm{B}}^p] = [\mathrm{B}^{n-p+1}]^{\mathrm{T}}, \tag{4 c}$$

$$(-1)^{p+1}[\underline{\mathrm{D}}^p] = [\mathrm{D}^{n-p-1}]^{\mathrm{T}}, \tag{4 d}$$

and from (4 a), (4 c) and (4 d)

$$(-1)^p[\underline{\mathrm{B}}^p] = [\mathrm{D}^{n-p}], \tag{5 a}$$

$$[\mathrm{B}^{n-p}] = (-1)^{p+1}[\underline{\mathrm{D}}^p]. \tag{5 b}$$

Equation (5 a) has an interesting consequence. It is well known that the discrete coboundary of a constant 0-cochain $\hat{F}$, $\{\hat{F}\}^1 = \{\hat{F}\}^2 = \cdots = \{\hat{F}\}^{n_0}$, is zero

$$[\mathrm{D}^0]\{\hat{F}\} = \{0\}. \tag{6}$$

The constant dual n-chain $\underline{\check{n}}$ represents all n-simplices in the dual complex. From (6) and (5 a) it follows that the boundary of the dual complex is empty,

$$[\underline{\mathrm{B}}^n]\{\underline{\check{n}}\} = \{0\}. \tag{7}$$

According to the standard definition, a complex which does not contain all sides of its simplices is not a complex. We shall therefore call the standard complex a *closed complex*, and a complex which does not contain its own boundary, i.e. $\mathbf{b}\,\check{n} = 0$ for constant, nonzero $\hat{n} \in \mathcal{C}_n$, an *open complex.*

Fortunately we can easily find a closure for the open barycentric dual complex. It is given by the barycentric dual of the primal boundary complex. Figure 1 (a) shows a 2-dimensional primal complex C and (b) its open dual $\underline{C}$. (c) and (d) show the boundary complex $C_{\mathrm{b}} = \mathbf{b}\,C$ and the dual of the boundary complex $\underline{C}_{\mathrm{b}}$. The closed dual complex is denoted $\underline{C}_{\mathrm{c}}$ and is given by $\underline{C}_{\mathrm{c}} = \underline{C} \oplus \underline{C}_{\mathrm{b}}$.

4 The Pairing Matrix

Pairing matrices map coefficients of primal p-cochains into coefficients of dual p-cochains on the open dual complex. We give a definition and discuss the properties of the mapping.

4.1 Definition

The definition of pairing matrices is based on the concept of interpolation of simplices [2]. Any p-simplex $\check{s}_p$ in the interior of a simplicial n-cell can be expressed as a weighted sum of p-simplices $\check{\sigma}_p^I$ in the n-cell's boundary:

$$\check{s}_p = \{\mu_1\check{\sigma}_p^1 + \cdots + \mu_{n_p}\check{\sigma}_p^{n_p} \,|\, 0 \le \mu_i \le 1\}, \tag{8}$$

where n_p denotes the number of p-cells in the n-cell's boundary. We assume that the n-simplex and its sides are canonically oriented.

For the definition of the coefficients μ_i we introduce the set of multiindices $\mathcal{I}_p$. An index $I \in \mathcal{I}_p$ is given by a set of ordered integers $I := \{i_1, \ldots, i_{p+1} \,|\, i_1 < \cdots < i_{p+1}\}$. $\mathcal{I}_p$ has n_p elements. Each multiindex specifies one p-simplex by listing the indices of its spanning nodes. We can write (8) for a p-simplex $\check{s}_p$ spanned by the nodes $\check{x}^1 \ldots \check{x}^{p+1}$ [2][3][4]

$$\check{s}_p = \sum_I \sum_\pi \mathrm{sgn}(\pi)\lambda_{i_1}(\check{x}^1)\ldots\lambda_{i_{p+1}}(\check{x}^{p+1})\check{\sigma}_p^I, \tag{9}$$

where the sums extend over multiindices $I \in \mathcal{I}_p$ and permutations of each multiindex $\pi \in \mathrm{Perm}(i_1, \ldots, i_{p+1})$. The 0-forms λ_{i_j} denote the barycentric coordinate function belonging to the node $\check{\sigma}_0^{i_j}$ in the n-simplex.

We will now interpolate the part of a jth dual cell $\underline{\check{\sigma}}_p^j$ that is contained in a primal n-cell, see Fig. 2 (a) and (b). We denote this part with a prime $\underline{\check{\sigma}}_p'^j$. Note that for $p \ge 2$, the part of a dual cell $\underline{\check{\sigma}}_p'^j$ is a chain of simplices,

$$\underline{\check{\sigma}}_p'^j = \sum_K \check{s}_p^K, \tag{10}$$

see Fig. 2 (c). We require that the dual cochain $\underline{\hat{F}}'$ fulfills

$$\langle \underline{\hat{F}}' \,|\, \underline{\check{\sigma}}_p'^j \rangle = \langle \hat{F} \,|\, \mu_1^j\check{\sigma}_p^1 + \cdots + \mu_{n_p}^j\check{\sigma}_p^{n_p} \rangle \tag{11}$$

with the coefficients

$$\mu_I^j = \sum_K \sum_\pi \mathrm{sgn}(\pi)\lambda_{i_1}(\check{x}^{k_1})\ldots\lambda_{i_{p+1}}(\check{x}^{k_{p+1}}). \tag{12}$$

Recall that the jth dual cell $\underline{\check{\sigma}}_p'^j$ is composed of simplices according to (10). The index j is therefore hidden in the

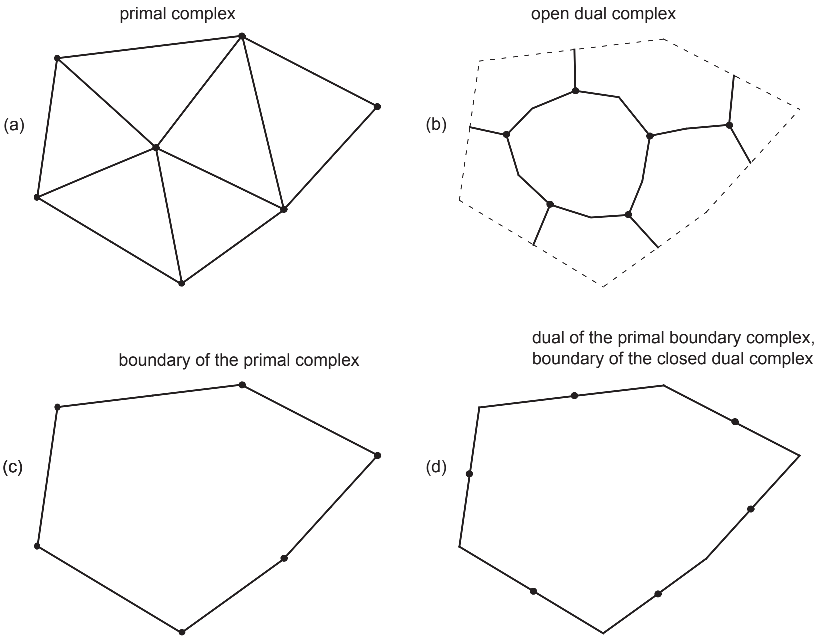


Fig. 1. (a) 2-dimensional simplicial complex. **(b)** Dual complex of the primal complex in (a). The complex is open, i.e. it has no boundary. **(c)** 1-dimensional boundary complex of the complex in (a). **(d)** Dual complex of the 1-dimensional boundary complex in (c). The dual boundary complex represents the closure of the open dual complex in (b).

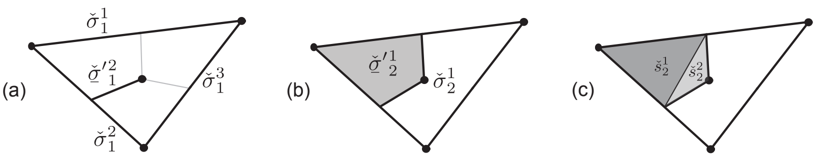


Fig. 2. (a) Part of a dual edge inside a primal n-cell. **(b)** Part of a dual face inside a primal n-cell. **(c)** Composition of the part of the dual 2-cell in (b) from 2-simplices.

sum over K, i.e. over multiindices of simplices. The coefficients of the dual cochain are obtained from

$$\{\hat{\underline{F}}'\} = [\mathrm{P}'^p]\{\hat{F}\}, \quad \text{with} \tag{13 a}$$

$$[\mathrm{P}'^p]^{ij} = \mu_I^j. \tag{13 b}$$

The prime-symbol in (13 b) denotes the element matrix. Its coefficients describe the coefficient of the ith primal p-cell in the interpolation of that part of the jth dual p-cell that lies inside the respective element. The global matrix $[\mathrm{P}^p]$ is obtained from the element matrices $[\mathrm{P}'^p]$ taking into account the global orientation of simplices in C. The matrix transfers coefficients of a p-cochain on the primal complex into coefficients of a p-cochain on the open barycentric dual complex

$$\{\hat{\underline{F}}\} = [\mathrm{P}^p]\{\hat{F}\}, \tag{14}$$

and is called the *pairing matrix* in [1].[1] We give the pairing matrices for $n = 3$, $0 \leq p \leq 3$:

$$[\mathrm{P}'^0] = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \tag{15 a}$$

$$[\mathrm{P}'^1] = \begin{pmatrix} 0 & 0 & \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} \\ 0 & -\frac{1}{12} & 0 & -\frac{1}{12} & 0 & \frac{1}{12} \\ -\frac{1}{12} & 0 & 0 & \frac{1}{12} & \frac{1}{12} & 0 \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & 0 & 0 \end{pmatrix}, \tag{15 b}$$

$$[\mathrm{P}'^2] = \begin{pmatrix} 0 & 0 & -\frac{1}{12} & \frac{1}{12} \\ 0 & -\frac{1}{12} & 0 & \frac{1}{12} \\ \frac{1}{12} & 0 & 0 & \frac{1}{12} \\ 0 & -\frac{1}{12} & \frac{1}{12} & 0 \\ \frac{1}{12} & 0 & \frac{1}{12} & 0 \\ \frac{1}{12} & \frac{1}{12} & 0 & 0 \end{pmatrix}, \tag{15 c}$$

$$[\mathrm{P}'^3] = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}. \tag{15 d}$$

A. Bossavit gives a geometric interpretation of the coefficients μ_I^j in (12) in terms of ratios of volumes [2]. Note that the definition of the matrix coefficients only uses barycentric coordinates of nodes that are the barycenters of volumes, faces, or edges. The coefficients therefore do not

[1] The matrix was called *transfer matrix* in [4] and [5].

depend on the actual shape of a simplex and need to be calculated only once. The pairing matrix is an affine concept - not a metric one.

The pairing matrix is the matrix representative of the pairing operator $\mathbf{p} : \mathcal{C}^p(C) \to \mathcal{C}^p(\underline{C})$, i.e. $\hat{\underline{F}} = \mathbf{p}\,\hat{F}$. The adjoint of the pairing operator maps dual p-chains into primal p-chains and is denoted $\mathbf{q} : \mathcal{C}_p(\underline{C}) \to \mathcal{C}_p(C)$. Its matrix representative is given by the transpose of the pairing matrix $[\underline{\mathrm{Q}}^p] = [\mathrm{P}^p]^{\mathrm{T}}$.

4.2 Properties

In this section we answer the important question whether the coboundary of a cochain is preserved under the action of the pairing operator. If a current-density cochain is defined on the primal complex to be divergence free, is it still free of divergence on the dual complex? In other words, does the pairing operator commute with the coboundary operator, and does it therefore represent a cochain map?

Before we are able to study the properties of the pairing matrix, we need to introduce more operators. The trace operator $\mathbf{t} : \mathcal{C}^p(C) \to \mathcal{C}^p(C_{\mathrm{b}})$ restricts a cochain on a complex to the complex' boundary. The coefficients of the matrix representative $[\mathrm{T}^p]$ are either 0 or 1. The adjoint of the trace operator $\mathbf{i} : \mathcal{C}_p(C_{\mathrm{b}}) \to \mathcal{C}_p(C)$ acts upon chains in the boundary complex and is called an immersion operator in [6]. Furthermore we introduce the closure operator $\mathbf{c} : \mathcal{C}_p(\underline{C}) \to \mathcal{C}_{p-1}(\underline{C}_{\mathrm{b}})$ which, as we will see, closes the boundary of a dual p-cell on the dual complex' boundary. Its adjoint is the coclosure operator on the dual complex $\mathbf{j} : \mathcal{C}^p(\underline{C}_{\mathrm{b}}) \to \mathcal{C}^{p+1}(\underline{C})$. We find that

$$[\mathrm{I}^p] = [\mathrm{T}^p]^{\mathrm{T}}, \quad (16)$$
$$[\underline{\mathrm{J}}^p] = [\mathrm{T}^{n-p-1}]^{\mathrm{T}}, \quad (17)$$
$$[\mathrm{I}^{n-p-1}] = [\underline{\mathrm{J}}^p], \quad (18)$$
$$[\underline{\mathrm{C}}^p] = [\underline{\mathrm{J}}^{p-1}]^{\mathrm{T}}. \quad (19)$$

Writing $[\mathrm{D}_{\mathrm{b}}^p]$ for the matrix representative of the coboundary operator on the boundary complex C_{b} we find

$$[\mathrm{T}^{p+1}][\mathrm{D}^p] = [\mathrm{D}_{\mathrm{b}}^p][\mathrm{T}^p], \quad (20)$$

which represents the analogue to the continuous equation $\mathrm{t}\,\mathrm{d} = \mathrm{d}\,\mathrm{t}$, i.e. the commutativity of continuous trace- and coboundary operators. We take the transpose of (20) and find with (4 d)

$$[\underline{\mathrm{D}}^p][\underline{\mathrm{J}}^{p-1}] = -[\underline{\mathrm{J}}^p][\underline{\mathrm{D}}_{\mathrm{b}}^{p-1}]. \quad (21)$$

With these prerequisites we can ask whether the operator $\mathbf{q}$, the adjoint of the pairing operator, represents a *chain map*. A chain map is a map between two complexes C_B and C_A, $\mathbf{q} : \mathcal{C}_p(C_B) \to \mathcal{C}_p(C_A)$ for all p, that sends cycles to cycles and boundaries to boundaries, i.e. that commutes with the boundary operator $\mathbf{q}\,\mathbf{b}_B = \mathbf{b}_A\,\mathbf{q}$. A chain map induces a cochain map $\mathbf{p} : \mathcal{C}_p(C_A) \to \mathcal{C}_p(C_B)$ (a discrete pull-back of the chain map) with $\mathbf{d}_B\,\mathbf{p} = \mathbf{p}\,\mathbf{d}_A$.

For $C_A = C$ and $C_B = \underline{C}$ we find

$$[\mathrm{P}^{p+1}][\mathrm{D}^p] = [\underline{\mathrm{D}}^p][\mathrm{P}^p] + [\underline{\mathrm{J}}^p][\mathrm{P}_{\mathrm{b}}^p][\mathrm{T}^p], \quad (22\,\mathrm{a})$$
$$[\mathrm{B}^{p+1}][\underline{\mathrm{Q}}^{p+1}] = [\underline{\mathrm{Q}}^p][\underline{\mathrm{B}}^{p+1}] + [\mathrm{I}^p][\underline{\mathrm{Q}}_{\mathrm{b}}^p][\underline{\mathrm{C}}^{p+1}]. \quad (22\,\mathrm{b})$$

which will be proven in Section 6.1. It follows directly that for $C_A = C_{\mathrm{b}}$ and $C_B = \underline{C}_{\mathrm{b}}$

$$[\mathrm{P}_{\mathrm{b}}^{p+1}][\mathrm{D}_{\mathrm{b}}^p] = [\underline{\mathrm{D}}_{\mathrm{b}}^p][\mathrm{P}_{\mathrm{b}}^p], \quad (23\,\mathrm{a})$$
$$[\mathrm{B}_{\mathrm{b}}^{p+1}][\underline{\mathrm{Q}}_{\mathrm{b}}^{p+1}] = [\underline{\mathrm{Q}}_{\mathrm{b}}^p][\underline{\mathrm{B}}_{\mathrm{b}}^{p+1}]. \quad (23\,\mathrm{b})$$

We can see that the (co)chain map condition is not met by the pairing matrix and its adjoint. Equations (22 a) and (22 b) feature terms from the boundary that break the symmetry. Only on the boundary complexes C_{b} and $\underline{C}_{\mathrm{b}}$ the (co)chain-map conditions are fulfilled. In other words, if we want to have the properties of a cochain preserved under the action of the pairing operator, we need to include a boundary term in the derivative on the dual complex, compare Fig. 3.

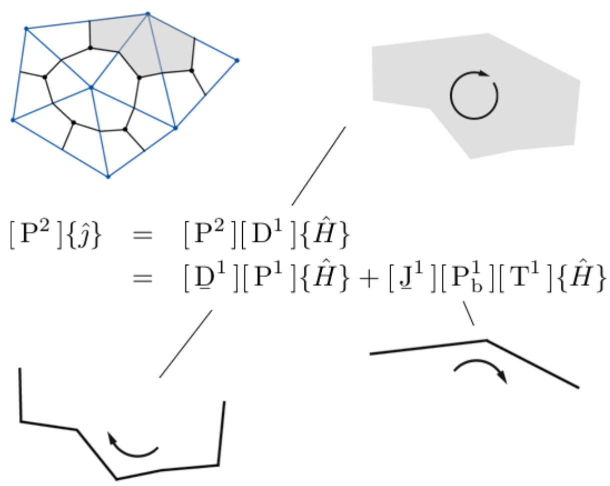


Fig. 3. Illustration of (22 a). An electric current density is mapped from the primal complex in Fig. 1 to the open dual complex. Applying first the pairing operator and then the dual derivative, we do not sum the circulation of the magnetic field along a closed path. The boundary term is needed in order to close the loop and recover the discrete Ampère's law on the dual complex.

5 The Pairing Matrix in FEM

We briefly introduce the finite-element method (FEM) with Whitney elements.

5.1 Whitney Element FEM

Following a convention, we denote $\bar{\alpha}$ a potential, e.g., the magnetic vector potential $\bar{A}$ or the electric scalar potential $\bar{\varphi}$, and $\bar{\eta}$ a source field, e.g., an electric current density $\bar{\jmath}$ or an electric charge density $\bar{\rho}$. ϑ denotes the material property. Consider the differential equation

$$\mathrm{d} *_{\vartheta} \mathrm{d}\,\bar{\alpha} = \bar{\eta}, \quad (24)$$

$\bar{\alpha} \in {}^{p}H_{\mathrm{d}}(\Omega)$, $\bar{\eta} \in {}^{n-p}H_{\mathrm{d}}(\mathrm{d}, \Omega)$, where $\bar{\eta}$ is chosen to lie in the Sobolev space of closed $(n-p)$-forms. For a definition of the Sobolev spaces ${}^{p}H_{\mathrm{d}}(\Omega)$ and ${}^{n-p}H_{\mathrm{d}}(\mathrm{d}, \Omega)$ see, e.g., [7]. For a given source $\bar{\eta}$, we search for a solution $\tilde{\alpha}$ in a finite-dimensional subspace $\mathcal{W}^p(C)$ of ${}^{p}H_{\mathrm{d}}(\Omega)$. An optimum solution is found by the weighted residual method: The residual is given by

$$R(\tilde{\alpha}) = \mathrm{d} *_{\vartheta} \mathrm{d}\, \tilde{\alpha} - \bar{\eta}. \tag{25}$$

We require that the weighted residual vanishes for all test functions $\tilde{w}_i^p$ in the space of Whitney forms,

$$\big\langle \mathrm{d} *_{\vartheta} \mathrm{d}\, \tilde{\alpha} \wedge \tilde{w}_i^p \,\big|\, \Omega \big\rangle = \big\langle \bar{\eta} \wedge \tilde{w}_i^p \,\big|\, \Omega \big\rangle, \quad i = 1, \ldots, n_p. \tag{26}$$

With $\big\langle \mathrm{d}\,(\bar{F} \wedge \bar{G}) \,\big|\, \Omega \big\rangle = \big\langle \mathrm{t}\, \bar{F} \wedge \mathrm{t}\, \bar{G} \,\big|\, \mathrm{b}\, \Omega \big\rangle$ and

$$\mathrm{d}\,(\bar{F} \wedge \bar{G}) = \mathrm{d}\, \bar{F} \wedge \bar{G} + (-1)^p \bar{F} \wedge \mathrm{d}\, \bar{G}, \tag{27}$$

where $\bar{F} \in \mathcal{F}^p(\Omega)$, $\bar{G} \in \mathcal{F}^{n-p-1}(\Omega)$, we can rewrite (26)

$$\begin{aligned} (-1)^{n-p} \big\langle *_{\vartheta} \mathrm{d}\, \tilde{\alpha} \wedge \mathrm{d}\, \tilde{w}_i^p \,\big|\, \Omega \big\rangle + \big\langle \mathrm{t} *_{\vartheta} \mathrm{d}\, \tilde{\alpha} \wedge \mathrm{t}\, \tilde{w}_i^p \,\big|\, \mathrm{b}\, \Omega \big\rangle \\ = \big\langle \bar{\eta} \wedge \tilde{w}_i^p \,\big|\, \Omega \big\rangle, \end{aligned} \tag{28}$$

$i = 1, \ldots, n_p$, where $\mathrm{t} *_{\vartheta} \mathrm{d}\, \tilde{\alpha}$ is the Neumann data of the field problem and $\mathrm{t}\, \tilde{\alpha}$ is the Dirichlet data. On a cell complex C that discretizes the domain Ω, and with Whitney p-forms available on the complex, (28) is the finite element formulation of (24).

5.2 Galerkin-Type Matrices

We introduce matrices $[\mathrm{P}^p]$ and $[\mathrm{M}_{\vartheta}^p]$ by

$$\big\langle \tilde{F} \wedge \tilde{G} \,\big|\, \Omega \big\rangle = \{\hat{G}\}^{\mathrm{T}} [\mathrm{P}^p] \{\hat{F}\}, \quad \text{and} \tag{29}$$

$$\big\langle *_{\vartheta} \tilde{F} \wedge \tilde{H} \,\big|\, \Omega \big\rangle = \{\hat{H}\}^{\mathrm{T}} [\mathrm{M}_{\vartheta}^p] \{\hat{F}\}, \tag{30}$$

$\tilde{F}, \tilde{H} \in \mathcal{W}^p(C)$, $\tilde{G} \in \mathcal{W}^{n-p}(C)$ with the element matrices of $[\mathrm{P}^p]$ and $[\mathrm{M}_{\vartheta}^p]$ defined by

$$[\mathrm{P'}^p]^{ij} = \big\langle \tilde{w}_j^p \wedge \tilde{w}_i^{n-p} \,\big|\, C \big\rangle \quad \text{and} \tag{31}$$

$$[\mathrm{M'}_{\vartheta}^p]^{ij} = \big\langle *_{\vartheta} \tilde{w}_j^p \wedge \tilde{w}_i^p \,\big|\, C \big\rangle. \tag{32}$$

Furthermore, we define

$$[\mathrm{P'}_{\mathrm{b}}^p]^{ij} = \big\langle \mathrm{t}\, \tilde{w}_j^p \wedge \mathrm{t}\, \tilde{w}_i^{n-p-1} \,\big|\, C_{\mathrm{b}} \big\rangle. \tag{33}$$

The "Galerkin-type" matrices are called Galerkin-Hodge matrix [8] and pairing matrix. It is insightful to realize that the pairing-matrix definitions (31) and (33) yield identical matrices as the geometrical definition by interpolation of simplices in (13 b). This identity is proven by verifying that both definitions yield identical matrices for $0 \le p \le n$ on one arbitrary simplicial n-cell. Now we can see that

$$[\mathrm{P}^p] = (-1)^{p(n-p)} [\mathrm{P}^{n-p}]^{\mathrm{T}}, \tag{34}$$

e.g. for $n = 3$, $[\mathrm{P}^p] = [\mathrm{P}^{3-p}]^{\mathrm{T}}$, which can be verified in (15 a)-(15 d).

5.3 DEM Interpretation of FEM

Eventually, we interpolate the source field $\bar{\eta} \in \mathcal{F}^{n-p}(\Omega)$ in (24) by a Whitney $(n-p)$-form $\tilde{\eta} \in \mathcal{W}^{n-p}(C)$ and the Neumann data $\mathrm{t} *_{\vartheta} \mathrm{d}\, \tilde{\alpha}$ by a Whitney $(n-p-1)$-form $\tilde{\gamma} \in \mathcal{W}^{n-p-1}(C_{\mathrm{b}})$. We can now rewrite (28):

$$\begin{aligned} (-1)^{n-p} [\mathrm{D}^p]^{\mathrm{T}} [\mathrm{M}_{\vartheta}^{p+1}] [\mathrm{D}^p] \{\hat{\alpha}\} \\ + [\mathrm{T}^p]^{\mathrm{T}} [\mathrm{P}_{\mathrm{b}}^{n-p-1}] \{\hat{\gamma}\} = [\mathrm{P}^{n-p}] \{\hat{\eta}\}, \end{aligned} \tag{35}$$

which reads with (4 d), (14) and (17)

$$[\underline{\mathrm{D}}^{n-p-1}] [\mathrm{M}_{\vartheta}^{p+1}] [\mathrm{D}^p] \{\hat{\alpha}\} + [\underline{\mathrm{J}}^{n-p-1}] \{\hat{\underline{\gamma}}\} = \{\hat{\underline{\eta}}\}. \tag{36}$$

We have shown that the finite-element method with Whitney forms uses an implicit definition of a barycentric dual complex.

The coclosure matrix $[\underline{\mathrm{J}}^{n-p-1}]$ is called jump matrix in [5]. The second term in (36) can be interpreted as a single layer of sources (charges or currents) that makes the Neumann data jump to zero on the domain boundary. These surface sources contribute to the total sources on the dual complex. This interpretation has advantages in the context of a coupling of discrete electromagnetism to the boundary-element method [5].

6 Discrete Energy Products

In this section we define energy products in discrete electromagnetism. One definition uses the discrete wedge product, whereas another one employs the duality product. We realize that there exist various ways to introduce a product of discrete quantities. With the above findings we also find that all these definitions are geometrically equivalent. This will lead to the introduction of discrete de Rham currents in the next section.

6.1 Discrete Wedge Product

The wedge product of cochains $\boldsymbol{\wedge} : \mathcal{C}^p(C) \times \mathcal{C}^q(C) \to \mathcal{C}^{p+q}(C)$ is defined using de Rham- and Whitney maps (interpolation and discretization) [9]. The discretization operation, called *de Rham map*, is defined as

$$\pi : \mathcal{F}^p(\Omega) \to \mathcal{C}^p(C), \quad \bar{F} \mapsto \sum_i^{n_p} \big\langle \bar{F} \,\big|\, \check{\sigma}_p^i \big\rangle\, \hat{\sigma}_i^p, \tag{37}$$

where $\hat{\sigma}_i^p$ is the ith p-cosimplex, i.e. the ith basis element of the cochain space $\mathcal{C}^p$. The interpolation operation, called *Whitney map*, is defined as

$$\boldsymbol{\omega} : \mathcal{C}^p(C) \to \mathcal{W}^p(C), \quad \hat{F} \mapsto \tilde{F} = \sum_i^{n_p} \{\hat{F}\}^i \tilde{w}_i^p. \tag{38}$$

The definition of the *discrete wedge product* reads[2]

$$\hat{G} \boldsymbol{\wedge} \hat{F} = \pi(\boldsymbol{\omega}\, \hat{G} \wedge \boldsymbol{\omega}\, \hat{F}). \tag{39}$$

[2] The discrete wedge product is not associative. This is felt as a problem by some, but it is of no concern for us here [10].

In particular for $\hat{G} \in \mathcal{C}^{n-p}$ and $\hat{F} \in \mathcal{C}^p$ we find

$$\begin{aligned}\langle \hat{G} \wedge \hat{F} \,|\, C \rangle &= \langle \boldsymbol{\omega}\, \hat{G} \wedge \boldsymbol{\omega}\, \hat{F} \,|\, C \rangle \\ &= \{\hat{F}\}^{\mathrm{T}} [\mathrm{P}^{n-p}] \{\hat{G}\},\end{aligned} \tag{40}$$

with the pairing-matrix definition of (31). We realize that the cochain product can be geometrically interpreted as a product of primal and dual cochains since

$$\begin{aligned}\{\hat{F}\}^{\mathrm{T}} [\mathrm{P}^{n-p}] \{\hat{G}\} &= \{\hat{F}\}^{\mathrm{T}} \{\hat{\underline{G}}\} \\ &= (-1)^{p(n-p)} \{\hat{\underline{F}}\}^{\mathrm{T}} \{\hat{G}\}.\end{aligned} \tag{41}$$

With the linear operator $\mathbf{w}$ defined by

$$\mathbf{w}\, \hat{F} = (-1)^p \hat{F} \quad \text{for} \quad \hat{F} \in \mathcal{C}^p(C) \tag{42}$$

we find in analogy to (27)

$$\begin{aligned}&\langle \mathbf{t}\, \hat{F} \wedge \mathbf{t}\, \hat{G} \,|\, C_{\mathrm{b}} \rangle = \\ &\qquad \langle \mathbf{d}\, \hat{F} \wedge \hat{G} \,|\, C \rangle + \langle \mathbf{w}\, \hat{F} \wedge \mathbf{d}\, \hat{G} \,|\, C \rangle,\end{aligned} \tag{43}$$

where $\hat{F} \in \mathcal{C}^p$ and $\hat{G} \in \mathcal{C}^{n-p-1}$. (43) reads in matrix notation

$$\begin{aligned}&\{\hat{G}\}^{\mathrm{T}} [\mathrm{T}^{n-p-1}]^{\mathrm{T}} [\mathrm{P}^p_{\mathrm{b}}] [\mathrm{T}^p] \{\hat{F}\} = \\ &\quad \{\hat{G}\}^{\mathrm{T}} [\mathrm{P}^{p+1}] [\mathrm{D}^p] \{\hat{F}\} + (-1)^p \{\hat{G}\}^{\mathrm{T}} [\mathrm{D}^{n-p-1}]^{\mathrm{T}} [\mathrm{P}^p] \{\hat{F}\}.\end{aligned} \tag{44}$$

With (4 d) and (17) we recover (22 a)

$$[\mathrm{P}^{p+1}][\mathrm{D}^p] = [\underline{\mathrm{D}}^p][\mathrm{P}^p] + [\underline{\mathrm{J}}^p][\mathrm{P}^p_{\mathrm{b}}][\mathrm{T}^p], \tag{45}$$

which was interpreted in Fig. 3.

As an example for a discrete wedge product in discrete electromagnetism we define the magnetostatic energy W in a complex filled with linear material

$$W = \frac{1}{2} \langle \hat{B} \wedge \hat{H} \,|\, C \rangle, \tag{46}$$

which can be written with (43)

$$\frac{1}{2} \langle \hat{B} \wedge \hat{H} \,|\, C \rangle = \frac{1}{2} \langle \hat{A} \wedge \hat{\jmath} \,|\, C \rangle + \frac{1}{2} \langle \mathbf{t}\, \hat{A} \wedge \mathbf{t}\, \hat{H} \,|\, C_{\mathrm{b}} \rangle. \tag{47}$$

Note that for the use of the discrete wedge product the Faraday- and Ampère-Maxwell fields must both be discretized on the primal complex.

6.2 The Chain-Cochain Interpretation of DEM

Discrete electromagnetism usually defines Faraday- and Ampère-Maxwell fields as cochains on the pair of a primal and a dual complex. In this section we propose to let primal $(n-p)$-chains take the role of dual p-cochains. The goal is that, while we loose in intuition, we gain in formalism, e.g., by introducing the duality product between fields and making use of Stokes' theorem.

The number of primal $(n-p)$-cells is identical to the number of dual p-cocells on the open dual complex $\underline{C}$. This, together with (5 b) enables us to use the canonical basis isomorphism and interprete the coefficients of a dual p-cochain as the coefficients of a primal $(n-p)$-chain

$$\mathcal{C}^p(\underline{C}) \xrightarrow{\cong} \mathbb{R}^{n_{n-p}} \xrightarrow{\cong} \mathcal{C}_{n-p}(C), \tag{48}$$

or simply $\{\hat{\underline{F}}\} = \{\check{F}\}$, and

$$\mathbf{d}_{\underline{C}} \xrightarrow{\cong} \mathbb{R}^{n_{n-p} \times n_{n-p-1}} \xrightarrow{\cong} \mathbf{w}\, \mathbf{b}_C, \tag{49}$$

where the linear operator $\mathbf{w}$ is defined on chains by

$$\mathbf{w}\, \check{F} = (-1)^{n-p} \check{F} \quad \text{for} \quad \check{F} \in \mathcal{C}_p(C), \tag{50}$$

compare (42). As for dual cochains on the open dual complex, a derivative of a primal chain is constituted of the boundary operator acting on the chain and an immersion from the boundary complex. As an example we take Ampère's law

$$\check{\jmath} = \mathbf{b}\, \check{H} + \mathbf{i}\, \check{H}_{\mathrm{b}}\,, \tag{51}$$

and Gauss' law

$$\check{\rho} = -\mathbf{b}\, \check{D} + \mathbf{i}\, \check{D}_{\mathrm{b}}\,. \tag{52}$$

It comes as no surprise, that the coefficients of a primal $(n-p)$-chain can be obtained from those of a primal p-cochain by means of the pairing matrix. The pairing operator is now defined as $\mathbf{p} : \mathcal{C}^p(C) \to \mathcal{C}_{n-p}(C)$ and from (22 a) we obtain

$$\mathbf{p}\, \mathbf{d} = \mathbf{w}\, \mathbf{b}\, \mathbf{p} + \mathbf{i}\, \mathbf{p}\, \mathbf{t}\,. \tag{53}$$

The magnetostatic energy in linear media can be calculated from

$$\frac{1}{2} \langle \hat{B} \,|\, \check{H} \rangle = \frac{1}{2} \langle \mathbf{d}\, \hat{A} \,|\, \check{H} \rangle, \tag{54 a}$$

$$= \frac{1}{2} \langle \hat{A} \,|\, \mathbf{b}\, \check{H} \rangle, \tag{54 b}$$

$$= \frac{1}{2} \langle \hat{A} \,|\, \check{\jmath} \rangle - \frac{1}{2} \langle \hat{A} \,|\, \mathbf{i}\, \check{H}_{\mathrm{b}} \rangle. \tag{54 c}$$

It can be easily verified that (47) and (54 c) have identical matrix representations, see (55), provided that the 2-chain $\check{H}$ and the 1-chain $\check{\jmath}$ are obtained from the 1-cochain $\hat{H}$ and the 2-cochain $\hat{\jmath}$ by $\check{H} = \mathbf{p}\, \hat{H}$ and $\check{\jmath} = \mathbf{p}\, \hat{\jmath}$.

7 Discrete de Rham Currents

In the last section we have seen that discrete energy products can be formulated in two different ways. The wedge-product formalism is derived in analogy to the continuous theory. The duality product between chains and cochains is a new and simple approach, but not quite intuitive. A third product of the same kind is used, e.g. in [10], which is defined as the product of primal p-cochains and dual $(n-p)$-cochains, compare (41). Of course, this latter product is isomorphic to the chain-cochain product by (48). It lacks, however, the convenience of Stoke's theorem. Even a primal-chain/dual-chain representation of discrete electromagnetism is conceivable.[3] This approach however does not seem to have merits for our present work. In what follows we shall only use wedge- and duality-product versions of energy products.

With the use of the pairing operator, all of the above-mentioned versions of energy products have identical matrix representations. For the magnetostatic energy it reads

$$\begin{aligned}&\frac{1}{2}\{\hat{B}\}^{\mathrm{T}}[\mathrm{P}^1]\{\hat{H}\}\\&=\tfrac{1}{2}\{\hat{A}\}^{\mathrm{T}}[\mathrm{P}^2]\{\hat{\jmath}\}-\tfrac{1}{2}\{\hat{A}\}^{\mathrm{T}}[\mathrm{T}^1]^{\mathrm{T}}[\mathrm{P}^1_{\mathrm{b}}][\mathrm{T}^1]\{\hat{H}\}.\end{aligned} \tag{55}$$

In the chain-cochain interpretation we read the pairing matrix as a mapping from cochain coefficients to chain coefficients. In the wedge-product interpretation we read it as the agent of the discrete wedge product, evaluated over the complex. Note that the wedge-product matrix representation strictly speaking is the transpose of the above equation according to (40), which explains the change of sign in the boundary term, see (34).

One might argue that, if the system of linear equations arising from a discrete-electromagnetism formulation is identical whatever concept we use, then there should exist a formalism that unifies all different approaches. This formalism, based on de Rham's continuous theory of currents [12], was proposed in [6].

7.1 Discrete Currents

A *discrete current* of dimension p and degree $n-p$ is a map of p-cochains into reals, i.e. currents are functionals on cochains. The linear space of discrete currents is denoted by $\mathcal{C}_p^{n-p}(C)$ or simply $\mathcal{C}_p^{n-p}$. $(n-p)$-cochains as well as p-chains qualify as currents $T\in\mathcal{C}_p^{n-p}$

$$\begin{aligned}T:\mathcal{C}^p\to\mathbb{R},\\ \hat{F}\mapsto T[\hat{F}]=\langle\hat{F}\,|\,\check{T}\rangle=\langle\hat{T}\wedge\hat{F}\,|\,C\rangle.\end{aligned} \tag{56}$$

The use of the pairing operator $\check{T}=\mathbf{p}\hat{T}$ ensures that $\check{T}$ and $\hat{T}$ in the above equation are the representatives of the same current T.

7.2 Operators on Currents

The *boundary operator* on currents is defined in accordance with the discrete Stokes theorem $\mathbf{b}:\ \mathcal{C}_p^{n-p}(C)\to\mathcal{C}_{p-1}^{n-p+1}(C)$

$$\mathbf{b}\,T[\hat{F}]=T[\mathbf{d}\hat{F}] \tag{57 a}$$
$$=\langle\hat{T}\wedge\mathbf{d}\hat{F}\,|\,C\rangle \tag{57 b}$$
$$=\langle\hat{F}\,|\,\mathbf{b}\check{T}\rangle=\langle\mathbf{d}\hat{F}\,|\,\check{T}\rangle. \tag{57 c}$$

The linear operator $\mathbf{w}$ acts as $\mathbf{w}\,T=(-1)^pT$ where p is the degree of the current. We (re)introduce the *immersion operator* as a mapping

$$\mathbf{i}:\mathcal{C}_p^{m-p}(B)\to\mathcal{C}_p^{n-p}(C),\quad p\le m\le n, \tag{58}$$

where B is an m-dimensional subcomplex of C. A discrete *trace operator on currents* is defined as a restriction to the boundary C_{b},

$$\begin{aligned}\mathbf{t}:\mathcal{C}_p^{n-p}(C)\to\mathcal{C}_{p-1}^{n-p}(C_{\mathrm{b}})\\ \mathbf{t}\,T[\hat{G}]=\langle\mathbf{t}\hat{T}\wedge\hat{G}\,|\,C_{\mathrm{b}}\rangle=\langle\hat{G}\,|\,\mathbf{t}\check{T}\rangle,\end{aligned} \tag{59}$$

with $\hat{G}\in\mathcal{C}^p(C_{\mathrm{b}})$. The trace of a chain is defined as

$$\mathbf{t}\check{T}=\mathbf{p}\,\mathbf{t}\hat{T}. \tag{60}$$

Furthermore the adjoint of an immersion acting upon a trace is a trace, i.e.

$$\mathbf{i}\,\mathbf{t}\,T[\hat{F}]=\mathbf{t}\,T[\mathbf{t}\hat{F}],\quad\hat{F}\in\mathcal{C}^p(C). \tag{61}$$

The definition of the *coboundary operator* reads

$$\begin{aligned}\mathbf{d}:\ \mathcal{C}_p^{n-p}(C)\to\mathcal{C}_{p-1}^{n-p+1}(C),\\ \mathbf{d}\,T[\hat{F}]=\langle\mathbf{d}\hat{T}\wedge\hat{F}\,|\,C\rangle=\langle\hat{F}\,|\,\mathbf{d}\check{T}\rangle,\end{aligned} \tag{62}$$

where the coboundary of a chain is defined by

$$\mathbf{d}\check{T}=\mathbf{p}\,\mathbf{d}\hat{T}. \tag{63}$$

It follows that

$$\mathbf{d}=\mathbf{w}\,\mathbf{b}+\mathbf{i}\,\mathbf{t}. \tag{64}$$

The discrete Hodge operator is a mapping from cochains to currents, $\bigstar:\mathcal{C}^p(C)\to\mathcal{C}_p^{n-p}(C)$. We set $H=\bigstar\hat{G}$ and define similarly to (39)

$$\hat{H}\wedge\hat{F}=\pi\,(*\,\boldsymbol{\omega}\,\hat{G}\wedge\boldsymbol{\omega}\,\hat{F}). \tag{65}$$

It follows that

$$\begin{aligned}\bigstar\hat{G}[\hat{F}]=\langle\hat{H}\wedge\hat{F}\,|\,C\rangle=\langle*\,\boldsymbol{\omega}\hat{G}\wedge\boldsymbol{\omega}\hat{F}\,|\,C\rangle,\\ =\langle\hat{F}\,|\,\check{H}\rangle,\end{aligned} \tag{66}$$

For $\hat{G}\in\mathcal{C}^p$ we find

$$\bigstar\hat{G}[\hat{F}]=\{\hat{F}\}^{\mathrm{T}}[\mathrm{M}^p]\{\hat{G}\}, \tag{67}$$

where the element matrix $[\mathrm{M}'^p]$ of a canonically oriented simplex is defined as

$$[\mathrm{M}'^p]^{ij}=\langle*\,\tilde{w}_j^p\wedge\tilde{w}_i^p\,|\,C\rangle. \tag{68}$$

The above defined operator is called the Galerkin Hodge operator, compare (32). Alternative definitions of a discrete Hodge operator do not obstruct the presented formalism.

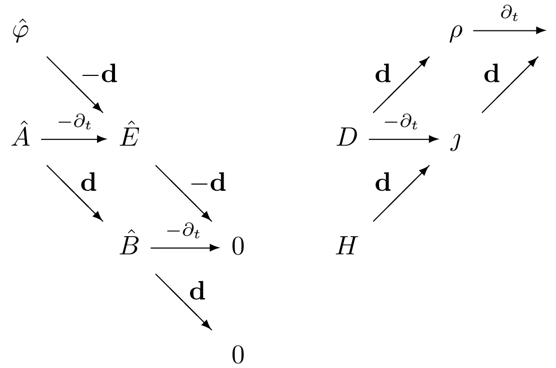
[3] In [11] 322ff. G. de Rham mentions the Kronecker index $I(\check{c}_1\underline{\check{c}}_2)$ of the cut prodcut between two chains $\check{c}_1$ and $\underline{\check{c}}_2$ of dimension p and $n-p$. In our context this operation defines a product of primal and dual chains that has the coefficient representation $I(\check{c}_1\underline{\check{c}}_2)=\{\check{c}_1\}^{\mathrm{T}}\{\underline{\check{c}}_2\}$. de Rham proceeds to illustrate the duality of chains and forms: he gives the index of the wedge-product of an $(n-p)$-form $\bar{F}_1$ and a p-form $\bar{F}_2$ as $I(\bar{F}_1\bar{F}_2)=\langle\bar{F}_1\wedge\bar{F}_2\,|\,\Omega\rangle$. He then sets out to show that this duality is rooted in a deeper identity of chains and forms, which is exhibited in the current formalism.

7.3 Currents in Discrete Electromagnetism

Objects that are discretized on the dual complex (dual cochains, primal chains) can be interpreted as currents. Generally, the Faraday complex is discretized on the primal complex and the Ampère-Maxwell complex on the dual complex. Hence, Ampère-Maxwell fields should be discretized by currents, mapping Faraday fields to energy quantities. For the magnetostatic energy in linear media this reads

$$\frac{1}{2}H\,[\,\hat{B}\,] = \frac{1}{2}\jmath\,[\,\hat{A}\,] - \frac{1}{2}\mathbf{t}\,H\,[\,\mathbf{t}\,\hat{A}\,]. \tag{69}$$

We can write the topological diagrams of discrete electromagnetism.



(70)

The functionals of the Ampère-Maxwell complex on the right-hand side map cochains of the Faraday complex on the same row of the left-hand side into energy-related quantities.

The pairing matrix has vanished from the visible formalism, as has the dual complex. Their definitions are implicit to the action of a current on a cochain.

8 Conclusion

We calculate the coefficients of the pairing matrix and see that it maps cochain coefficients on the simplicial primal complex into cochain coefficients on the open barycentric dual complex. We find that pairing matrices also figure in Whitney-element FEM and in the discrete wedge product. This finding allows for a geometric interpretation of the wedge product in terms of a duality product between primal cochains and primal chains. The equivalence of the matrix representations of these two formalisms leads us from the duality of chains and cochains to their unification in the formalism of de Rham's currents.

The practical importance of the pairing matrix lies in the possibility to avoid the explicit construction of a dual cell complex. It should be interesting to define pairing matrices for dual complexes other than barycentric duals. A further advantage of the presented theory lies in the clarification of apparent ambiguities in the definition of discrete energy products by means of de Rham's currents.

A A (Co)Chain Map between Primal and Closed Dual Complex

The operators $[\utilde{\mathrm{B}}^p]$ and $[\utilde{\mathrm{D}}^p]$, as well as $[\utilde{\mathrm{Q}}^p]$ and $[\mathrm{P}^p]$ are defined for the open dual complex $\utilde{C}$. Intuitively we understand that a map between an open and a closed complex cannot possibly preserve boundaries. To find a chain map we rather have to consider the closed dual complex $\utilde{C}_{\mathrm{c}} = \utilde{C} \oplus \utilde{C}_{\mathrm{b}}$, see Section 3, for C_B, and the primal complex for C_A. Then we find

$$\underbrace{\begin{pmatrix}[\mathrm{P}^{p+1}] & 0\\ 0 & [\mathrm{P}_{\mathrm{b}}^{p+1}]\end{pmatrix}\begin{pmatrix}[\,1\,]\\ [\mathrm{T}^{p+1}]\end{pmatrix}}_{\mathbf{p}*}\underbrace{[\mathrm{D}^p]}_{\mathbf{d}_A} = \underbrace{\begin{pmatrix}[\utilde{\mathrm{D}}^p] & [\utilde{\mathrm{J}}^p]\\ 0 & [\utilde{\mathrm{D}}_{\mathrm{b}}^p]\end{pmatrix}}_{\mathbf{d}_B}\underbrace{\begin{pmatrix}[\mathrm{P}^p] & 0\\ 0 & [\mathrm{P}^p_{\mathrm{b}}]\end{pmatrix}\begin{pmatrix}[\,1\,]\\ [\mathrm{T}^p]\end{pmatrix}}_{\mathbf{p}^*}, \tag{71 a}$$

$$\underbrace{[\mathrm{B}^{p+1}]}_{\mathbf{b}_A}\underbrace{\begin{pmatrix}[\,1\,] & [\mathrm{I}^{p+1}]\end{pmatrix}\begin{pmatrix}[\utilde{\mathrm{Q}}^{p+1}] & 0\\ 0 & [\utilde{\mathrm{Q}}_{\mathrm{b}}^{p+1}]\end{pmatrix}}_{\mathbf{q}^*} = \underbrace{\begin{pmatrix}[\,1\,] & [\mathrm{I}^p]\end{pmatrix}\begin{pmatrix}[\utilde{\mathrm{Q}}^p] & 0\\ 0 & [\utilde{\mathrm{Q}}_{\mathrm{b}}^p]\end{pmatrix}}_{\mathbf{q}^*}\underbrace{\begin{pmatrix}[\utilde{\mathrm{B}}^{p+1}] & 0\\ [\utilde{\mathrm{C}}^{p+1}] & [\utilde{\mathrm{B}}_{\mathrm{b}}^{p+1}]\end{pmatrix}}_{\mathbf{b}_B}, \tag{71 b}$$

where $[\,1\,]$ denotes the identity. We have therefore established that for $C_A = C$ and $C_B = \utilde{C}_{\mathrm{c}}$

$$\mathbf{p}^*\,\mathbf{d}_A = \mathbf{d}_B\,\mathbf{p}^*, \tag{72 a}$$

$$\mathbf{b}_A\,\mathbf{q}^* = \mathbf{q}^*\,\mathbf{b}_B. \tag{72 b}$$

Furthermore $\mathbf{d}_A\,\mathbf{d}_A = 0$ and $\mathbf{b}_A\,\mathbf{b}_A = 0$ follow from the discrete Poincaré lemma (a (co)boundary of a (co)boundary is empty). $\mathbf{d}_B\,\mathbf{d}_B = 0$ and $\mathbf{b}_B\,\mathbf{b}_B = 0$ follow from the Poincaré lemma and (21).

B Four Geometrically Equivalent Formulae

We list the geometrically equivalent expressions (43), (45), (53), and (64)

$$\langle \mathbf{d}\,\hat{F} \wedge \hat{G}\,|\,C\rangle = -\langle \mathbf{w}\,\hat{F} \wedge \mathbf{d}\,\hat{G}\,|\,C\rangle + \langle \mathbf{t}\,\hat{F} \wedge \mathbf{t}\,\hat{G}\,|\,C_{\mathrm{b}}\,\rangle, \tag{73 a}$$

$$[\mathrm{P}^{p+1}][\mathrm{D}^p] = [\utilde{\mathrm{D}}^p][\mathrm{P}^p] + [\utilde{\mathrm{J}}^p][\mathrm{P}^p_{\mathrm{b}}][\mathrm{T}^p], \tag{73 b}$$

$$\mathbf{p}\,\mathbf{d} = \mathbf{w}\,\mathbf{b}\,\mathbf{p} + \mathbf{i}\,\mathbf{p}\,\mathbf{t}\,, \tag{73 c}$$

$$\mathbf{d} = \mathbf{w}\,\mathbf{b} + \mathbf{i}\,\mathbf{t}\,. \tag{73 d}$$

- (73 a) is concerned with the wedge product between primal cochains. It represents the discrete version of an integration by parts.
- (73 b) defines the commutation of pairing operator and derivative operator. Here, the pairing operator is interpreted as a map from primal chochains to dual cochains. The boundary term of (73 b) is needed to "close the loops" in the dual derivative, compare Fig. 3. The relation between (73 a) and (73 b) was derived in Section 6.1.

– In (73 c) the pairing operator is interpreted as a mapping from cochains to chains on the primal complex. In matrix notation, (73 b) and (73 c) are equivalent due to (5 b), $[\mathrm{B}^{n-p}] = (-1)^{p+1}[\underline{\mathrm{D}}^{p}]$.
– (73 d) is defined for discrete de Rham currents. It is valid for the cochain representation of a current due to (73 a), as well as for the chain representation in (73 c).

All equations (73 a) to (73 d) are based on the same geometrical ground, which is best described by the Figure 3.

C Even and Odd Objects

As a last observation we take a look at the behavior of various mathematical objects of continuous and discrete electromagnetism under a change of (inner) orientation of the underlying cell complex.

We generally say that discrete Faraday- and Ampère-Maxwell fields are discretized on inner- and outer oriented complexes, respectively. This concept has difficulties: The theory of integration does not know outer orientation, but uses even and odd differential forms on inner oriented manifolds. De Rham maps can therefore only be applied on inner oriented complexes, by integrating even and odd differential forms of Faraday- and Ampère-Maxwell fields over cells.

This insight implies the solution to another problem: In a one-mesh method, such as the finite-element method or a discrete-electromagnetism approach with pairing matrices, all fields need to be discretized on the same complex. Is it inner- or outer oriented? Or do we need to foresee two copies of the same complex with inner- and outer orientation? The answer is that one inner oriented complex will do the trick for both types of fields, provided that we are clear about the transformation properties of discrete fields and operators under a global change of orientation which affects all cells in the complex, from nodes to volumes.

We shall call an object *even* if it keeps its sign, and *odd* if it changes its sign under a global change of orientation. For discrete objects the even and odd qualifiers apply to the respective coefficients. The results are displayed in Tables 1 and 2.

Table 1. Even and odd quantities and operators in continuous electromagnetism.

Object	Even / Odd
straight (even) differential form	even
twisted (odd) differential form	odd
continuous coboundary	even
continuous Hodge operator	odd

Table 2. Even and odd quantities and operators in discrete electromagnetism.

Object	Even / Odd
primal chain	odd
dual chain	even
primal cochain from even form[1]	odd
primal cochain from odd form[1]	even
primal chain (from odd form)[2]	odd
dual cochain (from odd form)[3]	odd
discrete de Rham current (from odd form)[2]	odd
discrete boundary operator	even
discrete coboundary operator	even
pairing operator	odd
discrete Hodge operator	even

[1] by de Rham map of even/odd form on odd primal complex.
[2] by action of the pairing operator on a primal cochain (from odd form).
[3] by action of the pairing operator on a primal cochain (from odd form), or by de Rham map of odd form on even dual chain.

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