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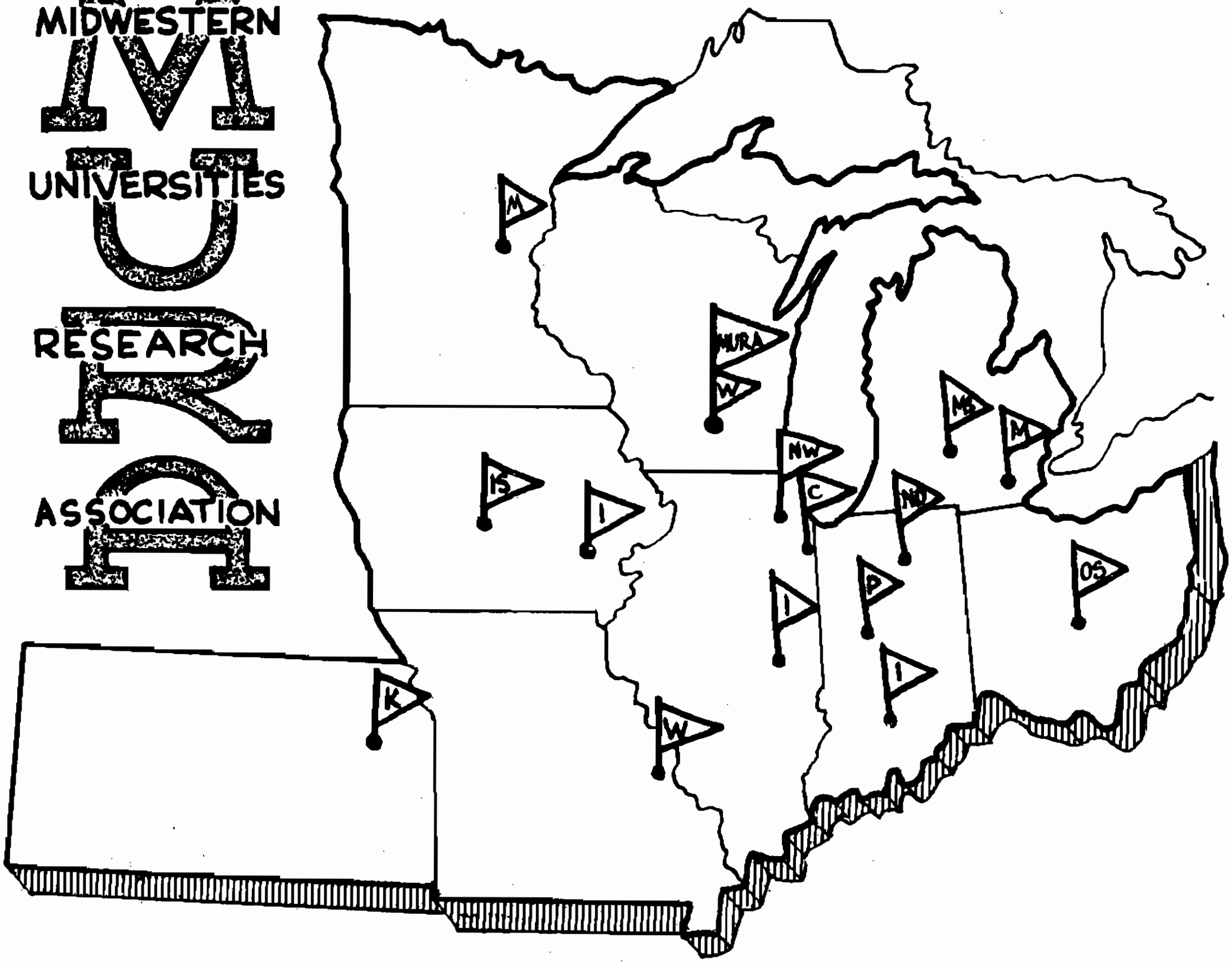

THE PRODUCTION BY VOLUME CURRENT DISTRIBUTIONS OF MAGNETIC FIELDS, WHICH ARE REPRESENTED BY SPHERICAL HARMONICS IN A CURRENT FREE REGION, WHICH ENCLOSES PART OF THE MEDIAN PLANE

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ABSTRACT

The scalar potential of a magnetic field is represented by a series of spherical harmonics in a current free region which encloses part of the median plane. The problem is treated of finding possible volume current distributions to produce this field, so that the field is everywhere zero outside a finite region that encloses the current distributions.

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** On leave from Purdue University, Lafayette, Indiana

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I. INTRODUCTION AND GENERAL PROCEDURE

Consider the magnetic field represented by the scalar potential:

$$V = -r_0 B_0 (1+s)^{k+1} L^{(m)}_{(o)(k+1)}(\sin\gamma)\sin m\phi \tag{1.1}$$

where (r, θ, ϕ) is a spherical coordinate system, $r = r_0(1+s)$, $\gamma = \frac{\pi}{2} - \theta$, and $L^{(m)}_{(o)(k+1)}(\sin\gamma)$ is a solution of the Legendre equation

$$\frac{d^2}{d\gamma^2} L^{(m)}_{(k+1)}(\sin\gamma) - \tan\gamma \frac{d}{d\gamma} L^{(m)}_{(k+1)}(\sin\gamma) + \left[(k+1)(k+2) - \frac{m^2}{\cos^2\gamma}\right] L^{(m)}_{(k+1)}(\sin\gamma) = 0 \tag{1.2}$$

which is an odd function of γ. This report is primarily concerned with the current distributions required to produce the field (1.1) in a finite closed region of space R_i which encloses a part of the median plane. R_i is surrounded by another region R_c finite in extent in which the current distributions are located. The region R_E is all space outside of R_c and is required to be field free.

Equation (1.2) has two independent solutions $L^{(m)}_{(e)(k+1)}(\sin\gamma)$ and $L^{(m)}_{(o)(k+1)}(\sin\gamma)$, which are respectively even and odd functions of γ. For small values of $|\gamma|$ and large values of $|\omega\gamma|$, the following expansions[1] represent rapidly convergent series for these functions:

$$L^{(m)}_{(e)(k+1)}\sin\gamma = \cos\omega\gamma + \text{higher order terms}$$

$$= \sin\omega\gamma + \frac{1}{4}\left[\frac{\gamma}{\omega}\cos\omega\gamma + \gamma^2\sin\omega\gamma\right] \tag{1.3}$$

$$+ \left\{\left[\frac{1}{4\omega}\left(\frac{1}{8} - m^2\right)\gamma^2 + \frac{\gamma^4}{32}\right]\sin\omega\gamma + \left[\frac{1}{4\omega^2}\left(\frac{1}{8} - m^2\right)\gamma + \frac{1}{6}\left(\frac{1}{8} + m^2\right)\gamma^3\right]\cos\omega\gamma\right\}$$

+ higher order terms.

where $\omega^2 = (k+1)(k+2) - m^2$

These functions are related to the associated Legendre functions $P^{(m)}_{(k+1)}(\sin y)$ and $Q^{m}_{k+1}(\sin y)$ as follows:

(1) m and k integral, $m+k+1$ odd, $m < \sqrt{(k+1)(k+2)}$ (1.4)

$$P^{(m)}_{(k+1)}(\sin y) = (\,) \underset{(o)}{L}{}^{m}_{(k+1)}(\sin y) \quad ; \quad Q^{(m)}_{k+1}(\sin y) = (\,) \underset{(e)}{L}{}^{(m)}_{(k+1)}(\sin y)$$

(2) m and k integral, $m+k+1$ even, $m < \sqrt{(k+1)(k+2)}$

$$Q^{(m)}_{(k+1)}(\sin y) = (\,) \underset{(o)}{L}{}^{(m)}_{(k+1)}(\sin y) \quad ; \quad P^{(m)}_{(k+1)}(\sin y) = (\,) \underset{(e)}{L}{}^{(m)}_{k+1} \qquad (1.5)$$

where the parentheses represent functions of m and k which have not been evaluated.

(3) m and k integral and $m > \sqrt{(k+1)(k+2)}$.

In this case, it is usually stated in the literature that $P^{(m)}_{(k+1)}$ and $Q^{(m)}_{(k+1)}$ do not exist. However, this statement is only correct if these functions are required to be single valued on the sphere. The functions $\underset{(e)}{L}{}^{m}_{(k+1)}$ and $\underset{(o)}{L}{}^{m}_{(k+1)}$ do exist in this case. If we write $\omega = i\omega_1$, then ω_1 is real and the series (1.3) become:

$$\underset{(e)}{L}{}^{(m)}_{(k+1)}(\sin y) = \cosh \omega_1 y + \text{higher order terms}$$
$$\qquad (1.6)$$
$$\underset{(o)}{L}{}^{(m)}_{(k+1)}(\sin y) = i \sinh \omega_1 y + \text{higher order terms}$$

These functions are not single valued on the sphere, but this causes no difficulties in the situations treated in this report.

(4) Either m or k or both are fractional. The relations between $\underset{(e)}{L}{}^{(m)}_{(k+1)}$, $\underset{(o)}{L}{}^{(m)}_{(k+1)}$ and $P^{(m)}_{(k+1)}$, $Q^{(m)}_{(k+1)}$ are much more complicated in these cases. One should use $\underset{(e)}{L}{}^{(m)}_{(k+1)}$ and $\underset{(o)}{L}{}^{(m)}_{(k+1)}$ instead of the $P^{(m)}_{(k+1)}$ and $Q^{(m)}_{(k+1)}$ as the former are respectively even and odd functions of y, while the latter are neither even or odd.

One is, however, greatly handicapped because of insufficient knowledge of the exact properties of the $L^{(m)}_{(e)(k+1)}$ and $L^{(m)}_{(o)(k+1)}$.

In this report we shall only consider the case where m and k are integral*. Due to the fact that one does not have exact relations for $L^{m}_{(E)(k+1)}$ and $L^{m}_{(O)(k+1)}$ and because of the large amount of information in the literature relative to the $\mathcal{P}^{(m)}_{(k+1)}$ and $\mathcal{Q}^{m}_{(k+1)}$ the latter functions are used in deriving exact relations. However, in working out special cases, it may be found more convenient to use approximations obtained from the series (1.3).

We shall further restrict the developments in this report to the case where m + k + 1 is odd**and shall require for the scalar potential in $\mathcal{R}_i$, the expression:

$$V = -r_0 B_0 (1+s)^{k+1} \mathcal{P}^{(m)}_{(k+1)}(\sin y) \sin m\phi \qquad (1.7)$$

We shall now consider the field in the volume current distributions of $\mathcal{R}_c$. The following function:

$$V = -r_0 B_0 \left[\nu_p \mathcal{P}^{(m)}_{(k+1)}(\sin y) + \nu_q \mathcal{Q}^{(m)}_{(k+1)}(\sin y)\right]\left[\nu_+ (1+s)^{k+1} + \nu_- (1+s)^{-(k+2)}\right]\left[\nu_c \cos m\phi + \nu_s \sin m\phi\right] \qquad (1.8)$$

where the v's are constants, represents a possible scalar potential for a magnetic field in free space. The field components are given by:

* The cases where this condition is violated can be treated in an approximate manner by using approximations obtained from the series (1.3). In this connection, one should refer to references (2) and (3).

** The case where m + k + 1 is even may be developed in a similar way. The $\mathcal{P}^{(m)}_{(k+1)}$ must be replaced by the $\mathcal{Q}^{(m)}_{(k+1)}$ in order that V be an odd function of y.

$$\frac{B_r}{B_0} = \left[v_P P^{(m)}_{(k+1)} + v_Q Q^{(m)}_{(k+1)}\right]\left[v_{(+)}(k+1)(1+s)^{k} - v_{(-)}(k+2)(1+s)^{-(k+3)}\right]\left[v_c \cos m\phi + v_s \sin m\phi\right]$$

$$\frac{B_y}{B_0} = \left[v_P \frac{d}{dy}P^{(m)}_{(k+1)} + v_Q \frac{d}{dy} Q^{(m)}_{(k+1)}\right]\left[v_{(+)}(1+s)^{k} + v_{(-)}(1+s)^{-(k+3)}\right]\left[v_c \cos m\phi + v_s \sin m\phi\right] \qquad (1.9)$$

$$\frac{B_\phi \cos y}{B_0} = m\left[v_P P^{(m)}_{(k+1)} + v_Q Q^{(m)}_{(k+1)}\right]\left[v_{(+)}(1+s)^{k} + v_{(-)}(1+s)^{-(k+3)}\right]\left[-v_c \sin m\phi + v_s \cos m\phi\right]$$

We shall now assume that in the region of the current distributions the field is represented by (1.9) where v_P and v_Q are functions of y, $v_{(+)}$ and $v_{(-)}$ are functions of s, v_c and v_s are functions of ϕ . In order that (1.9) represents a magnetic field $\nabla \cdot \vec{B} = 0$, or

$$\frac{\nabla \cdot \vec{B}}{B_0} = \left[\frac{dv_P}{dy}\frac{dP^{(m)}_{(k+1)}}{dy} + \frac{dv_Q}{dy}\frac{dQ^{(m)}_{(k+1)}}{dy}\right]\left[v_{(+)}(1+s)^{k-1} + v_{(-)}(1+s)^{-(k+4)}\right]\left[v_c \cos m\phi + v_s \sin m\phi\right]$$

$$+\left[v_P P^{(m)}_{(k+1)} + v_Q Q^{(m)}_{(k+1)}\right]\left[\frac{dv_{(+)}}{ds}(k+1)(1+s)^{k} - \frac{dv_{(-)}}{ds}(k+2)(1+s)^{-(k+3)}\right]\left[v_c \cos m\phi + v_s \sin m\phi\right]$$

$$+\frac{m}{\cos^2 y}\left[v_P P^{(m)}_{(k+1)} + v_Q Q^{(m)}_{(k+1)}\right]\left[v_{(+)}(1+s)^{k-1} + v_{(-)}(1+s)^{-(k+4)}\right]\left[-\frac{dv_c}{d\phi}\sin m\phi + \frac{dv_s}{d\phi}\cos m\phi\right] = 0 \qquad (1.10)$$

This will be called the divergence condition. It should be noted that the field (1.7) is of the type (1.9). Since surface and line current distributions will not be used, the boundary condition between any two continuous regions bounded by coordinate surfaces is the continuity of (B_r, B_y, B_ϕ), which is satisfied if and if only if v_P and v_Q are continous. The current density (i_r, i_y, i_ϕ) may be obtained from

$$4\pi \vec{i} = \nabla \times \vec{B}$$

and one finds: *

* Whenever $P^{(m)}_{(k+1)}(\sin y)$ and $Q^{(m)}_{(k+1)}(\sin y)$ are written without their argument "sin y", the argument "sin y" is to be understood.

$$\frac{4\pi r_0 \cos y}{B_0} i_r = \left[v_P P_{k+1}^{(m)} + v_Q Q_{k+1}^{m}\right]\left[v_{(+)}(1+s)^{k-1} + v_{(-)}(1+s)^{-(k+4)}\right]\left[\frac{dv_c}{d\phi}\cos m\phi + \frac{dv_s}{d\phi}\sin m\phi\right]$$
$$- m\left[\frac{dv_P}{dy} P_{(k+1)}^{(m)} + \frac{dv_Q}{dy} Q_{(k+1)}^{(m)}\right]\left[v_{(+)}(1+s)^{k-1} + v_{(-)}(1+s)^{-(k+4)}\right]\left[-v_c \sin m\phi + v_s \cos m\phi\right]$$

$$(1.10)$$

$$\frac{4\pi r_0 \cos y}{B_0} i_y = \left[v_P P_{(k+1)}^{(m)} + v_Q Q_{(k+1)}^{(m)}\right]\left[\frac{dv_{(+)}}{ds}(1+s)^{k} + \frac{dv_{(-)}}{ds}(1+s)^{-(k+3)}\right]\left[-m v_c \sin m\phi + m v_s \cos m\phi\right]$$
$$- \left[v_P P_{(k+1)}^{(m)} + v_Q Q_{(k+1)}^{(m)}\right]\left[v_{(+)}(k+1)(1+s)^{k-1} - v_{(-)}(k+2)(1+s)^{-(k+4)}\right]\left[\frac{dv_c}{d\phi}\cos m\phi + \frac{dv_s}{d\phi}\sin m\phi\right]$$

$$\frac{4\pi r_0}{B_0} i_\phi = \left[\frac{dv_P}{dy} P_{(k+1)}^{(m)} + \frac{dv_Q}{dy} Q_{k+1}^{(m)}\right]\left[v_{(+)}(k+1)(1+s)^{k-1} - v_{(-)}(k+2)(1+s)^{-(k+4)}\right]\left[v_c \cos m\phi + v_s \sin m\phi\right]$$
$$- \left[v_P \frac{d}{dy} P_{(k+1)}^{(m)} + v_Q \frac{d}{dy} Q_{k+1}^{m}\right]\left[\frac{dv_{(+)}}{ds}(1+s)^{k} + \frac{dv_{(-)}}{ds}(1+s)^{-(k+3)}\right]\left[v_c \cos m\phi + v_c \sin m\phi\right]$$

In order to illustrate the general procedure of finding current distributions which will produce the field (1.7) in $\mathcal{R}_i$, we shall outline the procedure for solving certain special cases.

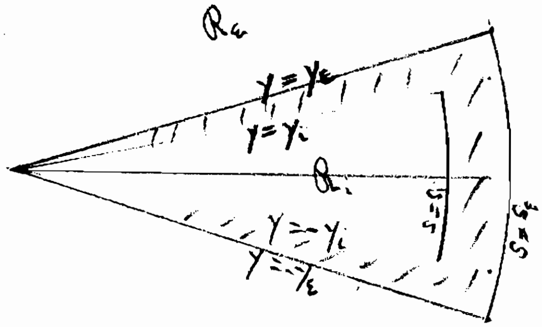

Fig. 1

Figure I represents a section made by an azimuthal plane of a figure of revolution. $\mathcal{R}_i$ is the region between the cones $y = \pm y_i$ where $-1 \leq s \leq s_i$. y_i and s_i are constants. $\mathcal{R}_\varepsilon$ is the region where $|y| \geq y_\varepsilon$ if $-1 \leq s \leq s_\varepsilon$ and the region where $s > s_\varepsilon$ for all y. $\mathcal{R}_c$ is the region outside of $\mathcal{R}_i$ and within $\mathcal{R}_\varepsilon$. It is represented by the shaded portion of the figure. We wish to produce the required field in $\mathcal{R}_i$ with zero field in $\mathcal{R}_\varepsilon$. This may, of course, be done in many different ways. One way will now be described. For this purpose $\mathcal{R}_c$ will be further subdivided as shown in Figure 2.

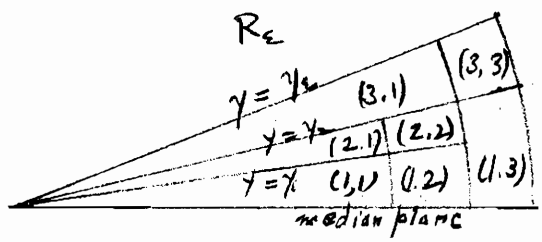

Fig. 2

In Figure 2, (1,1) together with its mirror image is identical with $\mathcal{R}_i$. $\mathcal{R}_\varepsilon$ is divided into the regions (2.1), (3.1), (1.2), (1.3), (2.2), (3.3) and their mirror images in the median plane. In the regions (2.1) and (3.1), we assume that ($\mathcal{U}_{(+)}, \mathcal{U}_{(-)}, \mathcal{V}_c, \mathcal{V}_s$) are constants, and that the current density in each of these regions depends on γ in a definite way. As soon as its dependence on γ is assumed, its dependence on s and ϕ is determined. The current density in these regions can not be made independent of γ , but it can be made small if γ is small. These conditions also require that $i_\gamma = 0$. We have introduced two conical regions (2.1) and (3.1) between $\mathcal{R}_i$ and $\mathcal{R}_\varepsilon$. It will be shown that two such regions are required to satisfy the above conditions, except for certain particular values of γ_i and γ_ε , for which only one such region is required. In a similar manner, ($\mathcal{V}_P, \mathcal{V}_Q, \mathcal{V}_c, \mathcal{V}_s$) are assumed constant in regions (1.2) and (1.3). The fields are assumed to satisfy the boundary conditions on the spherical surfaces $s = s_1, s = s_2, s = s_3$ and the dependence of the current density on s is specified in each of the regions. It will follow that $i_\mu = 0$. It can be shown that two but no more than two regions of type (s) are required between $\mathcal{R}_i$ and R_ε to satisfy these conditions. The fields on the boundaries of the regions (2.2) and (3.3) have now been determined. It remains

It remains to find the fields and the current densities in these regions, where only r_c and r_s can be assumed constant. Many different fields can be determined to satisfy the required conditions.

We may modify the above example as follows: γ_i is a given constant for $\varphi_1 \leq \varphi < \varphi_2$ and a different constant for $\varphi_2 \leq \varphi < \varphi_1 + 2\pi$. This would make it possible to introduce more free space between the copper windings in certain azimuthal regions for radio frequency and other devices. The regions (2.2) and (3.3) which were closed rings are now cut into sectors of rings by azimuthal planes at $\varphi = \varphi_1$, and $\varphi = \varphi_2$.

Another modification of the case illustrated in Figure 2 is illustrated in Figure 3. R_ε is here bounded by several cones of different γ_ε, connected by a spherical surfaces. In this way one can keep the region R_i narrow even for large s. In the figure, (1,11) with its mirror image is R_i. All the other numbered sections with their mirrored images form R_c. One could combine the above two modifications so that γ_i and γ_ε are functions of both s and φ.

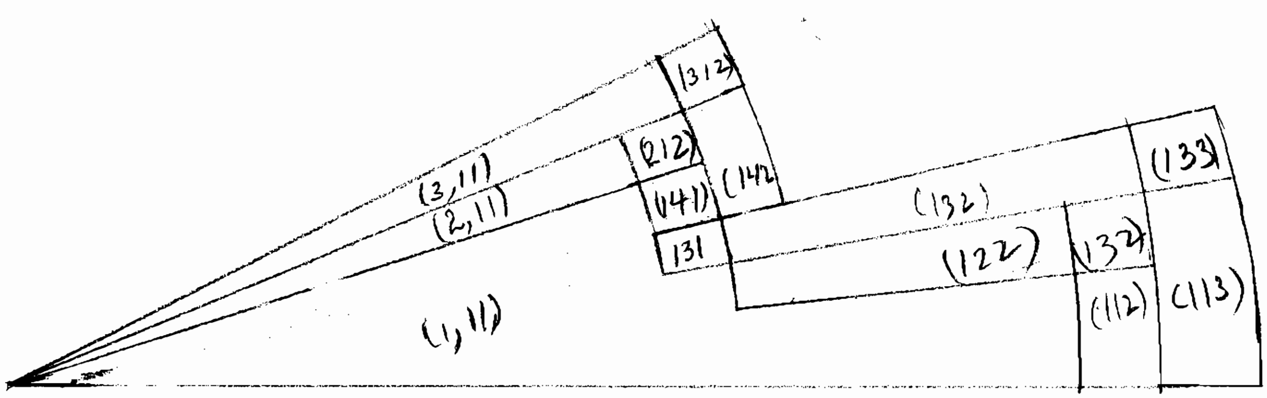

Figure 3

In the case illustrated in Figure 2, the current distributions required to produce the fields can be simplified if the angles γ that bound the current distributions are chosen properly. This is illustrated in Figure 4.

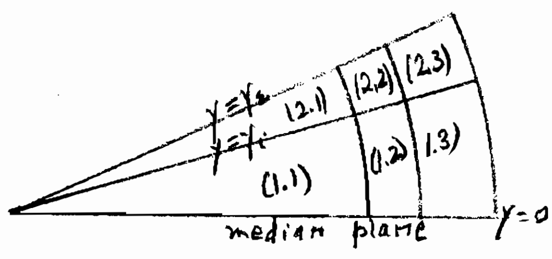

Fig. 4

The same result is achieved here as in the distributions represented in Figure 2, but the angles γ_1 and γ_2 must be given certain values.

The different regions into which $\mathcal{R}_c$ is subdivided can be classified according to which of the quantities ($\nu_p, \nu_k, \nu_{(+)}, \nu_{(-)}, \nu_c, \nu_s$) are not held constant. Thus if ($\nu_{(+)}, \nu_{(-)}, \nu_c, \nu_s$) are held constant, it is of type (γ). If none are held constant, it is of type (s, γ, ϕ). In Figure 2, (2.1) and 3.1) are of type (γ), (1.2) and (1.3) are of type (s), (2.2) and (3.3) of type (s, γ). In Figure 3, (2.11), (3.11), (1.22) and (1.32) are of type , (1.41) (1.42), (1.12) (1.13) are of type (γ), (2.12), (3.12), (1.31), (1.21), (1.32), (1.33) are of type (γs).

One may desire to produce in $\mathcal{R}_i$ not the magnetic field represented by (1.7) but a field represented by the sum of such fields corresponding to different values of m and k. Since the equations involved are linear, the current density can be determined for each term by the methods of this report, and the current densities are then added vectorally to find the resultant current density. It should be noted, however, that in this case one can not reduce the number of regions of type (y) between R_i and R_ε to one, as was done for the case represented by Figure 4. This is due to the fact that the y_i and y_ε required for this to be done are functions of m and k . However, one never needs more than two such regions between R_i and R_ε . This is true for regions of type (y), type (s) and type (ϕ).

II REGION OF TYPE (y).

v_p and v_Q are functions of y

($v_{(+)}$ $v_{(-)}$ v_c v_s) are constants.

Field:

$$\begin{aligned} \frac{B_r}{B_0} &= (k+1)(1+s)^k\left[v_p P^m_{k+1} + v_Q Q^m_{k+1}\right]\sin m\phi \\ \frac{B_y}{B_0} &= (1+s)^k\left[v_p \frac{d}{dy} P^m_{k+1} + v_Q \frac{d}{dy} Q^m_{k+1}\right]\sin m\phi \\ \frac{B_\phi \cos y}{B_0} &= m(1+s)^k\left[v_p P^m_{k+1} + v_Q Q^m_{k+1}\right]\cos m\phi \end{aligned} \tag{2.1}$$

Current density:

$$\begin{aligned} \frac{4\pi r_0}{B_0}\cos y\, i_r &= -m(1+s)^{k-1}\left[\frac{dv_p}{dy} P^{(m)}_{(k+1)} + \frac{dv_Q}{dy} Q^{(m)}_{(k+1)}\right]\cos m\phi \\ \frac{4\pi r_0}{B_0}\cos y\, i_y &= 0 \\ \frac{4\pi r_0}{B_0}\, i_\phi &= (k+1)(1+s)^{k-1}\left[\frac{dv_p}{dy} P^{(m)}_{(k+1)} + \frac{dv_Q}{dy} Q^m_{k+1}\right]\sin m\phi \end{aligned} \tag{2.2}$$

Divergence condition:

$$\frac{dv_p}{dy}\frac{dP^{(m)}_{(k+1)}}{dy} + \frac{dv_Q}{dy}\frac{dQ^{(m)}_{(k+1)}}{dy} = 0 \tag{2.3}$$

We shall define K by the relation:

$$\frac{dv_p}{dy} P^{(m)}_{(k+1)} + \frac{dv_Q}{dy} Q^{(m)}_{(k+1)} = \frac{K}{\cos y} \tag{2.4}$$

We shall make use of the relation:[4]

$$P^{(m)}_{(k+1)} \frac{dQ^{(m)}_{k+1}}{dy} - Q^{(m)}_{(k+1)} \frac{dP^{(m)}_{(k+1)}}{dy} = \frac{C^{(m)}_{(k+1)}}{\cos y} \tag{2.5}$$

where
$$C^{(m)}_{k+1} = 2^{2m} \frac{\Gamma\left(1+\frac{m+k+1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{m+k+1}{2}\right)}{\Gamma\left(1+\frac{k+1-m}{2}\right)\Gamma\left(\frac{1}{2}+\frac{k+1-m}{2}\right)} \tag{2.6}$$

where Γ is a symbol for the gamma function. Making use of equations (2.3), (2.4) and (2.5) one finds

$$\frac{d\nu_P}{dy} = \frac{K}{\cos y} \frac{\frac{dQ^{(m)}_{k+1}}{dy}}{P^{(m)}_{(k+1)} \frac{dQ^{(m)}_{k+1}}{dy} - Q^{(m)}_{(k+1)} \frac{dP^{(m)}_{k+1}}{dy}} = \frac{K}{C^{(m)}_{(k+1)}} \frac{dQ^{(m)}_{(k+1)}}{dy} \tag{2.7}$$

$$\frac{d\nu_Q}{dy} = -\frac{K}{C^{(m)}_{(k+1)}} \frac{dP^{(m)}_{k+1}}{dy}$$

We shall now require that K is a constant. In this case, equations (2.7) may be integrated, giving

$$\nu_P = \frac{K}{C^{(m)}_{(k+1)}} Q^{(m)}_{(k+1)} + K_{P,0} \tag{2.8}$$

$$\nu_Q = -\frac{K}{C^{(m)}_{(k+1)}} P^{m}_{(k+1)} + K_{Q,0} \tag{2.9}$$

where $K_{P,0}$ and $K_{Q,0}$ are constants. Substituting (2.8) in (2.1), we find

$$\frac{B_r}{B_0} = (k+1)(1+s)^k \left[K_{P,0} P^{(m)}_{(k+1)} + K_{Q_0} Q^{(m)}_{(k+1)}\right] \sin m\phi$$
$$\frac{B_y}{B_0} = (1+s)^k \left[K_{P,0} \frac{d}{dy} P^{(m)}_{(k+1)} + K_{Q,0} \frac{d}{dy} Q^{m}_{k+1}\right] \sin m\phi - \frac{K(1+s)^k}{\cos y} \sin m\phi \tag{2.10}$$
$$\frac{B_\phi}{B_0} = \frac{m(1+s)^k}{\cos y} \left[K_{P,0} P^{(m)}_{k+1} + K_{Q_0} Q^{m}_{k+1}\right] \cos m\phi$$

From (2.10) it follows that $\vec{B}$ can be split into two parts $\vec{B}^{(1)}$ and $\vec{B}^{(2)}$ such that:

$$\vec{B} = \vec{B}^{(1)} + \vec{B}^{(2)}, \quad \nabla \times \vec{B}^{(1)} = 0, \quad \vec{B}^{(2)} \cdot \vec{\iota} = 0 \tag{2.11}$$

and where $\vec{B}^{(2)} = -\vec{E}_\gamma \dfrac{K(1+s)^k}{\cos\gamma} \sin m\phi$

The current density is given by:

$$\frac{4\pi r_0}{B_0} i_r = -\frac{mK}{\cos^2\gamma}(1+s)^{k-1}\cos m\phi$$

$$\frac{4\pi r_0}{B_0} i_\gamma = 0 \tag{2.12}$$

$$\frac{4\pi r_0}{B_0} i_\phi = \frac{K}{\cos\gamma}(1+s)^{k-1}\sin m\phi$$

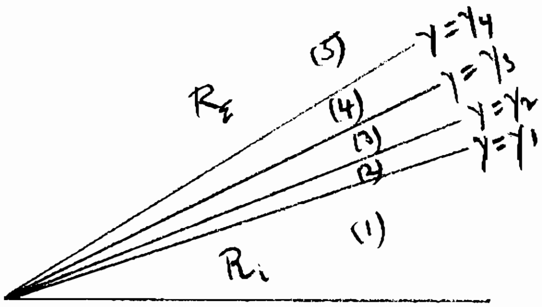

Figure 5

Let us now apply these results to the situation represented in Figure 5, where we have three (γ) type regions (2), (3) and (4) separating R_i and R_E. R_i is identical with (1) and its mirror image. R_E is identical with (5) and its mirror image. The field in R_i is given by (1.7) and therefore $V_P^{(1)} = 1$, $V_Q^{(1)} = 0$. R_E is field free and therefore $V_P^{(5)} = V_Q^{(5)} = 0$. The continuity of V_P and V_Q require that

$$\begin{aligned} 0 &= V_P^{(5)} = V_P^{(4)}(\gamma_4) & 0 &= V_Q^{(5)} = V_Q^{(4)}(\gamma_4) \\ V_P^{(4)}(\gamma_3) &= V_P^{(3)}(\gamma_3) & V_Q^{(4)}(\gamma_3) &= V_Q^{(3)}(\gamma_3) \\ V_P^{(3)}(\gamma_2) &= V_P^{(2)}(\gamma_2) & V_Q^{(3)}(\gamma_2) &= V_Q^{(2)}(\gamma_2) \\ 1 &= V_P^{(2)}(\gamma_1) & 0 &= V_Q^{(2)}(\gamma_1) \end{aligned} \tag{2.13}$$

Applying equations (2.7) to this case, we find

$$V_p^{(n)} = \frac{K^{(n)}}{C_{k+1}^{m}} Q_{(k+1)}^{(m)}(\sin\gamma) + K_{p,0}^{(n)} \qquad (n = 2, 3, 4)$$

and

$$V_p^{(n)}(\gamma_n) - V_p^{(n)}(\gamma_{n-1}) = \frac{K^{(n)}}{C_{(k+1)}^{(m)}}\left[Q_{(k+1)}^{(m)}(\sin\gamma_n) - Q_{(k+1)}^{(m)}(\sin\gamma_{n-1})\right] \qquad (2.14)$$

or

$$V_p^{(4)}(\gamma_4) - V_p^{(4)}(\gamma_3) = \frac{K^{(4)}}{C_{k+1}^{(m)}}\left[Q_{(k+1)}^{(m)}(\sin\gamma_4) - Q_{(k+1)}^{(m)}(\sin\gamma_3)\right]$$

$$V_p^{(3)}(\gamma_3) - V_p^{(3)}(\gamma_2) = \frac{K^{(3)}}{C_{(k+1)}^{(m)}}\left[Q_{(k+1)}^{(m)}(\sin\gamma_3) - Q_{(k+1)}^{(m)}(\sin\gamma_2)\right] \qquad (2.15)$$

$$V_p^{(2)}(\gamma_2) - V_p^{(2)}(\gamma_1) = \frac{K^{(2)}}{C_{(k+1)}^{(m)}}\left[Q_{(k+1)}^{(m)}(\sin\gamma_2) - Q_{(k+1)}^{m}(\sin\gamma_1)\right]$$

Adding equations (2.15) and making use of (2.13), we find

$$K^{(4)}\left[Q_{(k+1)}^{(m)}(\sin\gamma_4) - Q_{(k+1)}^{(m)}(\sin\gamma_3)\right] + K^{(3)}\left[Q_{(k+1)}^{(m)}(\sin\gamma_3) - Q_{(k+1)}^{(m)}(\sin\gamma_2)\right] + K^{(2)}\left[Q_{(k+1)}^{(m)}(\sin\gamma_2) - Q_{(k+1)}^{(m)}(\sin\gamma_1)\right] = -C_{(k+1)}^{(m)} \qquad (2.16)$$

Treating equations (2.8) in a similar way, we find

$$K^{(4)}\left[P_{(k+1)}^{(m)}(\sin\gamma_4) - P_{(k+1)}^{(m)}(\sin\gamma_3)\right] + K^{(3)}\left[P_{(k+1)}^{(m)}(\sin\gamma_3) - P_{(k+1)}^{(m)}(\sin\gamma_2)\right] + K^{(2)}\left[P_{(k+1)}^{(m)}(\sin\gamma_2) - P_{(k+1)}^{(m)}(\sin\gamma_1)\right] = 0 \qquad (2.17)$$

(2.16) and (2.17) are two simultaneous equations that $K^{(2)}$, $K^{(3)}$ and $K^{(4)}$ must satisfy. One of the $K^{(n)}$ may be chosen arbitrarily, and these two equations determine the other two Ks if the determinent of this coefficients is different from zero. This shows that no more than two current regions of the type considered are required between R_i and R_e. Let us consider when only one region is required. This case is illustrated in Figure 6.

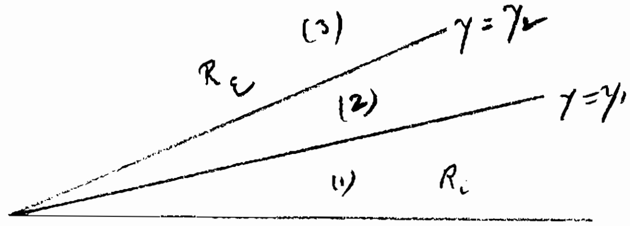

Figure 6

Equations (2.16) and (2.17) in this case reduce to

$$\begin{aligned} K^{(2)}\left[Q^{(m)}_{(k+1)}(\sin\gamma_2) - Q^{(m)}_{(k+1)}(\sin\gamma_1)\right] &= -C^{m}_{k+1} \\ K^{(2)}\left[P^{m}_{k+1}(\sin\gamma_2) - P^{m}_{(k+1)}(\sin\gamma_1)\right] &= 0 \end{aligned} \tag{2.18}$$

The second equation requires that

$$P^{m}_{(k+1)}(\sin\gamma_2) = P^{(m)}_{(k+1)}(\sin\gamma_1) \tag{2.19}$$

and the first equation gives

$$K^{(2)} = -\frac{C^{(m)}_{k+1}}{Q^{(m)}_{(k+1)}(\sin\gamma_2) - Q^{(m)}_{(k+1)}(\sin\gamma_1)}$$

Substituting this relation in (2.8) and (2.9), we have

$$\begin{aligned} V_P^{(2)}(\gamma) &= \frac{Q^{m}_{(k+1)}(\sin\gamma)}{Q^{(m)}_{(k+1)}(\sin\gamma_1) - Q^{(m)}_{(k+1)}(\sin\gamma_2)} + K_{P,0} \\ V_Q^{(2)}(\gamma) &= \frac{P^{m}_{(k+1)}(\sin\gamma)}{Q^{(m)}_{(k+1)}(\sin\gamma_2) - Q^{(m)}_{(k+1)}(\sin\gamma_1)} + K_{q,0} \end{aligned} \tag{2.20}$$

One can determine $K_{P,0}$ and $K_{Q,0}$ from the conditions $V_P^{(2)}(\gamma_1) = 1$, $V_Q^{(2)}(\gamma_1) = 0$, and one finds then

$$\mathcal{V}_P^{(2)} = \frac{Q_{(k+1)}^{(m)}(\sin y) - Q_{(k+1)}^{(m)}(\sin y_1)}{Q_{(k+1)}^{m}(\sin y_1) - Q_{(k+1)}^{m}(\sin y_2)} \tag{2.21}$$

$$\mathcal{V}_Q^{(2)} = \frac{P_{(k+1)}^{(m)}(\sin y_1) - P_{(k+1)}^{m}(\sin y)}{Q_{(k+1)}^{m}(\sin y_1) - Q_{(k+1)}^{(m)}(\sin y_2)} \tag{2.22}$$

The field and the current density may be easily obtained by substituting these equations into (2.1) and (2.2).

III REGION OF TYPE (s)

$\mathcal{V}_{(+)}$ and $\mathcal{V}_{(-)}$ are functions of s

$(\mathcal{V}_P, \mathcal{V}_Q, \mathcal{V}_t, \mathcal{V}_s)$ are constants

Field:
$$\frac{B_r}{B_0} = P_{(k+1)}^{(m)}\left[\mathcal{V}_{(+)}(k+1)(1+s)^k - \mathcal{V}_{(-)}(k+2)(1+s)^{-(k+3)}\right]\sin m\phi$$

$$\frac{B_y}{B_0} = \frac{dP_{(k+1)}^{(m)}}{dy}\left[\mathcal{V}_{(+)}(1+s)^k + \mathcal{V}_{(-)}(1+s)^{-(k+3)}\right]\sin m\phi \tag{3.1}$$

$$\frac{B_\phi}{B_0} = m\, P_{(k+1)}^{(m)}\left[\mathcal{V}_{(+)}(1+s)^k + \mathcal{V}_{(-)}(1+s)^{-(k+3)}\right]\cos m\phi$$

Current density:

$$\frac{4\pi r_0}{B_0}\cos y\; i_r = 0$$

$$\frac{4\pi r_0}{B_0}\cos y\; i_y = m\, P_{(k+1)}^{(m)}\left\{\frac{d\mathcal{V}_{(+)}}{ds}(1+s)^k + \frac{d\mathcal{V}_{(-)}}{ds}(1+s)^{-(k+3)}\right\}\cos m\phi \tag{3.2}$$

$$\frac{4\pi r_0}{B_0}\quad i_\phi = -\frac{d}{dy}P_{(k+1)}^{(m)}\left\{\frac{d\mathcal{V}_{+}}{ds}(1+s)^k + \frac{d\mathcal{V}_{(-)}}{ds}(1+s)^{-(k+3)}\right\}\sin m\phi$$

Divergence conditions:

$$\frac{dv_{(+)}}{ds}(k+1)(1+s)^{k} - \frac{dv_{(-)}}{ds}(k+2)(1+s)^{-(k+3)} = 0 \tag{3.3}$$

We shall write:

$$\frac{dv_{(+)}}{ds}(1+s)^{k} + \frac{dv_{(-)}}{ds}(1+s)^{-(k+3)} = L \tag{3.4}$$

We shall now require that L be a constant, so that ι_ρ and ι_y will be independent of s.

From (3.4) and (3.5), we find

$$\frac{dv_{(+)}}{ds} = \frac{L(k+2)}{(2k+3)}(1+s)^{-k} \quad ; \quad \frac{dv_{(-)}}{ds} = \frac{L(k+1)}{2k+3}(1+s)^{k+3} \tag{3.5}$$

and after integrating:

$$v_{(+)} = \frac{L(2k+3)(1+s)^{1-k}}{(2k+3)(1-k)} + L_{+,0}$$

$$v_{(-)} = \frac{L(k+1)(1+s)^{k+4}}{(2k+3)(k+4)} + L_{-,0} \tag{3.6}$$

where $L_{+,0}$ and $L_{-,0}$ are constants.

If one substitutes (3.6) in (3.1), one obtains expressions for the field components. We find that $\vec{B} = \vec{B}^{(1)} + \vec{B}^{(2)}$ where $\nabla \times \vec{B}^{(1)} = 0$, but $\vec{B}^{(2)}$ is not perpendicular to $\vec{\iota}$. This differs from a similar situation in a region of type (y) where $\vec{B}^{(2)}$ is perpendicular to $\vec{\iota}$. However, the same thing can be achieved in this case if one replaces (3.4) by

$$\frac{dv_{(+)}}{ds}(1+s)^{k} + \frac{dv_{(-)}}{ds}(1+s)^{-(k+3)} = L_1(1+s)^{-3} \tag{3.7}$$

and requires that L_1 is a constant.

From (3.3) and (3.7), we find

$$\frac{d\mathcal{V}_{(+)}}{ds} = \frac{L_1(k+2)}{(2k+3)}(1+s)^{-(k+3)}, \quad \frac{d\mathcal{V}_{(-)}}{ds} = \frac{L_1(k+1)}{(2k+3)}(1+s)^{k} \quad (3.8)$$

$$\text{and} \quad \mathcal{V}_{(+)} = -\frac{L_1(1+s)^{-(k+2)}}{(2k+3)} + L_{+,0} \quad (3.9)$$

$$\mathcal{V}_{(-)} = \frac{L_1(1+s)^{(k+1)}}{(2k+3)} + L_{1-,0} \quad (3.10)$$

where $L_{1+,0}$ and $L_{1-,0}$ are constants. Substituting (3.9) into (3.1), we find:

$$\begin{aligned}\frac{B_r}{B_0} &= -\frac{L_1}{(1+s)^2}P^{(m)}_{(k+1)} + P^{(m)}_{(k+1)}\left[L_{(+,0)}(k+1)(1+s)^k - L_{-,0}(k+2)(1+s)^{-(k+3)}\right]\sin m\phi \\ &= 0 \quad + \frac{dP^{(m)}_{(k+1)}}{dy}\left[L_{+,0}(1+s)^k + L_{-,0}(1+s)^{-(k+3)}\right]\sin m\phi \\ &= 0 \quad + mP^{(m)}_{(k+1)}\left[L_{+,0}(1+s)^k + L_{-,0}(1+s)^{-(k+3)}\right]\cos m\phi\end{aligned} \quad (3.11)$$

$$\text{so that now} \quad \vec{B} = \vec{B}^{(1)} + \vec{B}^{(2)}\ ; \ \nabla\times\vec{B}_1 = 0, \ \vec{B}_2\cdot\vec{i} = 0 \quad (3.12)$$

$$\vec{B}_2 = -\vec{\varepsilon}_r\frac{L_1}{(1+s)^2}P^{(m)}_{(k+1)}$$

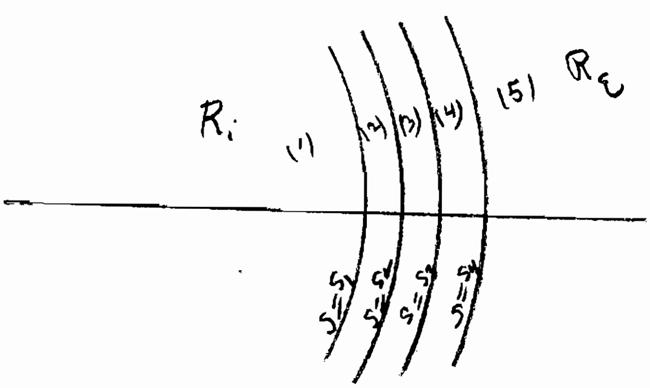

Figure 7

Consider the situation represented in Figure 7 which is analogous to the situation represented in Figure 5 in the case of a region of type (γ).

Comparing (3.1) and (2.7) and remembering that R_E is field free, it follows that:

$$v_+^{(1)} = 1, \quad v_+^{(5)} = 0, \quad v_{(-)}^{(5)} = v_{(-)}^{(1)} = 0 \tag{3.13}$$

Applying (3.9), (3.10) and (3.13) to this case and taking into account the continuity of the v's, we find

$$L_1^{(4)}\left[(1+s_4)^{-(k+3)} - (1+s_3)^{-(k+3)}\right] + L_1^{(3)}\left[(1+s_3)^{-(k+3)} - (1+s_2)^{-(k+3)}\right] + L_1^{(2)}\left[(1+s_2)^{-(k+3)} - (1+s_1)^{-(k+3)}\right] = (2k+3) \tag{3.14}$$

$$L_1^{(4)}\left[(1+s_4)^{(k+1)} - (1+s_3)^{(k+1)}\right] + L_1^{(3)}\left[(1+s_3)^{(k+1)} - (1+s_2)^{k+1}\right] + L_1^{(2)}\left[(1+s_2)^{(k+1)} - (1+s_1)^{(k+1)}\right] = 0 \tag{3.15}$$

These are analogous to relations (2.16) and (2.17) for a region of type (y). These are two simultaneous equations which $L^{(4)}, L^{(3)}, L^{(2)}$ must satisfy. One always requires two such regions between R_i and R_E, since (3.15) can never be identically satisfied by special values of s. The situation represented by Figure 6 in a region of type (y) does not appear for a region of type (s).

IV REGION OF TYPE (ϕ)

v_c and v_s are functions of ϕ.

(v_p v_q v_+ v_-) are constants .

Field:
$$\frac{B_r}{B_0} = (k+1)\,(1+s)^k P_{k+1}^m \left[v_c \cos m\phi + v_s \sin m\phi\right]$$
$$\frac{B_y}{B_0} = (1+s)^k \frac{dP_{k+1}^m}{dy} \left[v_c \cos m\phi + v_s \sin m\phi\right] \tag{4.1}$$
$$\frac{B_\phi}{B_0}\cos y = m\,(1+s)^k P_{k+1}^m \left[-v_c \sin m\phi + v_s \cos m\phi\right]$$

Current density

$$\frac{4\pi r_0}{B_0}\cos y\; i_r = (1+s)^{k-1}\frac{dP_{(k+1)}^{(m)}}{dy}\left[\frac{dV_c}{d\varphi}\cos m\phi + \frac{dV_s}{d\varphi}\sin m\phi\right] \tag{4.2}$$

$$\frac{4\pi r_0}{B_0}\cos y\; i_y = -(1+s)^{k-1}P_{(k+1)}^{(m)}\left[\frac{dV_c}{d\phi}\cos m\phi + \frac{dV_s}{d\varphi}\sin m\phi\right]$$

$$i_\phi = 0$$

Divergence

$$\frac{\nabla\cdot\vec{B}}{B_0} = \frac{m}{\cos^2 y}(1+s)^{(k-1)}P_{(k+1)}^{(m)}\left[-\frac{dV_c}{d\varphi}\sin m\phi + \frac{dV_s}{d\phi}\cos m\phi\right] \tag{4.3}$$

Divergence condition:

$$-\frac{dV_c}{d\varphi}\sin m\phi + \frac{dV_s}{d\varphi}\cos m\phi = 0 \tag{4.4}$$

We shall also write

$$\frac{dV_c}{d\phi}\cos m\phi + \frac{dV_s}{d\phi}\sin m\phi = N \tag{4.5}$$

and require that N be a constant in each region of type (ϕ).

Then, we find for the current density:

$$\frac{4\pi r_0}{B_0}\cos y\; i_r = N(1+s)^{k-1}\frac{dP_{(k+1)}^{(m)}}{dy}\sin m\phi$$

$$\frac{4\pi r_0}{B_0}\cos y = -N(1+s)^{k-1}P_{(k+1)}^{(m)}\sin m\phi \tag{4.6}$$

$$i_\phi = 0$$

so that the current density is independent of ϕ.

From (4.4) and (4.5), one finds

$$\frac{dV_c}{d\varphi} = N\cos m\phi \;;\; \frac{dV_s}{d\phi} = N\sin m\phi \tag{4.7}$$

and integrating (4.7), one has

$$V_c = \frac{N}{m} \sin m\phi + N_{C,0} \tag{4.8}$$

$$= - \frac{N}{m} \cos m\phi + N_{S,0} \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.1), we find

$$\frac{B_r}{B_0} = (k+1)\,(1+s)^k P^{(m)}_{(k+1)} \left[N_{C,0} \cos m\phi + N_{S,0} \sin m\phi \right]$$

$$\frac{B_y}{B_0} = (1+s)^k \,\frac{d P^{m}_{k+1}}{dy} \left[N_{C,0} \cos m\phi + N_{S,0} \sin m\phi \right] \tag{4.10}$$

$$\frac{B_\phi \cos y}{B_0} = m(1+s)^k P^{(m)}_{(k+1)} \left[- N_{C,0} \sin m\phi + N_{S,0} \cos m\phi \right] - N(1+s)^k P^{(m)}_{k+1}$$

and therefore $\vec{B} = \vec{B}^{(1)} + \vec{B}^{(2)}$, $\nabla \times \vec{B}^{(1)} = 0$, $\vec{B}^{(2)} \cdot \vec{l} = 0$

$$\text{and} \quad \vec{B}^{(2)} = - \vec{e}_\phi N(1+s)^k P^{(m)}_{k+1} \tag{4.11}$$

analogous to the situations in regions of type (y) and of type (s).

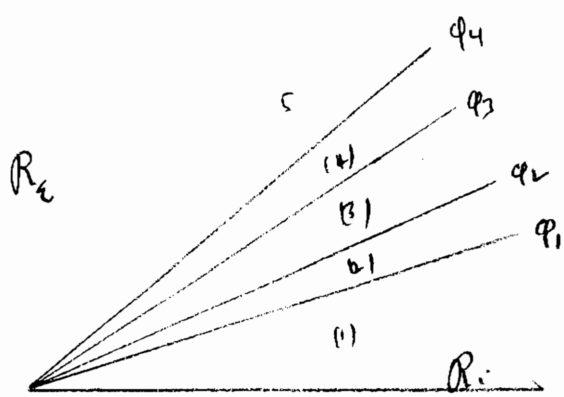

Figure 8

Let us now consider R_1 and R_2 separated by three type (ϕ) regions: (2), (3) and (4). This is illustrated in Figure 8, which represents a cross section through the median plane. It should be noted that R_1 and R_3 must be separated at at other values of ϕ . Using the same methods as were employed in regions of type (y) and type (s), we find the two equations:

$$N^{(4)}[\cos m\phi_4 - \cos m\phi_3] + N^{(3)}[\cos m\phi_3 - \cos m\phi_2] + N^{(2)}[\cos m\phi_2 - \cos m\phi_1] = m$$

$$N^{(4)}[\sin m\phi_4 - \sin m\phi_3] + N^{(3)}[\sin m\phi_3 - \sin m\phi_2] + N^{(2)}[\sin m\phi_2 - \sin m\phi_1] = 0 \qquad (4.12)$$

which the $N^{(m)}$ must satisfy. It follows that no more than two type (ϕ) regions are required, and for special values of ϕ , only one may be required. This case is illustrated in Figure 9. In this case, equations (4.12) become:

$$N^{(2)}[\cos m\phi_2 - \cos m\phi_1] = +m \qquad (4.13)$$

$$N^{(2)}[\sin m\phi_2 - \sin m\phi_1] = 0 \qquad (4.14)$$

From (4.14) it follows:

$$\sin m\phi_2 = \sin m\phi_1 \qquad (4.15)$$

Consider the special case when $m\phi_1$ is in the first quadrant. Then

$$m\phi_2 = \pi - m\phi_1 \qquad (4.16)$$

Then from (4.13),

$$N^{(2)} = -\frac{m}{2\cos m\phi_1} \qquad (4.17)$$

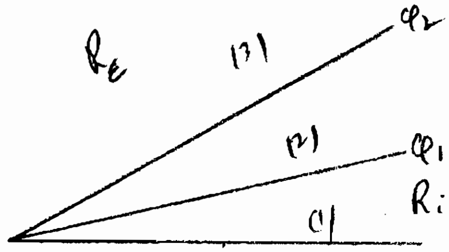

Figure 9

The boundary conditions for region (2) are

$$V_c^{(2)}(\varphi_1) = 0 \quad , \quad V_s^{(2)}(\varphi_1) = 1$$
$$V_c^{(2)}(\varphi_2) = 0 \quad , \quad V_s^{(2)}(\varphi_2) = 0 \qquad (4.18)$$

Then $N_{c,0} = \frac{1}{2}\frac{\sin m\phi_1}{\cos m\phi_1}$, $N_{s,0} = \frac{1}{2}$ (4.19)

Substituting (4.19) in (4.10), we find for the field

$$\frac{B_r}{B_0} = \frac{k+1}{2} (1+s)^k P^{(m)}_{(k+1)} \frac{\sin m(\phi_1+\phi)}{\cos m\phi_1}$$

$$\frac{B_y}{B_0} = (1+s)^k \frac{d}{dy} P^{(m)}_{(k+1)} \frac{\sin m(\phi_1+\phi)}{\cos m\phi_1}$$

$$\frac{B_\phi \cos y}{B_0} = \frac{m(1+s)^k}{2} P^{(m)}_{(k+1)} \frac{\cos m(\phi_1+\phi)}{\cos m\phi_1} + \frac{m}{2}(1+s)^k P^{(m)}_{(k+1)}$$

$$= \frac{m}{2}(1+s)^k P^{(m)}_{(k+1)} \left[\frac{\cos m(\phi_1+\phi)}{\cos m\phi_1} + 1\right]$$

V REGION OF TYPE (s, y)

$v_{(+)}$ and $v_{(-)}$ are functions of s

v_P and v_Q are functions of y

v_c and v_s are constants.

Field:
$$\frac{B_r}{B_0} = \left[v_P P^{(m)}_{(k+1)} + v_Q Q^{(m)}_{(k+1)}\right]\left[v_{(+)}(k+1)(1+s)^k - v_{(-)}(k+2)(1+s)^{-(k+3)}\right] \sin m\phi$$

$$\frac{B_y}{B_0} = \left[v_P \frac{d}{dy} P^{(m)}_{(k+1)} + v_Q \frac{d}{dy} Q^{(m)}_{(k+1)}\right]\left[v_{(+)}(1+s)^k + v_{(-)}(1+s)^{-(k+3)}\right] \sin m\phi \qquad (5.1)$$

$$\frac{B_\phi \cos y}{B_0} = m\left[v_P P^{(m)}_{(k+1)} + v_Q Q^{(m)}_{(k+1)}\right]\left[v_{(+)}(1+s)^k + v_{(-)}(1+s)^{-(k+3)}\right] \cos m\phi$$

Current Density

$$\frac{4\pi r_0}{B_0} \cos y \; i_r = -m\left[\frac{dv_P}{dy} P^{(m)}_{(k+1)} + \frac{dv_Q}{dy} Q^{(m)}_{(k+1)}\right]\left[v_{(+)}(1+s)^{k-1} + v_{(-)}(1+s)^{-(k+4)}\right] \cos m\phi$$

$$\frac{4\pi r_0}{B_0} \cos y \; i_y = m\left[v_P P^{(m)}_{(k+1)} + v_Q Q^{(m)}_{(k+1)}\right]\left[\frac{dv_{(+)}}{ds}(1+s)^k + \frac{dv_{(-)}}{ds}(1+s)^{-(k+3)}\right] \cos m\phi \qquad (5.2)$$

$$\frac{4\pi r_0}{B_0} i_\phi = \left[\frac{dv_P}{dy} P^{(m)}_{(k+1)} + \frac{dv_Q}{dy} Q^{(m)}_{(k+1)}\right]\left[v_{(+)}(k+1)(1+s)^{k-1} - v_{(-)}(k+2)(1+s)^{-(k+4)}\right] \sin m\phi$$

$$- \left[v_P \frac{d}{dy} P^{(m)}_{(k+1)} + v_Q \frac{d}{dy} Q^{(m)}_{(k+1)}\right]\left[\frac{dv_{(+)}}{ds}(1+s)^k + \frac{dv_{(-)}}{ds}(1+s)^{-(k+3)}\right] \sin m\phi$$

Divergence:

$$\frac{\nabla\cdot\vec{B}}{B_0} = \left[\frac{dv_P}{dy}\frac{dP^{(m)}_{(k+1)}}{dy} + \frac{dv_Q}{dy}\frac{dQ^{(m)}_{(k+1)}}{dy}\right]\left[v_{(+)}(1+s)^{k-1} + v_{(-)}(1+s)^{-(k+4)}\right]\sin m\phi$$
$$+\left[v_P P^{(m)}_{(k+1)} + v_Q Q^{(m)}_{(k+1)}\right]\left[\frac{dv_{(+)}}{ds}(k+1)(1+s)^{k} - \frac{dv_{(-)}}{ds}(k+2)(1+s)^{-(k+3)}\right] \quad (5.3)$$

The divergence condition can be written in the form

$$\frac{\frac{dv_P}{dy}\frac{dP^{(m)}_{(k+1)}}{dy} + \frac{dv_Q}{dy}\frac{dQ^{(m)}_{(k+1)}}{dy}}{v_P P^{(m)}_{(k+1)} + v_Q Q^{(m)}_{(k+1)}} = -\frac{\frac{dv_{(+)}}{ds}(k+1)(1+s)^{k} - \frac{dv_{(-)}}{ds}(k+2)(1+s)^{-(k+3)}}{v_{(+)}(1+s)^{k-1} + v_{(-)}(1+s)^{-(k+4)}} = f \quad (5.4)$$

where f is a constant. This follows since the left hand side of the equation is a function of y, and the right hand side is a function of s. We therefore obtain the two equations:

$$\frac{dv_P}{dy}\frac{dP^{(m)}_{(k+1)}}{dy} + \frac{dv_Q}{dy}\frac{dQ^{(m)}_{(k+1)}}{dy} = f\left[v_P P^{(m)}_{(k+1)} + v_Q Q^{m}_{k+1}\right] \quad (5.5)$$
$$\frac{dv_{(+)}}{ds}(k+1)(1+s)^{k} - \frac{dv_{(-)}}{ds}(k+2)(1+s)^{-(k+3)} = -f\left[v_{(+)}(1+s)^{k-1} + v_{(-)}(1+s)^{-(k+4)}\right]$$

From which, one obtains *

$$v_P = Q^{(m)}_{(k+1)}\left\{v_{P,0} + \frac{1}{C^{(m)}_{k+1}}\int dy \cos y\left[fK - \frac{dK}{dy}\frac{d\ln Q^{(m)}_{(k+1)}}{dy} + K\left(\frac{d\ln Q^{(m)}_{(k+1)}}{dy}\right)^2\right]\right\}$$

$$v_Q = \frac{K}{Q^{(m)}_{(k+1)}} - P^{(m)}_{(k+1)}\left\{v_{P,0} + \frac{1}{C^{(m)}_{k+1}}\int dy \cos y\left[fK - \frac{dK}{dy}\frac{d\ln Q^{(m)}_{(k+1)}}{dy} + K\left(\frac{d\ln Q^{m}_{k+1}}{dy}\right)^2\right]\right\}$$

* See Appendix I - II

where $\quad \mathcal{V}_p P^{(m)}_{(k+1)} + \mathcal{V}_Q Q^{(m)}_{(k+1)} = K, \quad$ and $\mathcal{V}_{P,0}$ is an arbitrary constant.

$$\mathcal{V}_{(+)} = (1+s)^{-(k+2)}\left\{\mathcal{V}_{+0} + \int ds\left[L(1+s)^{2}\frac{(k+2)(k+4)-f}{2k+3} + \frac{dL}{ds}(1+s)^{3}\frac{k+2}{2k+3}\right]\right\}$$

$$\mathcal{V}_{(-)} = L(1+s)^{k+4} - (1+s)^{k+1}\left\{\mathcal{V}_{(+0)} + \int ds\left[L(1+s)^{3}\frac{(k+2)(k+4)-f}{2k+3} + \frac{dL}{ds}(1+s)^{3}\frac{k+2}{2k+3}\right]\right\}$$

where $\mathcal{V}_+(1+s)^{k-1} + \mathcal{V}_{(-)}(1+s)^{-(k+4)} = L$, and $\mathcal{V}_{(+0)}$ is an arbitrary constant.

As soon as functions K and L have been chosen, the above equations determine ($\mathcal{V}_P, \mathcal{V}_Q, \mathcal{V}_{(+)}, \mathcal{V}_{(-)}$) in terms of the arbitrary constants ($\mathcal{V}_{P,0}, \mathcal{V}_{(+,0)}, f$), and arbitrary constants occuring in the functions K and L. When one has the problem of finding a current distribution in a region of type (y, s), one generally knows the values of ($\mathcal{V}_P, \mathcal{V}_Q, \mathcal{V}_{(+)}, \mathcal{V}_{(-)}$) on the boundary of the region. Sufficient number of arbitrary constants should be introduced in the functions K and L, so that there are a sufficient number of arbitrary constants to satisfy the boundary conditions. One must also choose the functions K and L in such a way as to advoid singularities in ($\mathcal{V}_P, \mathcal{V}_Q, \mathcal{V}_{(+)}, \mathcal{V}_{(-)}$) in the region considered.

VI REGION OF TYPE (s, y, ϕ)

$\mathcal{V}_{(+)}$ and $\mathcal{V}_{(-)}$ are functions of s

$\mathcal{V}_P$ and $\mathcal{V}_Q$ are functions of y

$\mathcal{V}_L$ and $\mathcal{V}_S$ are functions of ϕ

Field - given by (1.9)

Current density - given by (1.1

Divergence - given by (1.10)

The divergence conditions can be written in the form:

$$0 = \frac{\frac{dV_p}{dy}\frac{dP^{(m)}_{(k+1)}}{dy} + \frac{dV_Q}{dy}\frac{dQ^{(m)}_{(k+1)}}{dy}}{V_p P^{(m)}_{(k+1)} + V_Q Q^{m}_{k+1}} + \frac{\frac{dV_{(+)}}{ds}(k+1)(1+s)^{k} - \frac{dV_{(-)}}{ds}(k+2)(1+s)^{-(k+3)}}{V_{(+)}(1+s)^{k-1} + V_{(-)}(1+s)^{-(k+4)}} + \frac{m}{\cos^2 y}\,\frac{-\frac{dV_c}{d\phi}\sin m\phi + \frac{dV_s}{d\phi}\cos m\phi}{V_c \cos m\phi + V_s \sin m\phi} \qquad (6.1)$$

(s, y, ϕ)

It follows that

$$\frac{\frac{dV_{(+)}}{ds}(k+1)(1+s)^{k} - \frac{dV_{(-)}}{ds}(k+2)(1+s)^{-(k+3)}}{V_{(+)}(1+s)^{k-1} + V_{(-)}(1+s)^{-(k+4)}} = f_s \qquad (6.2)$$

$$\frac{\frac{dV_p}{dy}\frac{dP^{(m)}_{(k+1)}}{dy} + \frac{dV_Q}{dy}\frac{dQ^{(m)}_{(k+1)}}{dy}}{V_p P^{(m)}_{(k+1)} + V_Q Q^{(m)}_{(k+1)}} = \frac{f_y}{\cos^2 y} - f_s \qquad (6.3)$$

$$\frac{-\frac{dV_c}{d\phi}\sin m\phi + \frac{dV_s}{d\phi}\cos m\phi}{V_c \cos m\phi + V_s \sin m\phi} = -\frac{f_y}{m} \qquad (6.4)$$

where f_s and f_y are constants.

From which, we obtain*

$$V_{(+)} = (1+s)^{-(k+2)}\left\{V_{(+,0)} + \int ds\left[L(1+s)^{2}\frac{(k+2)(k+4)+f_s}{2k+3} + \frac{dL}{ds}(1+s)^{3}\frac{k+2}{2k+3}\right]\right\}$$

$$V_{(-)} = L(1+s)^{(k+4)} - (1+s)^{(k+1)}\left\{V_{(+,0)} + \int ds\left[L(1+s)^{2}\frac{(k+2)(k+4)+f_s}{2k+3} + \frac{dL}{ds}(1+s)^{3}\frac{k+2}{2k+3}\right]\right\}$$

*See Appendix I - II - III

$$v_P = P^{(m)}_{(k+1)} \left\{ v_{P,0} + \frac{1}{Q^{(m)}_{(k+1)}} \int dy \cos y \left[\left(\frac{f_y}{\cos^2 y} - f_s \right) K - \frac{dK}{dy} \frac{d \ln Q^{(m)}_{(k+1)}}{dy} + K \left(\frac{d \ln Q^{(m)}_{(k+1)}}{dy} \right)^2 \right] \right\}$$

$$v_Q = \frac{K}{Q^{(m)}_{k+1}} - P^{(m)}_{k+1} \left\{ v_{P,0} + \frac{1}{Q^{(m)}_{(k+1)}} \int dy \cos y \left[\left(\frac{f_y}{\cos^2 y} - f_s \right) K - \frac{dK}{dy} \frac{d \ln Q^{(m)}_{(k+1)}}{dy} + K \left(\frac{d \ln Q^{(m)}_{(k+1)}}{dy} \right)^2 \right] \right\}$$

$$v_c = \sin m\phi \left\{ v_{c,0} + \int \frac{d\phi}{\sin m\phi} \left[-\frac{f_y}{m} N \sin m\phi - m N \frac{\cos^2 m\phi}{\sin m\phi} + \frac{dN}{d\phi} \cos m\phi \right] \right\}$$

$$v_s = \frac{N}{\sin m\phi} - \cos m\phi \left\{ v_{c,0} + \int \frac{d\phi}{\sin m\phi} \left[-\frac{f_y N}{m} \sin m\phi - m N \frac{\cos^2 m\phi}{\sin m\phi} + \frac{dN}{d\phi} \cos m\phi \right] \right\}$$

The above equations determine the functions ($v_P, v_Q, v_{(+)}, v_{(-)}, v_c, v_s$) in terms of the arbitrary functions K, L, N, and the arbitrary constants f_y, f_s. Sufficient number of constants should be introudced in the functions K, L and N, so that the boundary conditions can be satisfied. The K, L and N must also be chosen so as to advoid singularities in v_s in the region under consideration.

VII CONCLUDING REMARKS

(1) The relations that have been derived in this report are exact. However, in working out particular cases, there may be mathematical difficulties in performing some of the calculations. In particular the integral occuring in (8.5) might be mentioned. In the case of such a difficulty, approximate results might be obtained as follows. Replace the $P^{(m)}_{(k+1)}$ and $Q^{(m)}_{(k+1)}$ by $L^{(k)}_{(c)(m+1)}$ and $L^{(k)}_{(s)(m+1)}$ respectively, and use the following approximations:

$$L^{(m)}_{(c)(k+1)} \simeq \cos wy, \quad L^{(m)}_{(s)(k+1)} \simeq \sin wy \qquad \cos y = 1$$

Then $$L^{(m)}_{(0)(k+1)} \frac{d L^{(m)}_{(\xi)k+1}}{dy} - L^{(m)}_{(\xi)(k+1)} \frac{d L^{(m)}_{0(k+1)}}{dy} \simeq -\omega \qquad (7.1)$$

Equation (7.1) replaces (2.5) and it will follow that if $C^{(m)}_{(k+1)}$ is replaced by $-\omega$ in any of the relations of this report, that valid results to this degree of approximation will be obtained. Futhermore, these results will be valid for $m+k+1$ even as well as odd, and in fact for nonintegeral values of k and m. This approximation is valid for large values of $|\omega y|$ and small values of y . The results should be quite accurate for large accelerators. For small accelerators, a better approximation may be required.

(2) It should be noted that regions of type ($\xi\phi$) and type (y, ϕ) have not been discussed. All problems that have been contemplated so far may be solved without their use. If, however, relations for these regions are required, they may easily be obtained using methods analogous to those used for a region of type (ξy) together with the results of the appendices.

(3) In another reference, another method was used for reducing the differential equations that appear in the appendix. In this method, these differential were solved, so that v_p was expressed as a function of v_ϕ , $v_{(+)}$ as a function of $v_{(-)}$, and v_c as a function of v_ξ . The functions (v_ϕ, $v_{(-)}$ v_ξ) could be chosen arbitrarily to a certain extent, and the (v_p, $v_{(+)}$ v_c) computed. One has to be careful to choose the (v_ϕ $v_{(-)}$ v_ξ) so that singularities do not appear in the (v_p, $v_{(+)}$ v_c).

The method used in this report would appear to be superior. It might be called the parametre method, the auxiliary functions (K, L, N) being the parameters.

(4) The fields and currents are determined uniquely in the regions of types (y), (s) and (ϕ). This is achieved by the introduction of auxiliary conditions. These conditions result in current distributions that are relatively simple, and should make the winding of coils quite practical. In the regions of type (s, y), and (s, y, ϕ) there is a great deal of freedom in the fields and current distributions. This freedom should be utilized to make current density distribution that would be practical for the winding of coils. Little is known at present about this. One would expect that the most practical current distribution would have a mathematically simple expression.

VIII APPENDIX I Reduction of the Differential Equation

$$\frac{\frac{dv_P}{dy}\frac{dP^{(m)}_{k+1}}{dy} + \frac{dv_Q}{dy}\frac{dQ^{(m)}_{k+1}}{dy}}{v_P P^{(m)}_{k+1} + v_Q Q^{m}_{k+1}} = g(y) \tag{8.1}$$

where $g(y)$ is a given function of y. Let us define K by:

$$v_P P^{(m)}_{(k+1)} + v_Q Q^{(m)}_{(k+1)} = K \tag{8.2}$$

then $$\frac{dv_P}{dy}\frac{dP^{(m)}_{k+1}}{dy} + \frac{dv_Q}{dy}\frac{dQ^{(m)}_{k+1}}{dy} = gK \tag{8.3}$$

Solving (8.2) for v_Q, substituting this value of v_Q into (8.3) and making use of (2.5), we find:

$$\frac{dv_P}{dy} - v_P \frac{d \ln Q^{(m)}_{k+1}}{dy} = \frac{\cos y}{C^{(m)}_{k+1}} \left[g K Q^{(m)}_{(k+1)} - \frac{dK}{dy} \frac{d Q^{(m)}_{k+1}}{dy} + K Q^{(m)}_{(k+1)} \left(\frac{d \ln Q^{(m)}_{k+1}}{dy} \right)^2 \right] \tag{8.4}$$

Noting that the left hand side of this equation has the integrating factor $\frac{1}{Q^{(m)}_{(k+1)}}$, we are able to integrate and we find

$$v_P = Q^{(m)}_{k+1} \left\{ v_{P,0} + \frac{1}{C^{(m)}_{(k+1)}} \int dy \cos y \left[g K - \frac{dK}{dy} \frac{d Q^{(m)}_{(k+1)}}{dy} + K \left(\frac{d \ln Q^{(m)}_{k+1}}{dy} \right)^2 \right] \right. \tag{8.5}$$

and from (8.2), we find

$$v_Q = \frac{K}{Q^{(m)}_{k+1}} - P^{(m)}_{k+1} \left\{ v_{P,0} + \frac{1}{C^{(m)}_{(k+1)}} \int dy \cos y \left[g K - \frac{dK}{dy} \frac{d Q^{(m)}_{k+1}}{dy} + K \left(\frac{d \ln Q^{(m)}_{k+1}}{dy} \right)^2 \right] \right. \tag{8.6}$$

where $v_{P,0}$ is a constant.

IX APPENDIX II Reduction of the Differential Equation

$$\frac{\frac{dv_{(+)}}{ds}(k+1)(1+s)^k - d v_{(-)}(k+2)(1+s)^{-(k+2)}}{v_{(+)}(1+s)^{k-1} + v_{(-)}(1+s)^{-(k+4)}} = f \tag{9.1}$$

Define L thus: (9.2)

$$v_+ (1+s)^{(k-1)} + v_{(-)} (1+s)^{-(k+4)} = L$$

Then,

$$\frac{dv_{(+)}}{ds}(k+1)(1+s)^{k} - \frac{dv_{(-)}}{ds}(k+2)(1+s)^{-(k+2)} = fL \tag{9.3}$$

Solve (9.2) for $v_{(-)}$, substitute in (9.3), and making use of (2.5) we find

$$\frac{dv_{(+)}}{ds} + \frac{(k+2)}{1+s}v_{(+)} = \frac{fL}{2k+3}(1+s)^{(1-k)} + L\frac{(k+2)(k+4)}{2k+3}(1+s)^{-k} + \frac{k+2}{2k+3}\frac{dL}{ds}(1+s)^{1-k} \tag{9.4}$$

Noting that the left hand side of this equation has the integrating factor $(1+s)^{(k+2)}$, we are able to integrate and find,

$$v_{(+)} = (1+s)^{-(k+2)}\left\{ v_{(+,0)} + \int ds\left[L(1+s)^{2}\frac{(k+2)(k+4)+f}{2k+3} + \frac{dL}{ds}(1+s)^{3}\frac{k+2}{2k+3}\right]\right. \tag{9.5}$$

and using (9.2) we find

$$v_{(-)} = L(1+s)^{(k+4)} - (1+s)^{k+1}\left\{v_{(+,0)} + \int ds\left[L(1+s)^{2}\frac{(k+2)(k+4)+f}{2k+3} + \frac{dL}{ds}(1+s)^{3}\frac{k+2}{2k+3}\right\}\right. \tag{9.6}$$

where $v_{+,0}$ is an arbitrary constant.

X APPENDIX III Reduction of the Differential Equation

$$\frac{-\frac{dv_c}{d\phi}\sin m\phi + \frac{dv_s}{d\phi}\cos m\phi}{v_c\cos m\phi + v_s\sin m\phi} = f \tag{10.1}$$

Define N thus:

$$v_c \cos\phi + v_s \sin m\phi = N \tag{10.2}$$

Then: $$-\frac{dV_c}{d\phi}\sin m\phi + \frac{dV_s}{d\phi}\cos m\phi = f N \tag{10.3}$$

Solving (10.2) for V_s, one finds

$$V_s = \frac{N - V_c \cos m\phi}{\sin m\phi} \tag{10.4}$$

Substituting (10.4) in (10.3) and making use of 2.5, we find

$$\frac{dV_c}{d\phi} - m\cot m\phi \; V = f N \sin m\phi - m N \frac{\cos^2 m\phi}{\sin m\phi} + \frac{dN}{d\phi}\cos m\phi \tag{10.5}$$

Noting that the left hand side of this equation has the integrating factor $\frac{1}{\sin m\phi}$, we are able to integrate and find:

$$V_c = \sin m\phi \left\{ V_{c,0} + \int \frac{d\phi}{\sin m\phi}\left[f N \sin m\phi - m N \frac{\cos^2 m\phi}{\sin m\phi} + \frac{dN}{d\phi}\cos m\phi \right] \tag{10.6}$$

Substituting (10.6) in (10.4), we find

$$V_s = \frac{N}{\sin m\phi} - \cos m\phi \left\{ V_{c,0} + \int \frac{d\phi}{\sin m\phi}\left[f N \sin m\phi - m N \frac{\cos^2 m\phi}{\sin m\phi} + \frac{dN}{d\phi}\cos m\phi \right] \right\} \tag{10.7}$$