Prof. Y. W. Lee
Prof. A. G. Bose
Prof. I. M. Jacobs
Prof. D. J. Sakrison
Prof. M. Schetzen
Prof. H. L. Van Trees, Jr.
V. R. Algazi
D. F. De Long, Jr.
C. L. Feldman
H. Hemami
W. S. Widnall

A. MEASUREMENT OF THE KERNELS OF A NONLINEAR SYSTEM BY CROSSCORRELATION WITH GAUSSIAN NON-WHITE INPUTS

In the Wiener theory a nonlinear system is characterized by the kernels, h_n, of the functionals, G_n.[1] For a Gaussian white-noise input, the functionals, G_n, are orthogonal and their kernels, h_n, can be determined by crosscorrelating the output with a multi-dimensional delay of the input.[2] In this report a procedure is developed for determining the kernels of a set of functionals which are orthogonal for a Gaussian non-white input process.

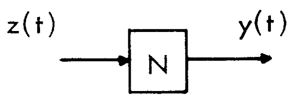

Fig. VIII-1. Nonlinear system with Gaussian non-white noise input.

Consider that the system N shown in Fig. VIII-1 is to be characterized with an input, z(t), which is a Gaussian non-white process. We shall assume that the power density spectrum, $\Phi_{zz}(\omega)$, of the input, z(t), is factorizable.[3] It then can be written

$$\Phi_{zz}(\omega) = \Phi^{+}_{zz}(\omega)\, \Phi^{-}_{zz}(\omega) \tag{1}$$

in which $\Phi^{+}_{zz}(\omega)$ is the complex conjugate of $\Phi^{-}_{zz}(\omega)$; also, all of the poles and zeros of $\Phi^{+}_{zz}(\omega)$ are in the upper half of the complex λ-plane. Thus $\Phi^{+}_{zz}(\omega)$ and $1/\Phi^{+}_{zz}(\omega)$ are each

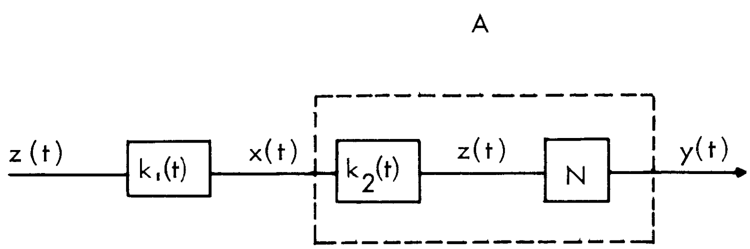

Fig. VIII-2. An equivalent form of the nonlinear system N.

realizable as the transfer function of a linear system. We can then consider the system of Fig. VIII-1 in the equivalent form shown in Fig. VIII-2, in which the transfer functions of the two linear systems, $k_1(t)$ and $k_2(t)$, are:

$$\left.\begin{aligned} K_1(\omega) &= \frac{1}{\Phi^+_{zz}(\omega)} \\ K_2(\omega) &= \Phi^+_{zz}(\omega) \end{aligned}\right\} \tag{2}$$

Also, as shown, the system A is the system formed by the tandem connection of the linear system $k_2(t)$ and the system N. We observe that the input to the system A is a Gaussian white process whose power density spectrum is 1 watt per radian per second. Thus, as previously described,[2] the kernels, h_n, of the orthogonal functionals, G_n, for the system A are

$$h_n(\tau_1, \ldots, \tau_n) = \frac{(2\pi)^n}{n!} \overline{y(t)\, x(t-\tau_1) \ldots x(t-\tau_n)} \tag{3}$$

except when, for $n > 2$, two or more τ's are equal.

We therefore need to know the crosscorrelation function

$$\phi_{yx^n}(\tau_1, \ldots, \tau_n) = \overline{y(t)\, x(t-\tau_1) \ldots x(t-\tau_n)} \tag{4}$$

in order to determine the kernels, h_n, of the system A. Since only z(t) is available to us, we shall express the desired correlation function in terms of the corsscorrelation between the output, y(t), and a multidimensional delay of the input, z(t). By substituting the relation

$$x(t) = \int_0^\infty k_1(\sigma)\, z(t-\sigma)\, d\sigma \tag{5}$$

in Eq. 4, the desired correlation function can be expressed as

$$\phi_{yx^n}(\tau_1, \ldots, \tau_n) = \int_0^\infty k_1(\sigma_1)\, d\sigma_1 \ldots \int_0^\infty k_1(\sigma_n)\, d\sigma_n \phi_{yz^n}(\tau_1-\sigma_1, \ldots, \tau_n-\sigma_n) \tag{6}$$

in which

$$\phi_{yz^n}(\tau_1, \ldots, \tau_n) = \overline{y(t)\, z(t+\tau_1) \ldots z(t+\tau_n)} \tag{7}$$

is the crosscorrelation between the output, y(t), and a multidimensional delay of the input, z(t). In the frequency domain, Eq. 6 can be expressed as

$$\Phi_{yx^n}(\omega_1, \ldots, \omega_n) = K_1(\omega_1)\, K_1(\omega_2) \ldots K_1(\omega_n)\, \Phi_{yz^n}(\omega_1, \ldots, \omega_n) \tag{8}$$

in which

$$\Phi_{yz^n}(\omega_1, \ldots, \omega_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} e^{-j\omega_1\tau_1} d\tau_1 \ldots \int_{-\infty}^{\infty} e^{-j\omega_n\tau_n} d\tau_n \phi_{yz^n}(\tau_1, \ldots, \tau_n) \tag{9}$$

and the transfer function $K_1(\omega)$ is given by

$$K_1(\omega) = \int_{-\infty}^{\infty} k_1(t)\, e^{-j\omega t}\, dt \tag{10}$$

Substituting Eq. 2 in Eq. 8, we have the desired relation in the frequency domain.

$$\Phi_{yx^n}(\omega_1, \ldots, \omega_n) = \frac{\Phi_{yz^n}(\omega_1, \ldots, \omega_n)}{\Phi_{zz}^+(\omega_1) \ldots \Phi_{zz}^+(\omega_n)}. \tag{11}$$

Either Eq. 6 or Eq. 11 can be used to determine the kernels, h_n, as given by Eq. 3 in terms of the measured crosscorrelation function between the output and a multidimensional delay of the input, z(t).

Once the kernels h_n have been determined, a representation of the system N is as given in Fig. VIII-3a which can be redrawn as shown in Fig. VIII-3b. We note from Fig. VIII-3b that the outputs of the parallel branches are orthogonal for the input z(t). Thus, we have expanded the nonlinear system N in a set of functionals that are orthogonal for Gaussian inputs with a power density spectrum of $\Phi_{zz}(\omega)$. Note that for this procedure, we never need construct either of the linear systems $k_1(t)$ or $k_2(t)$. Once the representation in the form of Fig. VIII-3b is known, the functionals that are orthogonal for the input z(t) can be calculated. We shall call these functionals L_n. The first few functionals are:

$$L_1[h_1, z(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\sigma_1)\, k_1(\tau_1-\sigma_1)\, x(t-\tau_1)\, d\sigma_1 d\tau_1 \tag{12}$$

$$L_2[h_2, z(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\sigma_1, \sigma_2)\, k_1(\tau_1-\sigma_1)\, k_1(\tau_2-\sigma_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\sigma_1 d\sigma_2 d\tau_1 d\tau_2$$

$$- 2\pi \int_{-\infty}^{\infty} h_2(\sigma_2, \sigma_2)\, d\sigma_2 \tag{13}$$

$$L_3[h_3, z(t)] = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_3(\sigma_1, \sigma_2, \sigma_3)\, k_1(\tau_1-\sigma_1)\, k_1(\tau_2-\sigma_2)\, k_1(\tau_3-\sigma_3)\, x(t-\tau_1)\, x(t-\tau_2)\, x(t-\tau_3)$$

$$d\sigma_1 d\sigma_2 d\sigma_3 d\tau_1 d\tau_2 d\tau_3$$

$$- 3(2\pi) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_3(\sigma_1, \sigma_2, \sigma_2)\, k_1(\tau_1-\sigma_1)\, x(t-\tau_1)\, d\sigma_1 d\sigma_2 d\tau_1 \tag{14}$$

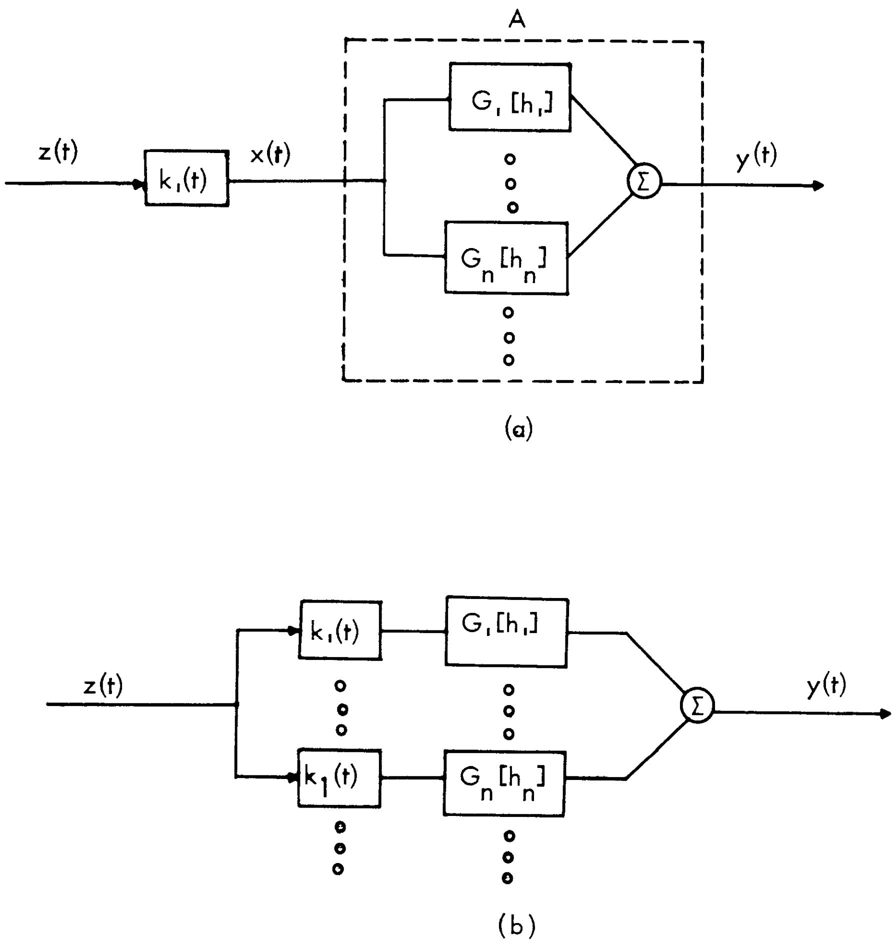

Fig. VIII-3. Representation of the expansion for system N.

in which

$$k_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{j\omega t}}{\Phi_{zz}^{+}(\omega)} d\omega. \tag{15}$$

In terms of these functionals, the output of the system N for an input z(t) can be written

$$y(t) = \sum_{n=1}^{\infty} L_n[h_n, z(t)]. \tag{16}$$

In this manner, we have characterized the nonlinear system N with a Gaussian non-white input in terms of a set of functionals, L_n, that are orthogonal for the given input process.

M. Schetzen

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2. Y. W. Lee and M. Schetzen, Measurement of the kernels of a nonlinear system by crosscorrelation, Quarterly Progress Report No. 60, Research Laboratory of Electronics, M.I.T., January 15, 1961, pp. 118-130.

3. Y. W. Lee, Statistical Theory of Communication (John Wiley and Sons, Inc., New York, 1960).