XVIII. STATISTICAL COMMUNICATION THEORY*

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RESEARCH OBJECTIVES

This group is interested in a variety of problems in statistical communication theory. Our current research is concerned primarily with: phase-locked oscillators, two-terminal nonlinear systems, location of noise sources in space by correlation methods, iterative methods in nonlinear theory, measurement of the kernels of a nonlinear system, and factors that influence the recording and reproduction of sound.

1. The threshold behavior of phase-locked oscillators in high noise-signal environments is being studied. A suitable model for the actual system has been shown to be a nonlinear time-variant system. This model is now being simulated on an analog computer to try to find out how the stability of the loop depends on the various system parameters.

2. Two-terminal nonlinear passive systems are being studied from the point of view of statistical theory. The study of these systems is producing some interesting results.

3. Noise sources in space can be located by means of higher-order correlation functions. A study of the errors in locating sources by this method is continuing.

4. The mean of a convex weighting function of the error of a system can be minimized by an iterative adjustment of certain parameters of the system. A study of some iterative methods is being made.

5. In the Wiener theory of nonlinear systems, a nonlinear system is characterized by a set of kernels. A method for the determination of these kernels was reported in Quarterly Progress Report No. 60 (pages 118-130). Some experimental results have indicated that the method is practical in many problems. Work on the method, both theoretical and experimental, is continuing.

6. We are also studying the factors that influence the accurate recording and reproduction of sound. In this study the tools of statistical communication theory are applied to spectral analysis under different methods of recording. In addition to the spectral studies, the transient behavior of the various links in the reproduction process will be investigated. Associated with this project, a filter of the Wiener-Lee type, which has controllable amplitude with a fixed phase over part of the audio spectrum, will be constructed as a tool for studying the effects of magnitude and phase perturbations on sound signals.

Y. W. Lee

A. POWER AMPLIFICATION WITH TWO-TERMINAL NONLINEAR SYSTEMS

Some results relating to the power absorbed by a two-terminal linear or nonlinear network, as depicted in Fig. XVIII-1, have been previously presented.[1] It was shown that the average power, P, absorbed can be expressed as

$$P = \overline{i(t)\ e(t)} = \phi_{ie}(0) = 2 \int_0^{\infty} \mathrm{Re}\left\{\Phi_{ie}(\omega)\right\} d\omega \tag{1}$$

in which $\mathrm{Re}\left\{\Phi_{ie}(\omega)\right\}$ is the real part of the Fourier transform of the crosscorrelation

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function, $\phi_{ie}(\tau) = \overline{i(t)\ e(t+\tau)}$. In this report, we shall present some results for passive systems for which $\text{Re}\ \{\Phi_{ie}(\omega)\} < 0$ over some range of frequencies.

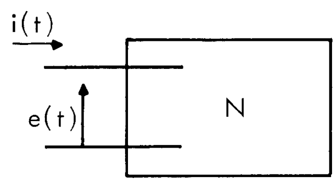

Fig. XVIII-1. A two-terminal linear or non-linear network.

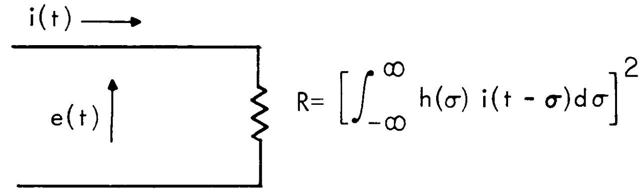

Fig. XVIII-2. A nonlinear resistive circuit.

If, for some excitation, $\text{Re}\ \{\Phi_{ie}(\omega)\} < 0$ over some range of frequencies, then the passive system can not be linear[1] and it can be used as a power amplifier. For example, consider the nonlinear resistive circuit depicted in Fig. XVIII-2. The value of the resistor can be expressed as

$$R = \left[\int_{-\infty}^{\infty} h(\sigma)\ i(t-\sigma)\ d\sigma\right]^2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\sigma_1)\ h(\sigma_2)\ i(t-\sigma_1)\ i(t-\sigma_2)\ d\sigma_1 d\sigma_2 \tag{2}$$

which is a quadratic function of the current flowing through the resistor. An

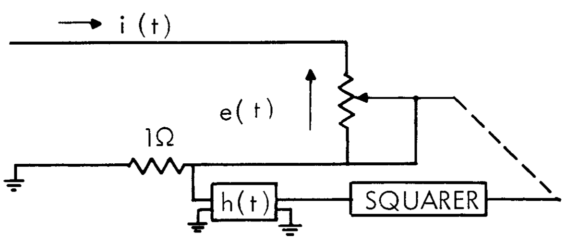

Fig. XVIII-3. An illustrative method of synthesizing the system of Fig. XVIII-2.

illustrative method by which such a system can be synthesized is depicted in Fig. XVIII-3. The voltage across the resistor is

$$e(t) = R\ i(t)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\sigma_1)\ h(\sigma_2)\ i(t-\sigma_1)\ i(t-\sigma_2)\ i(t)\ d\sigma_1 d\sigma_2. \tag{3}$$

Thus, the crosscorrelation function, $\phi_{ie}(\tau)$, is

$$\phi_{ie}(\tau) = \overline{e(t)\ i(t-\tau)}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\sigma_1)\ h(\sigma_2)\ \overline{i(t-\sigma_1)\ i(t-\sigma_2)\ i(t)\ i(t-\tau)}\ d\sigma_1 d\sigma_2. \tag{4}$$

For simplicity, we shall consider the case for which i(t) is a white Gaussian process whose autocorrelation function is

$$\phi_{ii}(\tau) = \overline{i(t)\ i(t+\tau)} = \mu_o(\tau). \tag{5}$$

Then[2]

$$\overline{i(t-\sigma_1)\ i(t-\sigma_2)\ i(t)\ i(t-\tau)} = \mu_o(\tau)\ \mu_o(\sigma_1-\sigma_2) + \mu_o(\sigma_1)\ \mu_o(\tau-\sigma_2) + \mu_o(\sigma_2)\ \mu_o(\tau-\sigma_1) \tag{6}$$

By substituting Eq. 6 in Eq. 4, we obtain

$$\phi_{ie}(\tau) = \mu_o(\tau) \int_{-\infty}^{\infty} h^2(\sigma)\ d\sigma + 2h(0)\ h(\tau) \tag{7}$$

The cross-power density spectrum is, then,

$$\Phi_{ie}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h^2(\sigma)\ d\sigma + \frac{1}{\pi} h(0)\ H(\omega) \tag{8}$$

in which the transfer function, $H(\omega)$, is

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\ e^{-j\omega t}\ dt \tag{9}$$

Thus, from Eq. 8, we obtain

$$\text{Re}\,\{\Phi_{ie}(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} h^2(\sigma)\ d\sigma + \frac{1}{\pi} h(0)\ \text{Re}\,\{H(\omega)\}. \tag{10}$$

We can make $\text{Re}\,\{\Phi_{ie}(\omega)\} < 0$ over a range of frequencies by choosing the impulse response, h(t), to be the first Laguerre function. That is, we let

$$h(t) = \begin{cases} \sqrt{2p}\ [2pt-1]\, e^{-pt} & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{11}$$

The network realization of this impulse response has been given by Lee.[3] The transfer function, $H(\omega)$, is, then,

$$H(\omega) = \sqrt{2p}\ \frac{p-j\omega}{(p+j\omega)^2} \tag{12}$$

Substituting Eq. 12 in Eq. 10, we obtain

$$\text{Re}\,\{\Phi_{ie}(\omega)\} = \frac{1}{2\pi}\left[1 - \frac{4p^2(p^2-3\omega^2)}{(p^2+\omega^2)^2}\right] \tag{13}$$

We note from this equation that $\text{Re}\,\{\Phi_{ie}(\omega)\} < 0$ for $\omega < 0.45\,p$. Such a nonlinear system, N, can be used as a power amplifier, as illustrated in Fig. XVIII-4. In

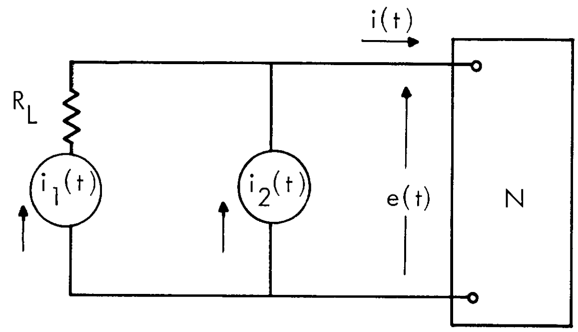

Fig. XVIII-4. A power amplifier having the circuit of Fig. XVIII-2.

this figure, the current, i(t), is obtained by summing the outputs of two separate current sources, $i_1(t)$ and $i_2(t)$, so that

$$i(t) = i_1(t) + i_2(t) \tag{14}$$

in which

$$i_1(t) = \int_{-\infty}^{\infty} k_1(\sigma)\, i(t-\sigma)\, d\sigma. \tag{15}$$

The total average power supplied to the nonlinear system, N, is, then,

$$P = P_1 + P_2 \tag{16}$$

in which

$$P_1 = \overline{e(t)\, i_1(t)}$$

and

$$P_2 = \overline{e(t)\, i_2(t)}.$$

By use of (15), we can express P_1 in terms of $\Phi_{ie}(\omega)$ as

$$P_1 = 2\int_0^{\infty} \text{Re}\,\{K_1(\omega)\Phi_{ie}(\omega)\}\, d\omega. \tag{17}$$

We thus observe that we can make $P_1 < 0$ by making $k_1(t)$ a lowpass filter so that $|K_1(\omega)|$ is small for $\omega > 0.45$ p. That is, $P_1 < 0$ if $i_1(t)$ is a Gaussian process for which little of its power is in the band of frequencies higher than 0.45 p. For such a current source, the average power dissipated in the load resistor, R_L, is

$$\overline{i_1^2(t)}\, R_L = P_s - P_1$$

in which P_s is the average power delivered by the current source, $i_1(t)$. But, since $P_1 < 0$, we have $P_s < \overline{i_1^2(t)}\, R_L$. That is, more power is delivered to the load resistor than is supplied by the source, $i_1(t)$. Of course, the additional power is being supplied by the other current source, $i_2(t)$. In this sense, we can consider the current source, $i_2(t)$, as the power supply and the nonlinear system, N, as a power amplifier.

In this manner, any two-terminal passive nonlinear system can be used as a power amplifier with any excitation for which $\text{Re}\{\Phi_{ie}(\omega)\} < 0$ over a range of frequencies. We also note that if, for some excitation, the passive nonlinear system can be used as a power amplifier, then $\text{Re}\{\Phi_{ie}(\omega)\} < 0$ over a range of frequencies. Thus the requirement that for some excitation $\text{Re}\{\Phi_{ie}(\omega)\} < 0$ over a range of frequencies is a necessary and sufficient condition for the passive nonlinear system to be used as a power amplifier.

M. Schetzen

References

1. M. Schetzen, Power absorbed by a nonlinear two-terminal network with a white Gaussian input, Quarterly Progress Report No. 62, Research Laboratory of Electronics, M.I.T., July 15, 1961, pp. 152-154.

2. M. Schetzen, Average of the product of Gaussian variables, Quarterly Progress Report No. 60, Research Laboratory of Electronics, M.I.T., January 15, 1961, pp. 137-141.

3. Y. W. Lee, Statistical Theory of Communication (John Wiley and Sons, Inc., New York, 1960).

B. ERRATA

In my report entitled "Measurement of the Kernels of a Nonlinear System by Cross-correlation with Gaussian Non-White Inputs," published in Quarterly Progress Report No. 63 (pages 113-117), each lower-case x in Eqs. 12-14 should be replaced by a lower-case z.

M. Schetzen