# XXXI. NETWORK SYNTHESIS


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## RESEARCH OBJECTIVES

The effectiveness of methods of approach to the synthesis problem as reported in last year's research objectives is being explored in greater detail. The detailed reports given below are specific steps in this evaluation process. The further development of an approach and its implementation will absorb our efforts for some time to come; and it is our expectation that many collateral problems will be generated in the process. Already the results of this work have clarified aspects of network theory which represent gaps in classical linear dynamics. The development of network theory seems to be on the threshold of a new phase that promises to be at least as significant as the one that yielded our present state of accomplishment in synthesis.

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## A. A NORMAL FORM FOR A MATRIX PERTINENT TO RLC NETWORKS WITHOUT MUTUAL INDUCTANCE

Guillemin[1] has shown how the problem of finding the normal coordinate transformation for a network with an arbitrary loss function can be reduced to that of simultaneously diagonalizing two matrices, and has indicated the value of this result for a more general approach to network synthesis. Desoer[2] pointed out that the equations for any linear circuit can always be cast in the Peano-Baker form

$$\dot{\overline{y}} = A\overline{y} + \overline{f} \tag{1}$$

and showed how the eigenvectors of the matrix $A$ can be identified with modal behavior.

Another formulation of the problem, which possesses certain advantages for synthesis applications, can be obtained from a combination of these methods. The familiar nodal-equilibrium equations

$$\left(sC + G + \frac{1}{s}\Gamma\right) \overline{e} = \overline{i} \tag{2}$$

are rewritten as

$$(sC+G)\, \overline{e} + \Gamma\overline{\lambda} = \overline{i} \tag{3}$$

$$\Gamma\overline{e} - s\Gamma\overline{\lambda} = \overline{0} \tag{4}$$

or, in partitioned form,

$$\left[\begin{array}{c:c} sC + G & \Gamma \\ \hdashline \Gamma & -s\Gamma \end{array}\right] \left[\begin{array}{c} \overline{e} \\ \hdashline \overline{\lambda} \end{array}\right] = \left[\begin{array}{c} \overline{i} \\ \hdashline \overline{0} \end{array}\right]. \tag{5}$$

Thus the network is described by the two real symmetric matrices

$$\left[\begin{array}{c:c} C & 0 \\ \hdashline 0 & -\Gamma \end{array}\right], \left[\begin{array}{c:c} G & \Gamma \\ \hdashline \Gamma & 0 \end{array}\right]$$

which are related to the network in a familiar way. If $\Gamma$ is nonsingular, this constitutes the desired formulation of the problem. In most cases, however, $\overline{\lambda}$ is not uniquely defined by Eq. 4 and thus the description is singular. This difficulty can be avoided by using the following technique:

(a) Number inductive branches consecutively from 1 to $b_\lambda$.

(b) Choose a tree that is such that all inductive branch voltages are determined by inductive tree-branch voltages.

With this choice of variables, it follows that $\Gamma$ can be partitioned into the form

$$\Gamma = \left[\begin{array}{c:c} \Gamma' & 0 \\ \hdashline 0 & 0 \end{array}\right] \tag{6}$$

in which $\Gamma'$ is nonsingular if $b_\lambda$ is not zero. When this result is substituted in Eq. 5 and the superfluous equations and variables are deleted, the following reduced description is obtained:

$$\left[\begin{array}{cc:c} \multicolumn{2}{c:}{\multirow{3}{*}{$sC + G$}} & \Gamma' \\ \cdashline{3-3} & & 0 \\ \hdashline \Gamma' & 0 & -s\Gamma' \end{array}\right] \left[\begin{array}{c} \overline{e} \\ \hdashline \overline{\lambda}' \end{array}\right] = \left[\begin{array}{c} \overline{i} \\ \hdashline \overline{0} \end{array}\right]. \tag{7}$$

Defining M(s) as

$$M(s) = sM_1 + M_2 \tag{8}$$

where

$$M_1 = \left[\begin{array}{c:c} C & 0 \\ \hdashline 0 & -\Gamma' \end{array}\right] \tag{9}$$

and

$$M_2 = \left[\begin{array}{cc:c} \multicolumn{2}{c:}{\multirow{3}{*}{$G$}} & \Gamma' \\ \cdashline{3-3} & & 0 \\ \hdashline \Gamma' & 0 & 0 \end{array}\right] \tag{10}$$

by the formulas of Schur (3), we have

$$|M(s)| = |-s\Gamma'|\,|sC + G + \frac{1}{s}\Gamma| \tag{11}$$

so that if the network is connected, M(s) is nonsingular. It follows from Eq. 7 that the inverse of M(s) can be partitioned in such a manner that its upper left-hand corner is a matrix of open-circuit impedance parameters (z parameters) for the network.

Let $M_n(s)$ denote the normal form of M(s) under a congruence transformation, and let A denote the nonsingular transformation matrix. Then

$$A_t M(s)\, A = M_n(s) \tag{12}$$

$$M^{-1}_{(s)} = AM_n^{-1}(s)\, A_t. \tag{13}$$

If

$$M^{-1}_{(s)} = [z_{ij}]$$

$$M_n^{-1}(s) = \left[z_{ij}^{(n)}\right]$$

and

$$A = [a_{ij}],$$

then

$$z_{ij} = \sum_{\mu} \sum_{\nu} a_{i\mu} a_{j\nu} z_{\mu\nu}^{(n)} \tag{14}$$

which reduces to a particularly simple form if $M_n(s)$ is diagonal.

The problem of reducing a pencil of symmetric matrices $sM_1 + M_2$ to a normal form by a congruence transformation is somewhat complicated if neither $M_1$ nor $M_2$ is definite or semidefinite. The general solution has been given by Gantmacher,[4] and a special case of this result pertains to the network situation. The following discussion of the normal form is limited to a brief description and physical interpretation.

The simplest situation occurs if the roots of the characteristic equation

$$|M(s)| = 0 \tag{15}$$

are distinct and equal in number to the order of M(s). For each root $s_i$ there exists an eigenvector $\overline{q}_i$ satisfying

$$M(s_i)\, \overline{q}_i = 0 \tag{16}$$

and having the properties

$$\overline{q}_{i_t} M_1 \overline{q}_j = \delta_{ij} \tag{17}$$

$$\bar{q}_{i_t} M_2 \bar{q}_j = -s_i \delta_{ij}. \tag{18}$$

These eigenvectors are directly associated with the modal behavior of the network, their components giving the distribution pattern of voltages and flux linkages corresponding to each natural mode. They appear as the columns of the transformation matrix A, resulting in the normal form

$$M_n(s) = \begin{bmatrix} s - s_1 & & 0 \\ & \ddots & \\ 0 & & s - s_m \end{bmatrix}. \tag{19}$$

The occurrence of multiple-order roots and/or the lack of a sufficient number of roots result in a more complicated normal form. Multiple-order roots arise from two distinctly different physical situations: (a) the presence of two "uncoupled" parts of the network which happen to have the same natural frequency, or (b) the presence of an impedance with a multiple-order pole. The former case leads to a normal form containing a submatrix of the form

$$\hat{M}_{d_i} = \begin{bmatrix} s - s_i & & 0 \\ & \ddots & \\ 0 & & s - s_i \end{bmatrix}, \tag{20}$$

while the latter case leads to a normal form that contains a submatrix of the form

$$\hat{M}_{m_i} = \begin{bmatrix} 0 & & s - s_i \\ & \cdot^{\cdot^{\cdot}} & 1 \\ & s - s_i \quad \cdot^{\cdot^{\cdot}} & \\ s - s_i & 1 & 0 \end{bmatrix}. \tag{21}$$

Since

$$\hat{M}_{d_i}^{-1} = \begin{bmatrix} \frac{1}{s - s_i} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{s - s_i} \end{bmatrix} \tag{22}$$

while

$$\hat{M}_{m_i}^{-1} = \begin{bmatrix} \frac{(-1)^{\mu_i+1}}{(s-s_i)^{\mu_i}} & \cdots & \frac{-1}{(s-s_i)^2} & \frac{1}{(s-s_i)} \\ \vdots & & \iddots & \\ \frac{-1}{(s-s_i)^2} & \frac{1}{(s-s_i)} & & \\ \frac{1}{(s-s_i)} & & & 0 \end{bmatrix} \tag{23}$$

the difference between these two cases is clearly reflected in the driving-point and transfer impedances given by Eq. 14.

If any of the driving-point or transfer impedances have resistive or inductive high-frequency behavior, then the degree of the characteristic equation is less than the order of M(s). As special cases, both purely resistive and purely inductive networks have zero-degree characteristic equations. The physical difference between resistive and inductive high-frequency behavior is reflected in the two different types of submatrices in the normal form,

$$\hat{M}_r = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \tag{24}$$

and

$$\hat{M}_\ell = \left[\begin{array}{cc:cc:c:cc} 0 & 1 & & & & & \\ 1 & s & & & & 0 & \\ \hdashline & & 0 & 1 & & & \\ & & 1 & s & & & \\ \hdashline & & & & \ddots & & \\ \hdashline & 0 & & & & 0 & 1 \\ & & & & & 1 & s \end{array}\right]. \tag{25}$$

The submatrices given by Eqs. 20, 21, 24, and 25 appear on the principal diagonal of the normal form, the remaining elements being zeros. Although a rigorous proof has not yet been obtained, it is believed that this represents a complete description of the normal form for the case of an RLC network without mutual inductance. Thus,

$$M_n(s) = \begin{bmatrix} \hat{M}_{d_1} & & & & & & & \\ & \ddots & & & & & & \\ & & \hat{M}_{d_k} & & & & & \\ & & & \hat{M}_{\mu_1} & & & & \\ & & & & \ddots & & & \\ & & & & & \hat{M}_{\mu_k} & & \\ & & & & & & \hat{M}_r & \\ & & & & & & & \hat{M}_\ell \end{bmatrix} \tag{26}$$

EXAMPLE 1

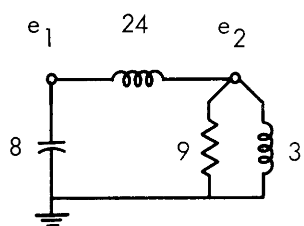


Fig. XXXI-1.

For the circuit shown in Fig. XXXI-1,

$$C = \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} 24 & -24 \\ -24 & 27 \end{bmatrix}$$

$$M(s) = \begin{bmatrix} 8s & 0 & 24 & -24 \\ 0 & 9 & -24 & 27 \\ 24 & -24 & -24s & 24s \\ -24 & 27 & 24s & -27s \end{bmatrix}$$

$$|M(s)| = 5184\ (s+1)^3$$

$$A = \begin{bmatrix} -\frac{1}{2}j & -\frac{1}{8}j & \frac{9}{64}j & 0 \\ -\frac{2}{3}j & -\frac{1}{2}j & -\frac{1}{16}j & \frac{1}{3} \\ \frac{1}{2}j & -\frac{3}{8}j & \frac{15}{64}j & 0 \\ \frac{2}{3}j & -\frac{1}{6}j & \frac{11}{48}j & 0 \end{bmatrix}$$

$$A_tMA = \begin{bmatrix} 0 & 0 & s+1 & 0 \\ 0 & s+1 & 1 & 0 \\ s+1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

EXAMPLE 2

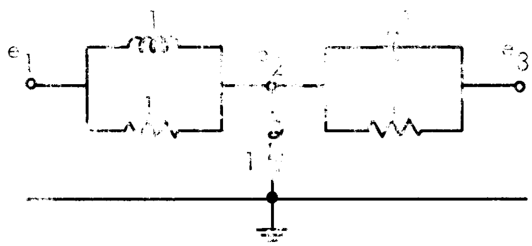


Fig. XXXI-2.

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M(s) = \begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ -1 & s+2 & -s-1 & -1 & 2 \\ 0 & -s-1 & s+1 & 0 & 0 \\ 1 & -1 & 0 & -s & s \\ -1 & 2 & 0 & s & -2s \end{bmatrix}$$

$$|M(s)| = (s+1)^2$$

$$\Lambda = \begin{bmatrix} 0 & j & 1 & j & 0 \\ 0 & 0 & 0 & j & 0 \\ 1 & 0 & 0 & j & 0 \\ 0 & -j & 0 & 0 & -j \\ 0 & 0 & 0 & 0 & -j \end{bmatrix}$$

$$A_tMA = \begin{bmatrix} s+1 & 0 & 0 & 0 & 0 \\ 0 & s+1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & s \end{bmatrix}.$$

These two examples illustrate the need for each of the submatrices $\hat{M}_d$, $\hat{M}_m$, $\hat{M}_r$, and $\hat{M}_\ell$. This need is associated with the partial fraction expansion

$$z_{ij} = \sum_\mu \sum_\nu a_{i\mu} a_{j\nu} z_{\mu\nu}^{(n)} \tag{14}$$

In Example 1,

$$z_{11} = \frac{s(s+3)}{8(s+1)^3} = \frac{-1}{4(s+1)^3} + \frac{1}{8(s+1)^2} + \frac{1}{8(s+1)}$$

$$z_{22} = \frac{s(s^2+3)}{9(s+1)^3} = \frac{-4}{9(s+1)^3} + \frac{2}{3(s+1)^2} - \frac{1}{3(s+1)} + \frac{1}{9}.$$

The third-order pole requires the presence of $\hat{M}_m$ in the normal form, while the resistive high-frequency behavior of $z_{22}$ accounts for $\hat{M}_r$. In Example 2,

$$z_{11} = \frac{s(s+2)}{s+1} = s + 1 - \frac{1}{s+1}$$

$$z_{22} = s = s$$

$$z_{33} = \frac{s^2+s+1}{s+1} = s + \frac{1}{s+1}$$

The multiple-order root in the characteristic equation is due to the two uncoupled circuits with the same natural frequency; thus $\hat{M}_d$ appears in the normal form. The constant term in the partial fraction expansion leads to $\hat{M}_r$, and the term proportional to frequency leads to $\hat{M}_\ell$.

While the normal form for RLC networks is more involved than the diagonal form in the two-element-kind case, these examples emphasize the need for a more complex normal form and illustrate its physical interpretation.

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References

1. E. A. Guillemin, The normal coordinate transformation of a linear system with an arbitrary loss function, J. Math. Phys. 39, 97-104.

2. C. A. Desoer, Modes in linear circuits, Trans. IRE, Vol. CT-7, pp. 211-225, 1960.

3. F. R. Gantmacher, Matrix Theory, Vol. I (Chelsea Publications, Inc., New York, 1959), p. 46.

4. F. R. Gantmacher, Matrix Theory, Vol. II, op. cit., Chapter XII.

## B. SYNTHESIS OF TWO-ELEMENT-KIND NETWORKS BY MEANS OF COORDINATE TRANSFORMATIONS

The ability to synthesize two-element-kind networks by means of coordinate transformations is greatly simplified by possession of the ability to diagonalize the nodal parameter matrices pertinent to such networks.[1] The existence of dynamical degeneracies in a network quite often leads to singularity of both the parameter matrices and renders them both positive semidefinite. Despite the statement that two real, symmetrical, singular matrices cannot, in general, be simultaneously diagonalized, we shall show that for two-element-kind electrical networks a simultaneous diagonalization can always be effected in a straightforward manner by means of a congruent, nonsingular, real, transformation.

### 1. Dynamical Degeneracies and Their Relation to Nodal Parameter Matrices

We shall concern ourselves with LC networks in which there are no tightly coupled mutual inductances. Furthermore, we shall assume an n independent node network consisting of a single part; that is, every node is connected to the network tree by at least one branch. As usual, we shall assume that each individual element constitutes a branch. Once certain results have been established for LC networks, extension to RC and RL networks will be trivial.

We recall that, on a nodal basis, the number of dynamically independent variables characterizing an LC network equals the number of topologically independent variables required to specify uniquely the equilibrium for the network, reduced by the number of independent constraints existing among these variables.

On a nodal basis, constraints among the dynamic variables are revealed through the presence of one-element-kind cut-sets in the network. Thus, the existence of a cut-set, all of whose branches are capacitors, leads to a constraint among the dynamic variables. Similarly, the presence of a pure inductive cut-set leads to a constraint. On a nodal basis, one-element-kind cut-sets furnish the only situations that give rise to dynamical constraints. Hence, if a network has n independent node pairs and contains

$\sigma$ independent, pure capacitive cut-sets and $\lambda$ independent, pure inductive cut-sets, the number of dynamically independent variables is exactly $(n-\sigma-\lambda)$.[2]

It is well known that the number of dynamically independent variables is just equal to the number of nonzero, noninfinite, natural frequencies displayed by the network. (Note that here the range of natural frequencies is defined from zero to plus infinity. For instance, a parallel LC tank circuit has one natural frequency.)

On physical grounds we may regard natural frequencies of a network as those frequencies at which a voltage may exist across branches in the network even though there is no current source exciting the network. If we consider from this point of view those branches in a network which comprise a pure capacitive cut-set, it is easy to see that when we place an appropriate charge on each of these capacitors, so that the voltages across these capacitors are all identical, no currents will flow and the charges will remain stationary. This can be seen to be true because at zero frequency the inductors appear as short circuits, and therefore all of the capacitors of a given cut-set appear in parallel with each other. Hence, if there are $\sigma$ independent, pure capacitive cut-sets in the network, there are evidently $\sigma$ different topological situations in which a zero frequency voltage can exist without any external current excitation. It seems reasonable, then, to assert that each independent capacitive cut-set gives rise to a natural frequency at zero.

If we should consider exciting the network across an inductive cut-set with a current source operating at as high a frequency as we please, it is clear that the capacitive branches essentially become short circuits and that the inductances associated with this inductive cut-set are all effectively in parallel. If we then increase the frequency and at the same time decrease the amplitude of the excitation in such a manner as to keep the voltage across these inductive branches essentially constant in amplitude, it is clear that in the limit, as the frequency approaches infinity, the amplitude of the excitation approaches zero, but the voltage remains nonzero and finite. Hence, in a limiting sense, we recognize that an inductive cut-set can be thought of as giving rise to a natural frequency at infinity. We see, just as before, that if there are $\lambda$ independent inductive cut-sets in the network, there are $\lambda$ different topological situations in which infinite natural frequencies arise. Thus it seems reasonable to assert that each independent inductive cut-set gives rise to a natural frequency at infinity.

If we agree to follow the line of reasoning presented above, we may state that a network containing n independent nodes, $\sigma$ independent, capacitive cut-sets and $\lambda$ independent, inductive cut-sets has $\sigma$ natural frequencies at zero, $\lambda$ natural frequencies at infinity, and $(n-\sigma-\lambda)$ nonzero, noninfinite natural frequencies. To confirm the correctness of such an interpretation, we point out that the natural frequencies of a network are the latent roots of the nodal admittance matrix of the network and appear as factors in the determinant of this matrix. If, as usual, we let s represent the complex frequency variable, and let C and $\Gamma$ represent the nodal capacitance and reciprocal inductance matrices, respectively, we may write Y, the nodal admittance matrix as

$$Y = sC + \frac{1}{s}\Gamma. \tag{1}$$

The properties of the determinant, $|Y|$, in which we are interested, may be found by inspection. $|Y|$ is the sum of all of the possible "tree values" associated to the network, where a "tree value" is the product of all of the branch admittances forming a particular tree. Since every tree connects to all of the nodes in the network, it is clear that every tree must contain at least one branch from every one of the $\sigma$ independent capacitive cut-sets, and at least one branch from every one of the $\lambda$ independent inductive cut-sets. Hence, every "tree value" contains the common factors $(s^{\sigma})$ and $(s^{-\lambda})$. If we denote by $t_j$ the various tree values, with the understanding that the common factors mentioned above have been factored out, we may write

$$|Y| = (s^{\sigma})\left(\frac{1}{s^{\lambda}}\right)\sum_{j=1}^{p} t_j \qquad p = \text{number of trees}. \tag{2}$$

We recognize that since there are n branches in every tree, a typical term such as $t_j$ must be the product of exactly $(n-\sigma-\lambda)$ admittances. We also recognize that $\left[\sum_{j=1}^{p} t_j\right]$, which is a ratio of polynomials in s, accounts for the nonzero, noninfinite natural frequency factors and may be written aside from a constant multiplier, as a product of factors of the form $\left(s+\frac{\gamma_j}{s}\right)$, where $\gamma_j$ are constants.

$$|Y| = k(s^{\sigma})\left(\frac{1}{s^{\lambda}}\right)\prod_{j=1}^{(n-\sigma-\lambda)}\left(s+\frac{\gamma_j}{s}\right). \tag{3}$$

Here, k = a constant.

Equation 3, provided that we do not make any cancellation of factors, shows that there are always exactly n factors in the determinantal expression. Since these factors represent the latent roots of the admittance matrix, Y, we should expect that if Y can be reduced to a diagonal canonic form by means of a nonsingular coordinate transformation, these same factors should appear as the diagonal entries in such a matrix. We shall show in the next section that, in fact, this is just what happens. However, first, we also wish to demonstrate how the presence of one-element-kind cut-sets relates to the rank of the individual C and $\Gamma$ matrices.

That which we wish to show is made evident by inspection when we use general cut-sets to define the nodal parameter matrices. Construction of the nodal parameter matrices in this general manner is quite similar to the construction on a node-to-datum basis, with which everyone is quite familiar. In the general cut-set formulation, coupling between one cut-set and another is brought about in two says. In forming a cut-set we can imagine that we pick up certain nodes in, say, our right hand and grasp all other nodes in our left hand. The voltage rise is from left hand to right hand. If we then pull our hands apart, certain branches are stretched; these branches form the cut-set. Hence, any two cut-sets may have some common "picked-up" nodes and some common branches. The latter may or may not connect to common "picked-up" nodes. The admittances of those common branches that do connect to such common "picked-up" nodes contribute to the mutual coupling with plus signs, while the admittances of the remaining common branches that do not connect to common nodes contribute with minus signs. As usual, the diagonal entries in the nodal admittance matrix represent the total admittance of the branches forming the cut-set, and the mutual coupling between cut-sets appears in the appropriate, symmetrical off-diagonal positions. One recognizes that the node-to-datum formulation is just a special case of the more general procedure.

Now, if we have an $n$ independent node network containing $\sigma$ independent capacitance cut-sets and $\lambda$ independent inductance cut-sets, there must also be exactly $(n-\sigma-\lambda)$ other independent cut-sets, which are comprised of both capacitive and inductive branches. Cut-sets containing both types of branches will be referred to as hybrid cut-sets. It is clear that the pure capacitive and pure inductive cut-sets are independent of each other and that their total number, $(\sigma+\lambda)$, must be less than or equal to $n$, the total number of independent cut-sets.

If we choose a set of $n$ independent cut-sets which includes all of the independent one-element-kind cut-sets, as well as $(n-\sigma-\lambda)$ hybrid cut-sets, we may number the independent capacitive cut-sets 1 through $\sigma$, the hybrid cut-sets $(\sigma+1)$ through $(n-\lambda)$, and the independent inductive cut-sets $(n-\lambda+1)$ through $n$. We are now in a position to write down the reciprocal inductance matrix, $\Gamma$, and the capacitance matrix, $C$, by inspection.

In the formation of $\Gamma$ the first $\sigma$ rows and columns must all be zeros, for the first $\sigma$ cut-sets are pure capacitance ones and hence involve no inductive branches. The remaining $(n-\sigma)$ rows and columns of $\Gamma$ pertain to hybrid and inductance cut-sets. Thus, the last $(n-\sigma)$ diagonal positions must be filled with positive quantities, while the off-diagonal entries in these last $(n-\sigma)$ rows and columns are filled in a symmetrical manner, representing the inductive coupling among the hybrid and inductive cut-sets.

In a similar manner, it is clear that the last $\lambda$ rows and columns of $C$ must be replete with zeros, for the last $\lambda$ cut-sets are pure inductive ones. The first $(n-\lambda)$ rows and columns of $C$ pertain to the capacitive and hybrid cut-sets and thus the first $(n-\lambda)$ diagonal positions are filled with positive quantities, while the off-diagonal terms represent the capacitive coupling between the capacitive and hybrid cut-sets.

Therefore we may write $C$ and $\Gamma$ in partitioned form as follows:

$$
\begin{aligned}
C &= \left[\begin{array}{c:c} C_{(n-\lambda)(n-\lambda)} & 0 \\ \hdashline 0 & 0_{\lambda\lambda} \end{array}\right] \\
\Gamma &= \left[\begin{array}{c:c} 0_{\sigma\sigma} & 0 \\ \hdashline 0 & \Gamma_{(n-\sigma)(n-\sigma)} \end{array}\right]
\end{aligned}
\tag{4}
$$

The cut-sets used to define the submatrix $C_{(n-\lambda)(n-\lambda)}$ are all independent ones and have nonzero branch capacitance values. Hence, $C_{(n-\lambda)(n-\lambda)}$ must be nonsingular and pertinent to a positive definite quadratic form. An analogous statement applies to the submatrix $\Gamma_{(n-\sigma)(n-\sigma)}$. Hence, it also is nonsingular and pertains to a positive definite quadratic form.

Another way of looking at this reveals that if all of the inductors in the independent inductive cut-sets were replaced by capacitors, then C would be modified only by the addition of terms in the partitioned spaces formerly occupied by zeros. This modified matrix is necessarily positive definite, since all nodes in the network would now be connected by capacitors. $C_{(n-\lambda)(n-\lambda)}$ would be a principal minor of this modified matrix, and hence it would be positive definite itself. Again, an analogous argument applies to the fact that $\Gamma_{(n-\sigma)(n-\sigma)}$ is also positive definite. Thus, we have the result that the rank of C is $(n-\lambda)$ and the rank of $\Gamma$ is $(n-\sigma)$.

A set of equilibrium equations defined on any independent group of cut-sets may be transformed to the set of equations defined on any other group of independent cut-sets by a real, nonsingular congruent transformation. Such a transformation does not change the rank of the nodal parameter matrices. Hence, we may draw the valid conclusion that for an n node network containing $\sigma$ independent capacitive cut-sets and $\lambda$ independent inductive cut-sets:

(a) The rank of the nodal capacitance matrix is determined by the number of independent inductive cut-sets, and is $(n-\lambda)$.

(b) The rank of the nodal reciprocal inductance matrix is determined by number of independent capacitive cut-sets, and is $(n-\sigma)$.

(c) The nodal parameter matrices may always be reduced simultaneously to the forms given in Eq. 4 by a real, nonsingular congruent transformation.

## 2. Reduction of the Nodal Parameter Matrices to Diagonal Form

Let us assume an n independent node LC network containing no tightly coupled mutual inductances. The network is assumed to contain $\sigma$ independent capacitive cut-sets and $\lambda$ independent inductive cut-sets. The nodal equilibrium equation based on an arbitrary set of topologically independent variables may be written in matrix notation as follows:

$$\left[sC + \frac{1}{s}\Gamma\right] V = I, \tag{5}$$

where V and I are column voltage and current matrices, respectively.

We have seen from the previous section that the presence of dynamical degeneracies, which implies and is implied by the existence of one-element-kind cut-sets, leads to the fact that both C and $\Gamma$ are singular. Quite often the statement is heard that in general two singular, real symmetric matrices cannot be simultaneously diagonalized. The objective here is to show that, for two-element-kind networks having no tightly coupled mutual inductances, a simultaneous reduction to diagonal form can always be carried out, irrespective of the dynamical degeneracies that exist in the network.

The first step in the reduction is to render C and $\Gamma$ simultaneously into the form of Eq. 4. As we have shown, this can always be accomplished by a real, nonsingular congruent transformation. Such a transformation really amounts to a redefinition of the voltage and current variables of expression (5). Denoting the transformation matrix by A, we have

$$\begin{aligned} C_1 = A^tCA &= \left[\begin{array}{c:c} C_{1_{(n-\lambda)(n-\lambda)}} & 0 \\ \hdashline 0 & 0_{\lambda\lambda} \end{array}\right] \\ \Gamma_1 = A^t\Gamma A &= \left[\begin{array}{c:c} 0_{\sigma\sigma} & 0 \\ \hdashline 0 & \Gamma_{1_{(n-\sigma)(n-\sigma)}} \end{array}\right] \end{aligned} \tag{6}$$

where $|A| \neq 0$. We note that the symmetrical submatrices $C_{1_{(n-\lambda)(n-\lambda)}}$ and $\Gamma_{1_{(n-\sigma)(n-\sigma)}}$ are both pertinent to positive definite quadratic forms and hence nonsingular.

It is convenient to repartition $C_1$ and $\Gamma_1$ in the following manner:

$$\begin{aligned} C_1 &= \left[\begin{array}{c:c:c} C_{1_{\sigma\sigma}} & C_{1_{\sigma(n-\sigma-\lambda)}} & 0 \\ \hdashline C^t_{1_{\sigma(n-\sigma-\lambda)}} & C_{1_{(n-\sigma-\lambda)(n-\sigma-\lambda)}} & 0 \\ \hdashline 0 & 0 & 0_{\lambda\lambda} \end{array}\right] \\ \Gamma_1 &= \left[\begin{array}{c:c:c} 0_{\sigma\sigma} & 0 & 0 \\ \hdashline 0 & \Gamma_{1_{(n-\sigma-\lambda)(n-\sigma-\lambda)}} & \Gamma_{1_{(n-\sigma-\lambda)\lambda}} \\ \hdashline 0 & \Gamma^t_{1_{(n-\sigma-\lambda)\lambda}} & \Gamma_{1_{\lambda\lambda}} \end{array}\right] \end{aligned} \tag{7}$$

Here, the superscript "t" denotes the transpose.

The total number of independent one-element-kind cut-sets cannot be greater than the number of independent nodes in the network. Hence,

$$(n-\sigma-\lambda) \geq 0 \qquad (n-\sigma) \geq \lambda \qquad (n-\lambda) \geq \sigma. \tag{8}$$

Therefore, it is clear that when the partitioning indicated in (7) is carried out,

$$C_{1_{\sigma\sigma}} \text{ and } C_{1_{(n-\sigma-\lambda)(n-\sigma-\lambda)}}, \quad \Gamma_{1_{\lambda\lambda}} \text{ and } \Gamma_{1_{(n-\sigma-\lambda)(n-\sigma-\lambda)}}$$

are all symmetrical positive definite matrices. This follows because these submatrices are principal minors of $C_{1_{(n-\lambda)(n-\lambda)}}$ and $\Gamma_{1_{(n-\sigma)(n-\sigma)}}$, respectively, which themselves are symmetrical positive definite matrices.

The immediate objective of the next step in the procedure is to reduce $C_1$ and $\Gamma_1$ simultaneously to diagonal partitioned forms by means of a congruent transformation by using a matrix Q.

$$Q = \begin{bmatrix} U_\sigma & Q_{\sigma(n-\sigma-\lambda)} & 0 \\ 0 & U_{(n-\sigma-\lambda)} & 0 \\ 0 & Q_{\lambda(n-\sigma-\lambda)} & U_\lambda \end{bmatrix} \tag{9}$$

where $U_\sigma$, $U_{(n-\sigma-\lambda)}$, and $U_\lambda$ are diagonal unit matrices.

$$Q_{\sigma(n-\sigma-\lambda)} = -C_{1_{\sigma\sigma}}^{-1} C_{1_{\sigma(n-\sigma-\lambda)}}$$

$$Q_{\lambda(n-\sigma-\lambda)} = -\Gamma_{1_{\lambda\lambda}}^{-1} \Gamma_{1_{(n-\sigma-\lambda)\lambda}}^{t}$$

$$|Q| = |U_\sigma|\,|U_{(n-\sigma-\lambda)}|\,|U_\lambda| = 1$$

The superscript "-1" denotes the inverse.

In particular, it will be noted that since $C_{1_{\sigma\sigma}}$ and $\Gamma_{1_{\lambda\lambda}}$ are nonsingular, we are guaranteed that the definitions given above will always be valid.

Performing the congruent transformation simultaneously on $C_1$ and $\Gamma_1$ yields $C_2$ and $\Gamma_2$ as follows:

$$C_2 = Q^t C_1 Q = \left[\begin{array}{c:c:c} C_{2_{\sigma\sigma}} & 0 & 0 \\ \hdashline 0 & C_{2_{(n-\sigma-\lambda)(n-\sigma-\lambda)}} & 0 \\ \hdashline 0 & 0 & 0_{\lambda\lambda} \end{array}\right]$$

$$\Gamma_2 = Q^t \Gamma_1 Q = \left[\begin{array}{c:c:c} 0_{\sigma\sigma} & 0 & 0 \\ \hdashline 0 & \Gamma_{2_{(n-\sigma-\lambda)(n-\sigma-\lambda)}} & 0 \\ \hdashline 0 & 0 & \Gamma_{2_{\lambda\lambda}} \end{array}\right] \tag{10}$$

where

$$C_{2_{\sigma\sigma}} = C_{1_{\sigma\sigma}}$$

$$C_{2_{(n-\sigma-\lambda)(n-\sigma-\lambda)}} = C_{1_{(n-\sigma-\lambda)(n-\sigma-\lambda)}} - \left[C^t_{1_{\sigma(n-\sigma-\lambda)}} C^{-1}_{1_{\sigma\sigma}} C_{1_{\sigma(n-\sigma-\lambda)}}\right]$$

$$\Gamma_{2_{\lambda\lambda}} = \Gamma_{1_{\lambda\lambda}}$$

$$\Gamma_{2_{(n-\sigma-\lambda)(n-\sigma-\lambda)}} = \Gamma_{1_{(n-\sigma-\lambda)(n-\sigma-\lambda)}} - \left[\Gamma_{1_{(n-\sigma-\lambda)\lambda}} \Gamma^{-1}_{1_{\lambda\lambda}} \Gamma^t_{1_{(n-\sigma-\lambda)\lambda}}\right].$$

The correctness of the results expressed in Eq. 10 may be verified by a direct expansion of the indicated matrix products.

In view of the fact that the congruent transformation performed in (10) cannot change the rank, nor can it change the positive semidefinite character of the resulting matrices $C_2$ and $\Gamma_2$, it is evident that $C_{2_{\sigma\sigma}}$, $C_{2_{(n-\sigma-\lambda)(n-\sigma-\lambda)}}$, $\Gamma_{2_{\lambda\lambda}}$, and $\Gamma_{2_{(n-\sigma-\lambda)(n-\sigma-\lambda)}}$ are all positive definite matrices and hence nonsingular.

The final step in the diagonalization process is now reduced to routine simplicity, for the form of (10) shows that we may treat each of the three corresponding diagonal pairs of submatrices in $C_2$ and $\Gamma_2$ separately. The ultimate diagonal form obtained depends on just how we choose to do this. Here we shall do this in a manner that lends simplicity to the interpretation of the final result.

We may form the nonsingular transformation matrix, P, as follows:

$$P = \left[\begin{array}{c:c:c} P_{\sigma\sigma} & 0 & 0 \\ \hdashline 0 & P_{(n-\sigma-\lambda)(n-\sigma-\lambda)} & 0 \\ \hdashline 0 & 0 & P_{\lambda\lambda} \end{array}\right] \tag{11}$$

Since $C_{2_{\sigma\sigma}}$ and $\Gamma_{2_{\lambda\lambda}}$ are both positive definite, $P_{\sigma\sigma}$ and $P_{\lambda\lambda}$ may be formed by routine methods so that

$$P^t_{\sigma\sigma} C_{2_{\sigma\sigma}} P_{\sigma\sigma} = U_\sigma \qquad |P_{\sigma\sigma}| \neq 0$$

and

$$P^t_{\lambda\lambda} \Gamma_{2_{\lambda\lambda}} P_{\lambda\lambda} = U_\lambda \qquad |P_{\lambda\lambda}| \neq 0.$$

Since the center diagonal terms in $C_2$ and $\Gamma_2$ are real, symmetrical and positive definite, it is always possible by standard techniques[3] to construct a nonsingular real matrix $P_{(n-\sigma-\lambda)(n-\sigma-\lambda)}$ that is such that

$$P^t_{(n-\sigma-\lambda)(n-\sigma-\lambda)} C_{2_{(n-\sigma-\lambda)(n-\sigma-\lambda)}} P_{(n-\sigma-\lambda)(n-\sigma-\lambda)} = U_{(n-\sigma-\lambda)}$$

$$P^t_{(n-\sigma-\lambda)(n-\sigma-\lambda)} \Gamma_{2_{(n-\sigma-\lambda)(n-\sigma-\lambda)}} P_{(n-\sigma-\lambda)(n-\sigma-\lambda)} = \begin{bmatrix} \gamma_{(\sigma+1)} & & \\ & \ddots & \\ & & \gamma_{(n-\lambda)} \end{bmatrix} \tag{12}$$

where the terms $\gamma_{(\sigma+1)}$, $\gamma_{(\sigma+2)}$, and so forth are real nonzero, positive quantities.

Performing the final transformation yields $C_3$ and $\Gamma_3$ in the following manner:

$$\begin{gathered} C_3 = P^tC_2P = \left[\begin{array}{c:c:c} U_\sigma & 0 & 0 \\ \hdashline 0 & U_{(n-\sigma-\lambda)} & 0 \\ \hdashline 0 & 0 & 0_{\lambda\lambda} \end{array}\right] \\ \Gamma_3 = P^t\Gamma_2P = \left[\begin{array}{c:c:c} 0_{\sigma\sigma} & 0 & 0 \\ \hdashline 0 & \begin{matrix} \gamma_{(\sigma+1)} & & \\ & \ddots & \\ & & \gamma_{(n-\lambda)} \end{matrix} & 0 \\ \hdashline 0 & 0 & U_\sigma \end{array}\right] \end{gathered} \tag{13}$$

where

$$|P| = |P_{\sigma\sigma}|\,|P_{(n-\sigma-\lambda)(n-\sigma-\lambda)}|\,|P_{\lambda\lambda}| \neq 0.$$

All of the congruent transformations performed have been real, nonsingular ones. We may cascade these operations into a single nonsingular matrix denoted by M.

$$M = AQP, \quad M^t = P^tQ^tA^t, \quad |M| \neq 0. \tag{14}$$

From Eq. 5, we have

$$YV = \left[sC + \frac{1}{s}\Gamma\right] V = I$$

and by inversion we find the open-circuit impedance matrix Z.

$$Y^{-1}I = \left[sC + \frac{1}{s}\Gamma\right]^{-1} I = V \tag{15}$$

$$Z = Y^{-1} = \left[sC + \frac{1}{s}\Gamma\right]^{-1}$$

Applying the transformation M congruently to Y yields a diagonal matrix $Y_d$.

$$Y_d = M^tYM = M^t\left[sC + \frac{1}{s}\Gamma\right] M = s[M^tCM] + \frac{1}{s}[M^t\Gamma M]$$

$$Y_d = s\left[\begin{array}{c:c:c} U_\sigma & 0 & 0 \\ \hdashline 0 & U_{(n-\sigma-\lambda)} & 0 \\ \hdashline 0 & 0 & 0_{\lambda\lambda} \end{array}\right] + \frac{1}{s}\left[\begin{array}{c:c:c} 0_{\sigma\sigma} & 0 & 0 \\ \hdashline 0 & \begin{matrix} \gamma_{(\sigma+1)} & & \\ & \ddots & \\ & & \gamma_{(n-\lambda)} \end{matrix} & 0 \\ \hdashline 0 & 0 & U_\lambda \end{array}\right]$$

$$Y_d = \left[\begin{array}{c:c:c} sU_\sigma & 0 & 0 \\ \hdashline 0 & \begin{matrix} \left(s + \frac{\gamma_{(\sigma+1)}}{s}\right) & & \\ & \ddots & \\ & & \left(s + \frac{\gamma_{(n-\lambda)}}{s}\right) \end{matrix} & 0 \\ \hdashline 0 & 0 & \frac{1}{s}U_\lambda \end{array}\right] \tag{16}$$

Since

$$Y_d = M^tYM \quad ; \qquad Y^{-1} = MY_d^{-1}M^t,$$

we find that

$$Z = Y^{-1} = M \left[\begin{array}{c:ccc:c} \frac{1}{s}U_\sigma & & 0 & & 0 \\ \hdashline 0 & \left(\dfrac{1}{s + \dfrac{\gamma_{(\sigma+1)}}{s}}\right) & & & 0 \\ & & \ddots & & \\ & & & \left(\dfrac{1}{s + \dfrac{\gamma_{(n-\lambda)}}{s}}\right) & \\ \hdashline 0 & & 0 & & sU_\lambda \end{array}\right] M^t \tag{17}$$

Equation 16 bears out the earlier contention that even though the network may exhibit dynamical degeneracies, which cause both parameter matrices to be singular, a diagonalization may always be effected by means of a real, nonsingular transformation. Also, comparison of Eqs. 3 and 16 reveals that all of the factors in Eq. 3 are diagonal entries in $Y_d$ as we should expect, since the determinant of $Y_d$ is just equal to some nonzero constant times the determinant of Y. In a situation in which both parameter matrices are nonsingular it is correct to regard the frequency (that is, the value of s) at which a particular diagonal entry in $Y_d$ is zero as being a natural frequency of the system. It seems reasonable to extend this concept to the general situation and regard zero and infinity as legitimate natural frequencies also.

Going a step further, we may denote the entries in M, which are all real quantities, by $m_{jk}$ and write

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & & & \vdots \\ \vdots & & & \vdots \\ m_{n1} & \cdots & \cdots & m_{nn} \end{bmatrix} \tag{18}$$

From the form of Eq. 17 it can be seen that the open-circuit impedance matrix, Z, has the form of a Gramian. Hence, if we make the identification

$$m_{jq}m_{kq} = k_{jk}^{(q)}, \tag{19}$$

a typical term in Z, $z_{jk}$, can be written as follows:

$$z_{jk}(s) = \sum{}' + \sum_{q=(\sigma+1)}^{(n-\lambda)}{}' \frac{k_{jk}^{(q)}}{\left(s + \frac{q}{s}\right)} + \sum_{q=(n-\lambda+1)}^{n} sk_{jk}^{(q)}. \tag{20}$$

Expression 20 is recognized as a partial fraction expansion of the impedance $z_{jk}$. In particular, we note that the diagonal terms of $Y_d$ are the pole factors of the expression (20) and that the terms $k_{jk}^{(q)}$ are the coefficients associated to these factors in the partial fraction expansion.

One should not necessarily conclude that the $(n-\sigma-\lambda)$ nonzero, noninfinite natural frequencies, which are placed in evidence by the diagonal entries in $Y_d$, are numerically distinct. On a physical basis, it is quite clear that certain topological situations, combined with a rather special assignment of element values, can lead to the existence of several numerically equal natural frequencies in a network. For example, the network shown in Fig. XXXI-3 is a very simple example of such a situation. It is evident

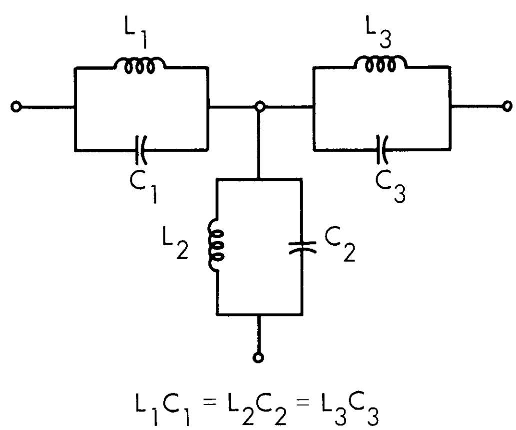


Fig. XXXI-3.

that each of the tank circuits can be excited separately by an appropriately placed current source. The fact that the three resonant frequencies coincide has nothing to do with the ability to excite these frequencies separately. The normal coordinates describing such a system are just the rows in the matrix $Y_d$; it is clear that the rows of $Y_d$ are independent even though the diagonal entries in $Y_d$ may be numerically equal. In general networks such situations arise in much more devious ways, and lead to the existence of several numerically equal roots. The reader should not confuse repeated roots with the idea of multiple-order poles. Reference to Eq. 20 shows directly that, even though a particular term may be repeated many times in $Y_d$, the expression for the open-circuit impedances can never have a multiple-order pole.

Extension of these results to RC and RL networks is now perfectly straightforward.

In an RC network the nodal parameter matrices are C and G, representing capacitance and conductance. Thus

$$Y V = [sC+G] \, V = I. \tag{21}$$

If we follow the same line of reasoning that we used in the discussion of LC networks, it is clear that independent capacitive cut-sets lead to natural frequencies at zero and account for the singularity of the G matrix. It is also clear that independent conductance cut-sets lead to singularity of the C matrix. However, these conductance cut-sets certainly do not lead to infinite natural frequencies. Rather, they are responsible for the appearance of constants on the main diagonal in the diagonalized form of Y. In turn, these constants lead to constants in the partial fraction expansions of the open-circuit impedances.

If we assume an n independent node RC network containing $\sigma$ independent capacitance cut-sets and $\rho$ independent conductance cut-sets, the rank of C is $(n-\rho)$ and the rank of G is $(n-\sigma)$. The network will have $(n-\sigma-\rho)$ nonzero, noninfinite natural frequencies. C and G may simultaneously be reduced to diagonal form by a real, nonsingular congruent transformation M in a manner exactly analogous to the LC case. Thus, for the RC case, we have

$$Y_d = M^tYM = M^t[sC+G]\,M = s[M^tCM] + [M^tGM]$$

$$Y_d = s\left[\begin{array}{c:c:c} U_\sigma & 0 & 0 \\ \hdashline 0 & U_{(n-\sigma-\rho)} & 0 \\ \hdashline 0 & 0 & 0_{\rho\rho} \end{array}\right] + \left[\begin{array}{c:c:c} 0_{\sigma\sigma} & 0 & 0 \\ \hdashline 0 & \begin{matrix} g_{(\sigma+1)} & & \\ & \ddots & \\ & & g_{(n-\rho)} \end{matrix} & 0 \\ \hdashline 0 & 0 & U_\rho \end{array}\right] \tag{22}$$

where $g_{(\sigma+1)}$, $g_{(\sigma+2)}$, and so on, are real positive, nonzero quantities.

$$Y_d = \left[\begin{array}{c:c:c} sU_\sigma & 0 & 0 \\ \hdashline 0 & \begin{matrix} (s+g_{(\sigma+1)}) & & \\ & \ddots & \\ & & (s+g_{(n-\rho)}) \end{matrix} & 0 \\ \hdashline 0 & 0 & U_\rho \end{array}\right] \tag{23}$$

$$Z = Y^{-1} = MY_d^{-1}M^t = M \left[\begin{array}{c:ccc:c} \frac{1}{s}U_\sigma & & 0 & & 0 \\ \hdashline 0 & \left(\frac{1}{s+g_{(\sigma+1)}}\right) & & & 0 \\ & & \ddots & & \\ & & & \left(\frac{1}{s+g_{(n-\rho)}}\right) & \\ \hdashline 0 & & 0 & & U_\rho \end{array}\right] M^t \tag{24}$$

Using the same types of definitions as those in Eqs. 18 and 19, gives a typical term, $z_{jk}$, in the open-circuit impedance matrix:

$$z_{jk}(s) = \sum_{q=1}^{\sigma} \frac{k_{jk}^{(q)}}{s} + \sum_{q=(\sigma+1)}^{(n-\rho)} \frac{k_{jk}^{(q)}}{(s+g_q)} + \sum_{q=(n-\rho+1)}^{n} \frac{k_{jk}^{(q)}}{1}. \tag{25}$$

The summation on the extreme right of (25) shows the existence of the predicted constant terms in the partial fraction expansion.

In the RL situation the nodal parameter matrices are G and $\Gamma$, representing conductance and reciprocal inductance. We have

$$YV = \left[G + \frac{1}{s}\Gamma\right] V = I.$$

Independent inductive cut-sets lead to natural frequencies at infinity and account for the singularity of the G matrix. Independent conductance cut-sets lead to the appearance of constants on the main diagonal in the diagonalized form of Y and also account for the singularity of the $\Gamma$ matrix. These constants will be seen to lead, once again, to constants in the partial fraction expansions of the open-circuit impedances.

Assuming an n independent node network containing $\rho$ independent conductance cut-sets and $\lambda$ independent inductive cut-sets, we have the rank of G equal to $(n-\lambda)$ and the rank of $\Gamma$ equal to $(n-\rho)$. There will be $(n-\rho-\lambda)$ nonzero, noninfinite natural frequencies. Reducing to diagonal form with a nonsingular, real transformation M, we have, in the RL case,

$$Y_d = M^tYM = M^t\left[G + \frac{1}{s}\Gamma\right] M = [M^tGM] + \frac{1}{s}[M^t\Gamma M]$$

$$Y_d = \left[\begin{array}{c:c:c} 0_{\lambda\lambda} & 0 & 0 \\ \hdashline 0 & U_{(n-\rho-\lambda)} & 0 \\ \hdashline 0 & 0 & U_\rho \end{array}\right] + \frac{1}{s}\left[\begin{array}{c:ccc:c} U_\rho & & 0 & & 0 \\ \hdashline 0 & \gamma_{(\lambda+1)} & & & 0 \\ & & \ddots & & \\ & & & \gamma_{(n-\rho)} & \\ \hdashline 0 & & 0 & & 0_{\rho\rho} \end{array}\right] \tag{26}$$

where $\gamma_{(\lambda+1)}$, $\gamma_{(\lambda+2)}$, etc., are real, nonzero, positive quantities.

$$Y_d = \left[\begin{array}{c:ccc:c} \frac{1}{s}U_\lambda & & 0 & & 0 \\ \hdashline 0 & \left(1+\frac{\gamma_{(\lambda+1)}}{s}\right) & & & 0 \\ & & \ddots & & \\ & & & \left(1+\frac{\gamma_{(n-\rho)}}{s}\right) & \\ \hdashline 0 & & 0 & & U_\rho \end{array}\right] \tag{27}$$

$$Z = Y^{-1} = MY_d^{-1}M^t = M\left[\begin{array}{c:ccc:c} sU_\lambda & & 0 & & 0 \\ \hdashline 0 & \left(\frac{1}{1+\frac{\gamma_{(\lambda+1)}}{s}}\right) & & & 0 \\ & & \ddots & & \\ & & & \left(\frac{1}{1+\frac{\gamma_{(n-\rho)}}{s}}\right) & \\ \hdashline 0 & & 0 & & U_\rho \end{array}\right] \tag{28}$$

Once again, following the definitions outlined in (18) and (19), we may write a typical open-circuit impedance, $z_{jk}$, as

$$z_{jk}(s) = \sum_{q=1}^{\lambda} s\,k_{jk}^{(q)} + \sum_{q=(\lambda+1)}^{(n-\rho)} \frac{k_{jk}^{(q)}}{\left(1+\frac{\gamma_q}{s}\right)} + \sum_{q=(n-\rho+1)}^{n} \frac{k_{jk}^{(q)}}{1}. \tag{29}$$

The ultimate objective in all of this work is the development of techniques for the synthesis of two-element-kind networks by means of normal coordinate transformations. Such techniques revolve around the ability to effect a diagonalization of the parameter matrices. The difficulty formally encountered when both matrices were singular has been resolved.

W. C. Schwab

References

1. E. A. Guillemin, The normal coordinate transformation of a linear system with an arbitrary loss function, Quarterly Progress Report No. 60, Research Laboratory of Electronics, M.I.T., January 15, 1960, pp. 235-245.

2. E. A. Guillemin, Synthesis of Passive Networks (John Wiley and Sons, Inc., New York, 1957), pp. 157-176.

3. F. B. Hildebrand, Methods of Applied Mathematics (Prentice-Hall, Inc., Englewood Cliffs, N.J., 1952), pp. 74-80.

## C. CANONIC REALIZATIONS OF RC DRIVING-POINT ADMITTANCE. PART II

A particular structure has been found to play a key role in canonic two-element-kind networks. This structure is called the "complementary tree structure." Its definition runs as follows: When the resistors of an n+1 node RC structure form a tree, and the same is true of the capacitors, then the structure is said to be a complementary tree structure of order n.

It has been shown[1] that a necessary condition for an RC network to canonically realize an admittance of the form

$$Y(s) = \frac{a_n s^n + a_{n-1} s^{n-1} \dots + a_1 s}{s^n + b_{n-1} s^{n-1} \dots + b_1 s + b_o} \qquad (1)$$

is that the network become a complementary tree structure of order n when the driving terminals are shorted.

Many people who have done research in network theory believe that natural frequencies provide a criterion for the equivalence of two-element driving-point impedances. That is, it is widely believed that if a two-element-kind network with one pair of terminals has the same number of open- and short-circuit natural frequencies as a given impedance, and moreover exhibits the same zero and infinite frequency behavior as the impedance, then the impedance can be realized by properly selecting the element values on the network.

Guided by this belief, the writer expected that any RC network, generated by a pliers type of entry into a connected complementary tree structure of order n, would realize any admittance of the form (1). Had this been the case, such networks would have provided new canonic forms equally as good as those of Foster and Cauer.

Considerable effort was devoted to proving this conjecture, but no proof resulted. In the course of working out particular examples, counterexamples began to appear.

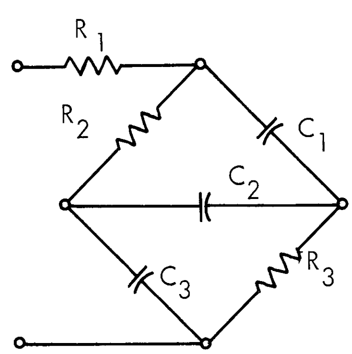


Fig. XXXI-4.

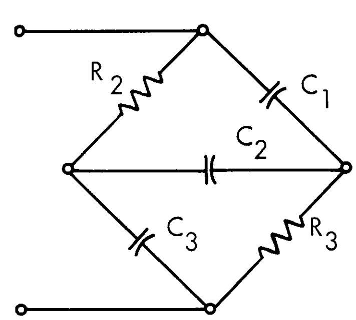


Fig. XXXI-5.

The simplest counterexample was provided by the network of Fig. XXXI-4. According to the natural-frequency line of reasoning, this network should be able to realize any RC admittance of the form

$$Y(s) = \frac{s^3 + a_2 s^2 + a_1 s}{s^3 + b_2 s^2 + b_1 s + b_o} = \frac{1}{1 + \dfrac{b_2' s^2 + b_1' s + b_o}{s^3 + a_2 s^2 + a_1 s}}. \tag{2}$$

For this to be the case, the right-hand member of (2) shows that $R_1$ must equal 1 ohm, and the network of Fig. XXXI-5 must be capable of realizing any RC admittance of the form

$$Y'(s) = \frac{s^3 + a_2 s^2 + a_1 s}{b_2' s^2 + b_1' s + b_o'}. \tag{3}$$

It is shown in the appendix that the network of Fig. XXXI-5 cannot realize all admittances of the form (3), however. Thus, the network of Fig. XXXI-4 does not realize all admittances of the form (2).

The network of Fig. XXXI-4 fails to do all of the things predicted by natural-frequency reasoning. The writer has found it convenient to call such networks "bad" networks." Similarly, it has been found convenient to call networks that live up to natural-frequency predictions "good networks."

The purpose of this report is to show that a complementary tree structure must have a definite series-parallel flavor if the pliers type of entry into it is to produce "good" networks. Before plunging into details it is necessary to agree on notation.

Single resistors and capacitors will be denoted by small r's and c's. Each resistor r of a complementary tree structure is a link of a capacitor tree, and as such possesses a single capacity path between its terminals; a capital C will be used to indicate both this path, and the value of its terminal capacity. Similarly, each capacitance of a complementary tree structure possesses a single resistive path between its terminals provided by the resistive tree; a capital R will be used to indicate both this path, and the value of its terminal resistance.

We begin by proving two preliminary theorems.

THEOREM I. If $s_{max}$ and $s_{min}$ are, magnitudewise, the largest and smallest natural frequencies of a complementary tree structure, and r is a resistor of the structure, having associated a capacitive path C, then

$$|s_{max}| > \frac{1}{rC} > |s_{min}|.$$

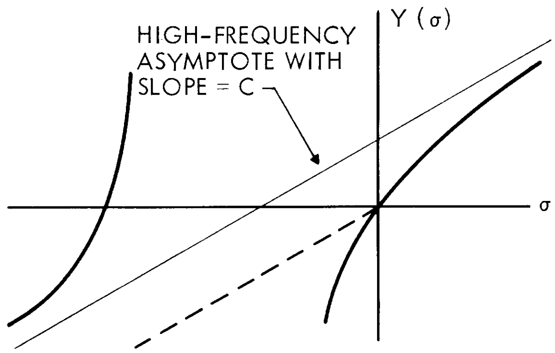


Fig. XXXI-6.

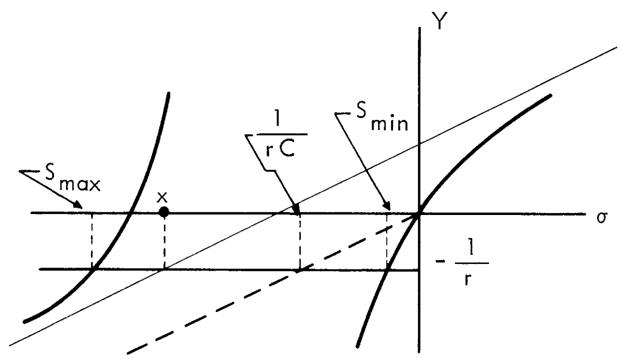


Fig. XXXI-7.

PROOF. Let Y denote the admittance into which r looks. The $\sigma$-axis plot of Y has the behavior shown in Fig. XXXI-6. The zero at s = 0 is due to the fact that r cannot look into a resistive path (if r did look into a resistive path, the over-all structure would contain a resistive tie set, and thus would contradict the hypothesis that the resistors form a tree). The pole at $s = \infty$ is due to the capacitive path C, and for this reason the high-frequency asymptote has slope equal to C. The dotted line in Fig. XXXI-6 is the parallel to the high-frequency asymptote which passes through the origin. The natural frequencies of the structure are given by the intersection between the Y vs $\sigma$ curve and the line $Y = -\frac{1}{r}$, as shown in Fig. XXXI-7. From the familiar properties of the Y vs $\sigma$ curve, it is clear that

$$|s_{max}| > x > \frac{1}{rC} > |s_{min}|$$

so that

$$|s_{max}| > \frac{1}{rC} > |s_{min}| \qquad \text{Q. E. D.}$$

THEOREM II. This is the complement of Theorem I. It may be stated: If, magnitudewise, $s_{max}$ and $s_{min}$ are the maximum and minimum natural frequencies of a complementary tree structure, and c is a capacitor of the structure, having associated a resistive path R, then

$$|s_{max}| > \frac{1}{Rc} > |s_{min}|.$$

The proof of this theorem is omitted since it is merely the complement of the proof to Theorem I.

In the rest of this report we shall assume that all RC networks that are considered have been frequency-scaled so that $|s_{min}|$ becomes equal to 1 radian per second. If the ratio $|s_{max}/s_{min}|$ for an unscaled network is denoted by $\rho$, Theorems I and II for the corresponding scaled network read

$$\rho > \frac{1}{rC} > 1 \tag{4}$$

and

$$\rho > \frac{1}{Rc} > 1, \tag{5}$$

where r, R, c, and C refer to the element values on the scaled network. And, in particular, from (4),

$$r > \frac{1}{\rho C} \tag{6}$$

and from (5),

$$\frac{1}{c} > R. \tag{7}$$

The key theorem follows.

THEOREM III. Let N be a frequency-scaled complementary tree structure possessing n capacitors and n resistors that have fixed positive values. If all tie sets of N contain at least three branches, then $\rho > 4$. (This theorem implies that $\left|\frac{s_{max}}{s_{min}}\right| > 4$ for the corresponding unscaled network.)

PROOF. Select any resistor from N and let this be denoted $r^1$. Let $C^1$ be the capacity path associated with $r^1$ and assume that $C^1$ contains $n_{c1}$ branches. Since, by hypothesis, the tie set consisting of $r^1$ and $C^1$ contains at least three branches, it follows that $n_{c1} \geqslant 2$.

From (6), it is clear that

$$r^1 > \frac{1}{\rho C^1}. \tag{8}$$

If $c^1$ denotes the largest capacity in $C^1$, then $c^1$ and $C^1$ satisfy the inequality

$$\frac{1}{n_{c1}C^1} \geqslant \frac{1}{c^1}$$

or

$$\frac{1}{C^1} \geqslant \frac{n_{c1}}{c^1} \tag{9}$$

since the minimum elastance in $C^1$ cannot exceed the average elastance in $C^1$. Substitution of (9) in (8) gives

$$r^1 > \frac{n_{c1}}{\rho c^1}. \tag{10}$$

Consider next the tie set formed by $c^1$ and its associated resistive path $R^1$. Since this tie set must contain at least three branches, the number of resistors in $R^1$, $n_{R1}$, must satisfy $n_{R1} \geqslant 2$. Also, from (7),

$$\frac{1}{c^1} > R^1. \tag{11}$$

If $r^2$ denotes the smallest resistor in $R^1$, then

$$\frac{R^1}{n_{R1}} \geqslant r^2$$

or

$$R^1 \geqslant n_{R1} r^2 \tag{12}$$

since the smallest resistance in $R^1$ does not exceed the average resistance in $R^1$. Substitution of (12) in (11) gives

$$\frac{1}{c^1} > n_{R1} r^2$$

or

$$\frac{n_{c1}}{\rho c^1} > \frac{n_{c1} n_{R1}}{\rho} r^2. \tag{13}$$

Let us review the cycle that has just been carried out. A resistor $r^1$ was selected from the network at random. The choice of $r^1$ singled out a capacitor $c^1$ (the largest capacitor in $C^1$). The capacitor $c^1$, in turn, singled out a resistor $r^2$ (the smallest resistor in $R^1$). It has been shown that the following relations among $r^1$, $n_{c1}$, $c^1$, $n_{R1}$ hold

$$r^1 > \frac{n_{c1}}{\rho c^1}$$

and

$$\frac{n_{c1}}{\rho c^1} > \frac{n_{c1} n_{R1}}{\rho} r .$$

We can now repeat this cycle, starting with $r^2$. When this is done, $r^2$ singles out a capacitor $c^2$, which, in turn, singles out a resistor $r^3$. The quantities $r^2$, $n_{c2}$, $c^2$, $n_{Rc}$, and $r^3$ will satisfy the relations

$$r^2 > \frac{n_{c2}}{\rho c^2}$$

$$\frac{n_{c2}}{\rho c^2} > \frac{n_{c2} n_{R2}}{\rho} r^3 .$$

It might happen that some of the physical elements denoted by the symbols $r^2$, $r^3$, and $c^2$ are also denoted by the symbols $r^1$ and $c^1$; however, this need not be the case.

If this cycle is repeated until it has been carried out n times, one obtains the following set of inequalities

$$\text{cycle 1} \begin{cases} r^1 > \dfrac{n_{c1}}{\rho c^1} & \text{(1a)} \\[2ex] \dfrac{n_{c1}}{\rho c^1} > \dfrac{n_{c1} n_{R1}}{\rho} r^2 & \text{(1b)} \end{cases}$$

$$\text{cycle 2} \begin{cases} r^2 > \dfrac{n_{c2}}{\rho c^2} & \\[2ex] \dfrac{n_{c2}}{\rho c^2} > \dfrac{n_{c2} n_{R2}}{\rho} r^3 & \text{(2b)} \end{cases}$$

$$\vdots$$

$$\text{cycle n} \begin{cases} r^n > \dfrac{n_{cn}}{\rho c^n} & \text{(na)} \\[2ex] \dfrac{n_{cn}}{\rho c^n} > \dfrac{n_{cn} n_{Rn}}{\rho} r^{n+1} & \text{(nb)} \end{cases}$$

Because there are only n different resistors in the network, whereas reference is made to these by n+1 distinct $r^i$ in the set of inequalities above, reference must be made to the same physical resistor by at least two distinct $r^i$ in the inequalities. If it happens that $r^1$ and $r^2$ refer to the same physical resistor, substitution of (1b) in (1a) shows that

$$r^1 > \frac{n_{c1}n_{R1}}{\rho} r^2$$

or

$$1 > \frac{n_{c1}n_{R1}}{\rho}$$

from which it follows that

$$\rho > n_{c1}n_{R1} \geqslant 4.$$

If, on the other hand, $r^1$ and $r^j$ refer to the same physical element, then the substitution

$$(j-1, b) \rightarrow (j-1, a) \rightarrow (j-2, b) \rightarrow (j-2, a) \ldots (1b) \rightarrow (1a)$$

gives

$$r^1 > \frac{n_{c1}n_{R1}}{\rho} \frac{n_{c2}n_{R2}}{\rho} \cdots \frac{n_{c,j-1}n_{R,j-1}}{\rho} r^j$$

or

$$1 > \frac{n_{c1}n_{R1}}{\rho} \frac{n_{c2}n_{R2}}{\rho} \cdots \frac{n_{c,j-1}n_{R,j-1}}{\rho}$$

which implies

$$\rho \quad [(n_{c1}n_{R1}) \cdots (n_{c,j-1}n_{R,j-1})]^{\frac{1}{j-1}} \geqslant 4. \tag{14}$$

If no other $r^i$ denotes the same physical resistor as $r^1$, but rather $r^a$ and $r^b$ denote the same resistor, where $a < b$, then the set of n cycles may be repeated by taking $r^a$ as the initial resistor instead of $r^1$. After this has been done, inequality (14) again applies. Thus, the inequality $\rho > 4$ holds in any case. Q. E. D.

Note: The point may be raised that the proof of Theorem III does not carry through at face value if several elements on a capacitive or resistive path have numerically equal values. A review of the proof, however, will show that it still holds if an arbitrary choice is made between numerically equal elements that qualify as being the smallest elements on a resistive or elastive path.

Theorem III shows that any particular selection of positive real numbers for the element values on a complementary tree structure of the type described leads to the result

$$\left| \frac{s_{max}}{s_{min}} \right| > 4.$$

Thus, it is impossible to find a set of positive real values for the elements of such a network, so as to obtain

$$\left|\frac{s_{max}}{s_{min}}\right| \leq 4.$$

An important consequence of Theorem III is the following theorem.

THEOREM IV. If the pliers type of entries to a complementary tree structure N are to produce "good" networks, then N must contain the tie set of Fig. XXXI-8.

PROOF. To be good, N must be able to realize all distinct sets of n natural frequencies, which lie on the negative $\sigma$-axis of the s-plane. According to Theorem III, N must have at least one two-branch tie set for this to be possible. Since two-branch tie sets consisting of two resistors or two capacitors are prohibited, the contained tie set must have the form shown in Fig. XXXI-8.

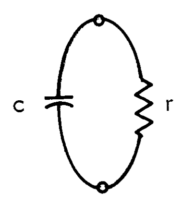


Fig. XXXI-8.

The complementary tree structures for which the pliers type of entries lead to "good" networks, will now be pinned down further. To accomplish this, we assume that

(a) N is a complementary tree structure containing a tie set of the form of Fig. XXXI-8, and

(b) N' is the network obtained by shorting out the tie set (Fig. XXXI-8), and prove the following theorems concerning N and N'.

THEOREM V. N' is a complementary tree structure.

PROOF. The proof of this theorem consists of noticing that

(a) The r's and c's and N each form a tree, and

(b) When a branch of a tree t on a network is shorted, the remaining branches of t form a tree on the simpler network that results.

THEOREM VI. The ratio $|s_{max}/s_{min}|$ is greater for N than it is for N'.

PROOF. Let Y denote the admittance into which the tie set (Fig. XXXI-8) looks. The $\sigma$-axis behavior of Y will be as shown in Fig. XXXI-9. The zero at s = 0 is due to the fact that the tie set (Fig. XXXI-8) cannot look into a resistive path (the tie set cannot look into a resistive path because if it did, then the path and r would form a resistive loop which is not allowed). The constant behavior at s = $\infty$ is due to the fact that the tie set (Fig. XXXI-8) does not look into a capacitive path. The extreme finite

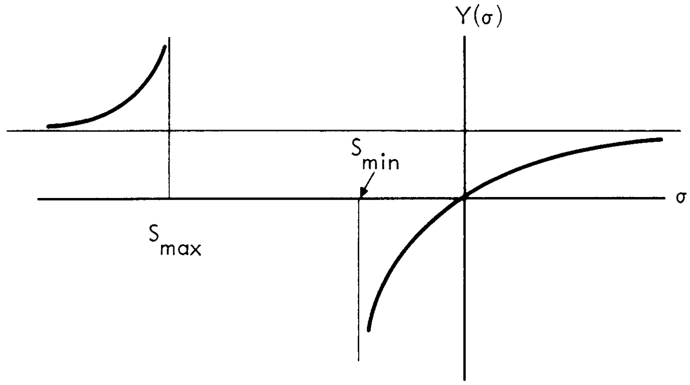


Fig. XXXI-9.

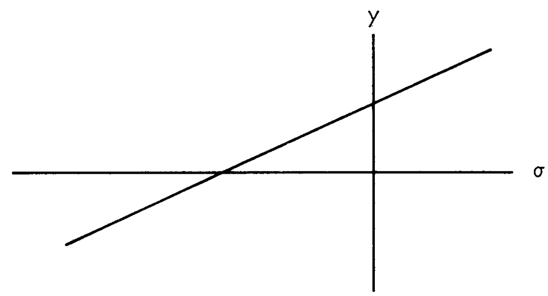


Fig. XXXI-10.

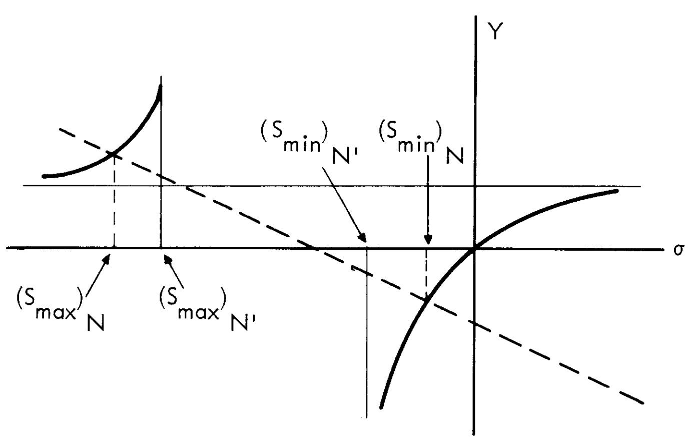


Fig. XXXI-11.

poles of Y correspond to $s_{max}$ and $s_{min}$ for N'. The admittance y of the tie set (13) will have the $\sigma$-axis behavior shown in Fig. XXXI-10. The natural frequencies of N will occur when Y = -y as shown in Fig. XXXI-11. Clearly,

$$\left|s_{max}\right|_N > \left|s_{max}\right|_{N'}$$

and

$$\left|s_{min}\right|_N < \left|s_{min}\right|_{N'}$$

so that

$$\left|\frac{s_{max}}{s_{min}}\right|_N > \left|\frac{s_{max}}{s_{min}}\right|_{N'}$$

Q. E. D.

THEOREM VII. A necessary condition for N to realize all sets of n distinct negative real natural frequencies is that the topology of N' must impose no lower bound, in excess of 1, on the ratio

$$\left|\frac{s_{max}}{s_{min}}\right|_{N'}.$$

PROOF. If the topology of N' did impose a lower bound $\gamma$, in excess of 1, upon

$$\left|\frac{s_{max}}{s_{min}}\right|_{N'},$$

then by Theorem VI, $\left|\frac{s_{max}}{s_{min}}\right|_N$ would also possess $\gamma$ as a lower bound, and N would be unable to realize frequency sets having $\gamma > \left|\frac{s_{max}}{s_{min}}\right| > 1$. Thus if N is to realize all possible frequency sets, no lower bound on $\left|\frac{s_{max}}{s_{min}}\right|_{N'}$ can exist.

With the help of these theorems, we can now proceed to the main result of this report. If the pliers type of entries to the complementary tree structure N are to produce "good" networks, then N certainly must be able to realize all sets of n distinct negative real natural frequencies. Theorem IV shows that N must contain a tie set of the form of Fig. XXXI-8 for this to be possible. Theorem VII shows that if this tie set is shorted, $|s_{max}/s_{min}|$ on the resulting network N' must be free of ratio problems. Theorems IV and VII show that this will be the case only if

(a) N' possesses a tie set of the form of Fig. XXXI-8, and one finds that

(b) $|s_{max}/s_{min}|$ on the network N" obtained by shorting this tie set is free of ratio problems.

By continuing in this manner it becomes clear that the pliers type of entries to N can yield "good" networks only if N can be reduced to a network of the form of Fig. XXXI-8 by successively shorting out subnetworks of this form.

This result implies that all complementary tree structures that lead to "good" networks are obtainable by the reverse of the reduction above. That is, any structure leading to a "good" network can be obtained by breaking a short circuit on the network (Fig. XXXI-8), inserting a network of this form, repeating the procedure on the resulting network, etc. This is the series-parallel flavor, mentioned earlier, which complementary tree structures must have if the pliers type of entries into them are to produce "good" networks.

It remains to be seen whether the structures generated by reversing this reduction, always yield "good" networks at the pliers type of entry.

## APPENDIX

To show that the network of Fig. XXXI-5 cannot realize all admittances of the form (3), it suffices to show that there is one RC admittance of the form (3) which it cannot realize. Such an admittance is

$$Y_o = \frac{s(s+2)(s+4)}{(s+1)(s+3)}.$$

From physical considerations, it is clear that the network degenerates to the series combination of $c_1$, $c_2$, and $c_3$ at high frequencies. If the capacitance of this path is to be 1 farad, as required by $Y_o$, then each of the capacitors $c_1$, $c_2$, and $c_3$ must have capacitance in excess of 1 farad, and their parallel connection must have capacitance in excess of 3 farads. Since the parallel connection of $c_1$, $c_2$, and $c_3$ accounts for the low-frequency behavior at the indicated terminal pair, this means that the low-frequency capacitance of the network cannot be made as small as that required by $Y_o$ (8/3 farads). Therefore the network cannot realize $Y_o$.

It might be thought that $Y_o$ could be realized by allowing the elements of Fig. XXXI-5 to assume negative values. It can be shown, however, that this is not the case.

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## References

1. H. B. Lee, Canonic realizations of RC driving-point admittances, Quarterly Progress Report No. 62, Research Laboratory of Electronics, M.I.T., July 15, 1961, pp. 287-290.