

## VI. MICROWAVE ELECTRONICS\*

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### A. EXACT SOLUTION OF THE SMALL-SIGNAL, ONE-DIMENSIONAL, GAP INTERACTION FOR NEUTRALIZED, RELATIVISTIC, ELECTRON BEAMS

In our studies of gap interaction in electron-beam waveguides, we have achieved a formulation and solution to the problem of a circuit field weakly coupled to an electron-beam waveguide for arbitrary relative charge densities ( $\omega_p/\omega$ ) in the beam.<sup>1, 2</sup> The exact solution (that is, without the weak coupling assumption) of the two-dimensional gap interaction problem, although it is readily formulated, requires a considerable amount of computation for evaluating the results.

The one-dimensional problem, on the other hand, can be solved and evaluated exactly, and also under the assumption of weak coupling. We have already presented the weak-coupling solution of the one-dimensional gap interaction problem, for arbitrary ( $\omega_p/\omega$ ).<sup>3</sup> In this report we shall give the exact solution for arbitrary ( $\omega_p/\omega$ ), and compare the results with those of the weak-coupling theory.

The system under consideration consists of a cold, collision-free, electron stream that has its motion constrained to one direction only, the z-direction; this constraint is

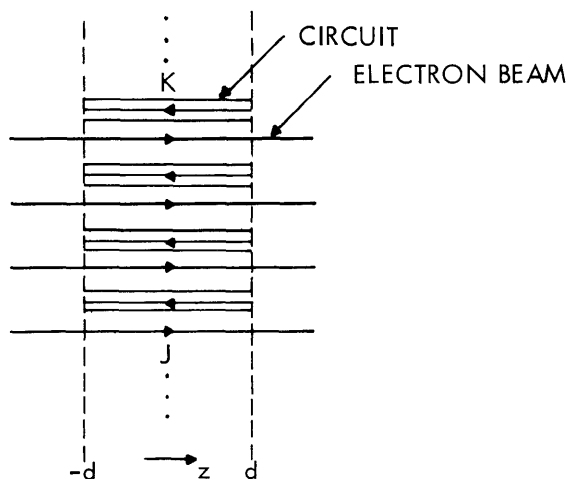


Fig. VI-1. One-dimensional electron-beam and gap circuit.

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assumed to be provided by a z-directed magnetic focusing field of infinite strength. Furthermore, we assume that: (a) the electron stream is uniform in all planes transverse to the z-direction; (b) in the absence of perturbations the space charge is neutralized by stationary ions that are unaffected by perturbations; and (c) in the unperturbed state the electron stream has a uniform velocity  $v_o$ , and a uniform charge density  $\rho_o$ . Consistently with the one-dimensional character of the electron beam, the gap circuit must also be chosen as one-dimensional. Thus we assume a circuit that is uniformly distributed throughout space in a region  $|z| < d$ , as shown in Fig. VI-1, having a uniformly distributed current density  $K(\text{amp}/\text{m}^2)$  flowing through it. The electron beam is assumed to pass, without interception, through this "permeable" distributed circuit. Only the small-signal interaction problem will be considered here.

We assume a time dependence of  $e^{j\omega t}$  for all small-signal quantities and proceed to describe the small-signal system for  $|z| < d$ . From Maxwell's equation of Ampere's law we have

$$j\omega\epsilon_o E(z) + J(z) = K. \quad (1)$$

From the relativistic small-signal force equation and Eq. 1 we obtain

$$\left(j\beta_e + \frac{\partial}{\partial z}\right) U(z) = jZ_o\beta_p J(z) + \frac{K}{j\omega\epsilon_o}; \quad (2)$$

and from the small-signal equation of conservation of charge we obtain

$$\left(j\beta_e + \frac{\partial}{\partial z}\right) J(z) = jY_o\beta_p U(z). \quad (3)$$

In Eqs. 2 and 3 we have introduced the following symbols and auxiliary definitions:

$$\beta_e = \frac{\omega}{v_o} \quad (4)$$

$$\beta_p = \frac{\omega_p}{v_o} \quad (5)$$

$$\omega_p^2 = \frac{e\rho_o}{m_\ell\epsilon_o} \quad (6)$$

$$m_\ell = m_o R^3 \quad (7)$$

$$R = \left[1 - \left(\frac{v_o}{c}\right)^2\right]^{-1/2} \quad (8)$$

$$Y_o = \omega\epsilon_o\beta_p = \frac{|J_o|}{R(R+1)} \frac{\omega}{v_o} \frac{1}{\omega_p} \quad (9)$$

$$U(z) = -\frac{m_0}{e} v_0 v(z) \quad (10)$$

where  $v(z)$  is the complex amplitude of the small-signal electron velocity,  $e$  is the electron charge, and  $m_0$  is the electron rest mass.

Equations 2 and 3 can be regarded as the equations describing the beam driven by the circuit. An equation describing the circuit is obtained by integrating Eq. 1 over the gap length, which gives,

$$j\omega c_0 V_c - K_i = K, \quad (11)$$

where

$$V_c = \int_{-d}^d E(z) dz \quad (12)$$

$$K_i = \frac{1}{2d} \int_{-d}^d J(z) dz \quad (13)$$

$$c_0 = \frac{\epsilon_0}{2d} \quad (14)$$

The circuit relationship of Eq. 11 is illustrated in Fig. VI-2.

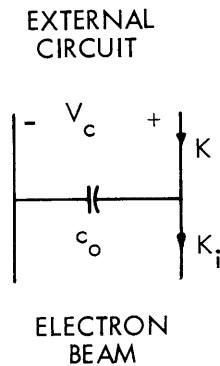


Fig. VI-2. The circuit variables of Eq. 11.

The interaction described by Eqs. 2, 3, 11, 12, and 13 is that of a linear three-port. Two of the ports are associated with the electron beam at  $z = -d$  and  $z = d$ , and the third port is the circuit. Because of the assumed unidirectional flow of the unperturbed beam, all signals on a one-dimensional beam travel in this same direction. Hence for the electron beam the independent variables are the beam excitation at  $z = -d$ ,

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$$\underline{\underline{B}}_1 = \begin{bmatrix} U_1 \\ J_1 \end{bmatrix}, \quad (15)$$

and the dependent variables are the beam excitations at  $z = d$ ,

$$\underline{\underline{B}}_2 = \begin{bmatrix} U_2 \\ J_2 \end{bmatrix}. \quad (16)$$

At the circuit terminals we have a freer choice of dependent and independent variables. For the three quantities in Eq. 11 we have a total choice of six sets of variables. The transformation from any set of variables to any other is obtained with the aid of Eq. 11. We shall first choose one set of circuit variables which will give the simplest form for the solution, and then transform to another set of circuit variables for which this exact solution can be easily interpreted and also easily compared with the weak-coupling solution obtained before.

Consider  $(K/j\omega c_o)$  to be the independent circuit variable, and  $K_1$  the dependent variable. Comparing Eqs. 2 and 3 with our previous report<sup>3</sup> we find that we can write the solution of Eqs. 2, 3, and 13 at once

$$\begin{bmatrix} \underline{\underline{B}}_2 \\ K_1 \end{bmatrix} = \begin{bmatrix} \underline{\underline{D}} & \underline{\underline{0}} \\ \underline{\underline{0}} & 1 \end{bmatrix} \begin{bmatrix} \underline{\underline{I}} & \underline{\underline{K}} \\ \underline{\underline{\Gamma}} & \underline{\underline{Y}} \end{bmatrix} \begin{bmatrix} \underline{\underline{D}} & \underline{\underline{0}} \\ \underline{\underline{0}} & 1 \end{bmatrix} \begin{bmatrix} \underline{\underline{B}}_1 \\ K/j\omega c_o \end{bmatrix} \quad (17)$$

where  $\underline{\underline{I}}$  is the two-by-two identity matrix, and

$$\underline{\underline{D}} = \begin{bmatrix} \cos \beta_p d & jZ_o \sin \beta_p d \\ jY_o \sin \beta_p d & \cos \beta_p d \end{bmatrix} e^{-j\beta_e d} \quad (18)$$

$$\underline{\underline{K}} = \begin{bmatrix} M \\ Y_o N \end{bmatrix} \quad (19)$$

$$\underline{\underline{\Gamma}} = [Y_o N \quad M] \quad (20)$$

$$Y = G + jB \quad (21)$$

$$G = Y_o MN = \frac{1}{4} Y_o (M_+^2 - M_-^2) \quad (22)$$

$$B = \frac{1}{4} Y_o \left[ \frac{M_+ \cos \beta_+ d - 1}{\beta_+ d} - \frac{M_- \cos \beta_- d - 1}{\beta_- d} \right] \quad (23)$$

$$M = \frac{1}{2}(M_+ + M_-) \quad (24)$$

$$N = \frac{1}{2}(M_- - M_+) \quad (25)$$

$$M_{\pm} = \frac{\sin \beta_{\pm} d}{\beta_{\pm} d} \quad (26)$$

$$\beta_{\pm} = \beta_e \mp \beta_p. \quad (27)$$

The circuit variables chosen for Eq. 17 are interpretable from a Thévenin equivalent of Fig. VI-2, as shown in Fig. VI-3. This makes them rather inconvenient. With the

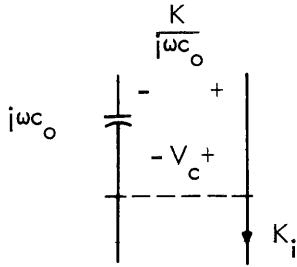


Fig. VI-3. Thévenin equivalent of the circuit in Fig. VI-2.

aid of Eq. 11 we can transform Eq. 17 so that the circuit variables are the more natural ones,  $K_i$  and  $V_c$ . We obtain

$$\begin{bmatrix} \underline{B}_2 \\ \underline{K}_i \end{bmatrix} = \begin{bmatrix} \underline{D} & \underline{0} \\ \underline{0} & 1 \end{bmatrix} \begin{bmatrix} \underline{G}_c & \underline{K}_c \\ \underline{\Gamma}_c & \underline{Y}_{el} \end{bmatrix} \begin{bmatrix} \underline{D} & \underline{0} \\ \underline{0} & 1 \end{bmatrix} \begin{bmatrix} \underline{B}_z \\ \underline{V}_c \end{bmatrix} \quad (28)$$

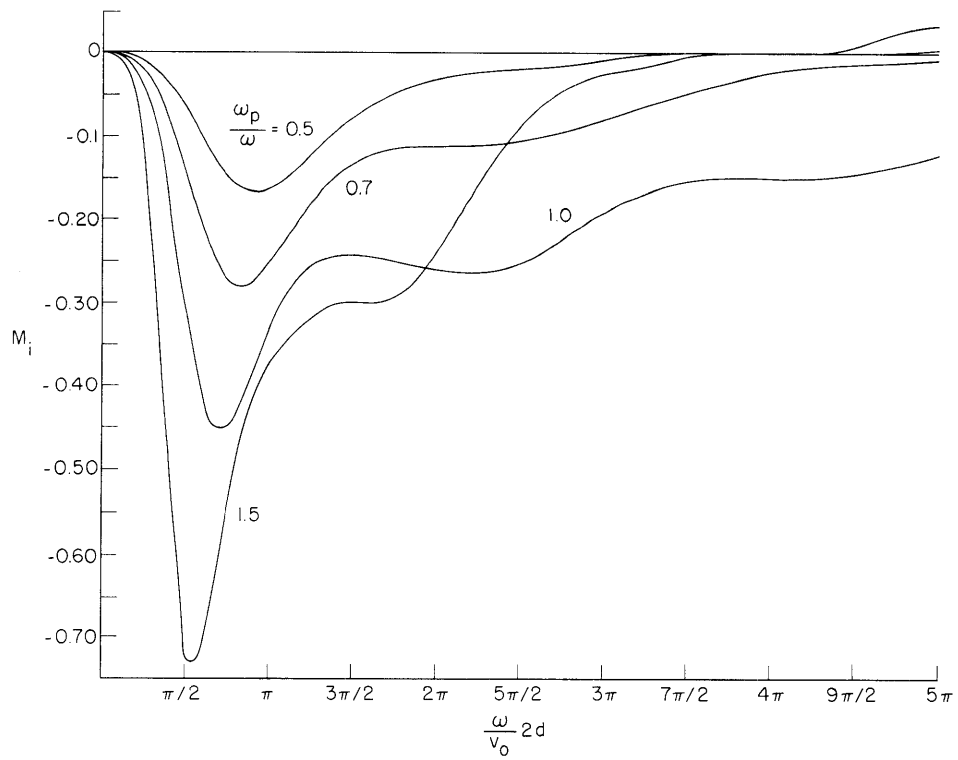
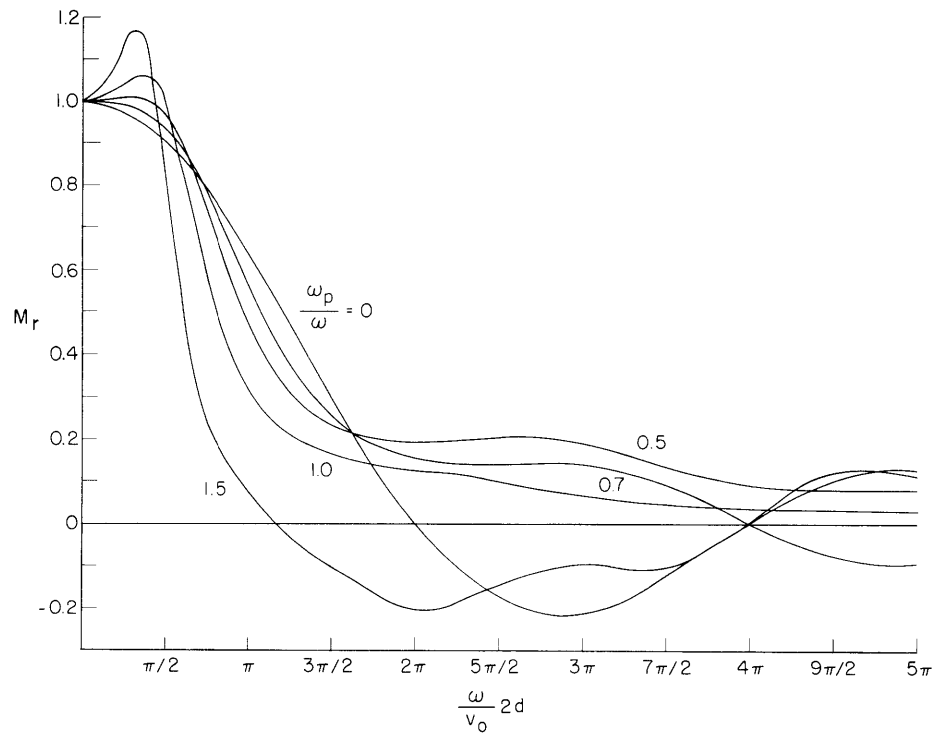
in which we have written

$$\begin{bmatrix} \underline{G}_c & \underline{K}_c \\ \underline{\Gamma}_c & \underline{Y}_{el} \end{bmatrix} = \begin{bmatrix} \underline{I} & \underline{K} \\ \underline{\Gamma} & \underline{Y} \end{bmatrix} + y \begin{bmatrix} \underline{ZK\Gamma} & \underline{K} \\ \underline{\Gamma} & \underline{Y} \end{bmatrix} \quad (29)$$

$$y = \frac{Y/j\omega c_0}{1 - (Y/j\omega c_0)} = \frac{Y_{el}}{j\omega c_0} = \frac{Y_{el}}{Y} - 1. \quad (30)$$

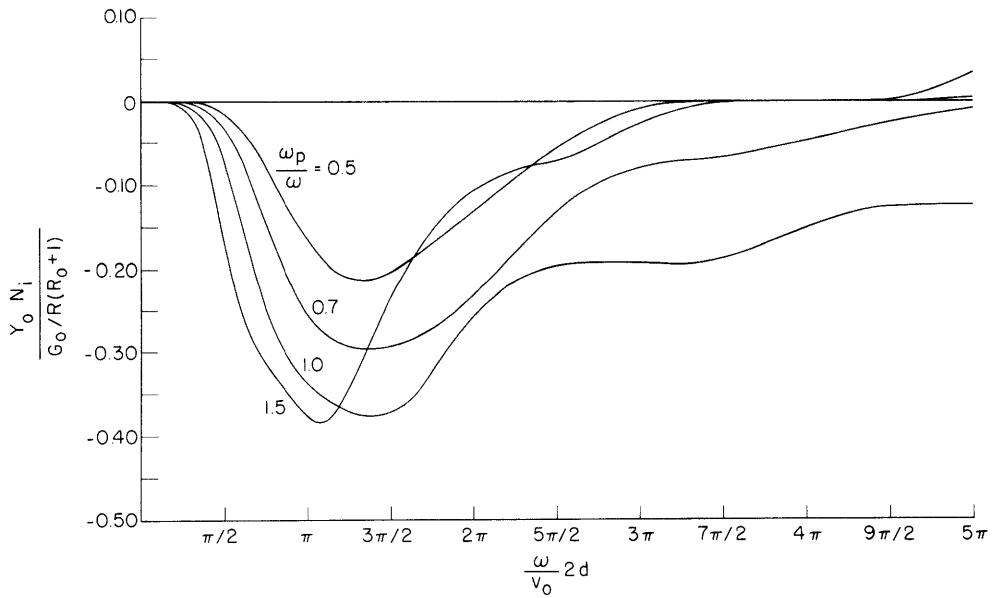
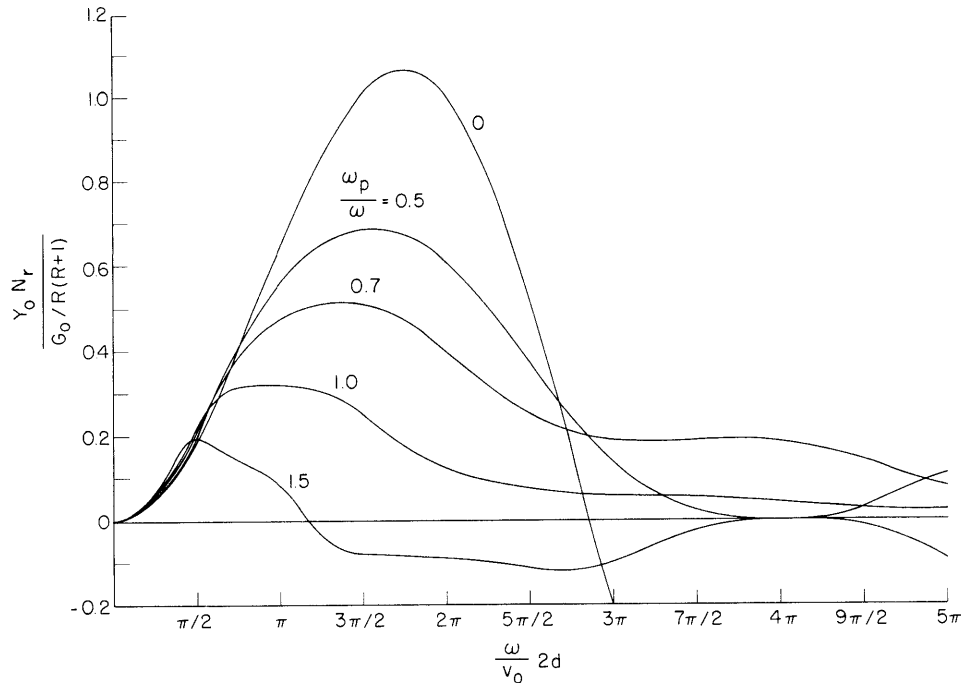
Equations 28-30 give the exact solution of the one-dimensional gap interaction problem. All of the matrix elements are determined by Eqs. 18-21. The parameters  $M$ ,  $Y_0 N$ , and  $Y$ , which determine Eqs. 18-21, have been given<sup>3</sup> as a function of  $(\beta_e 2d)$  with  $(\omega_p/\omega)$  a parameter. From these, the matrix elements of Eq. 29 can

Fig. VI-4. Gap parameters of the interaction matrix.



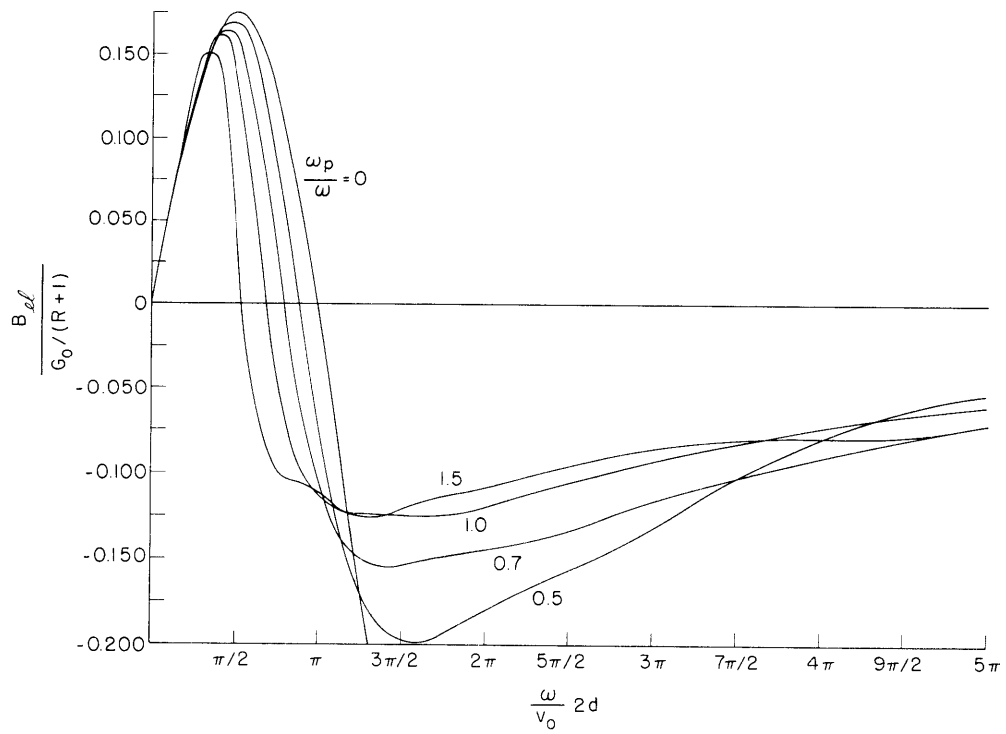
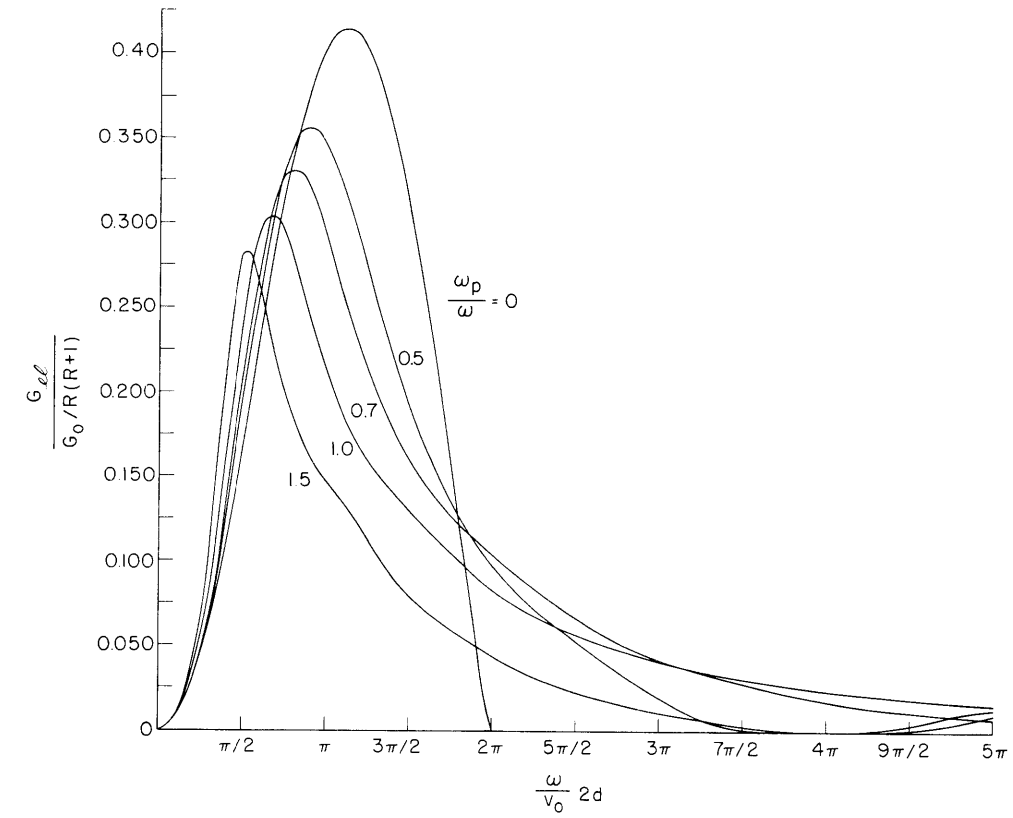
$$M_c = M_r + jM_i$$

Fig. VI-4. Gap parameters of the interaction matrix.



$$N_c = N_r + jN_i$$

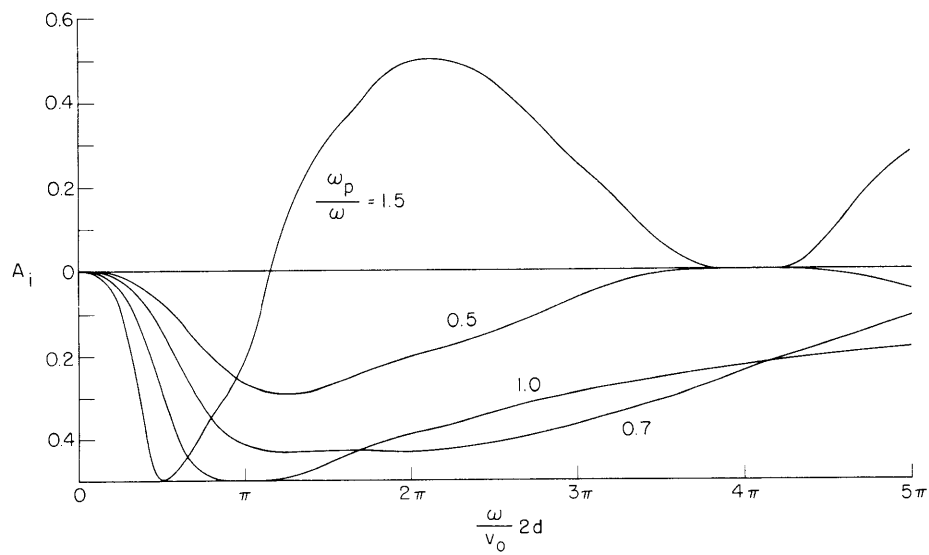
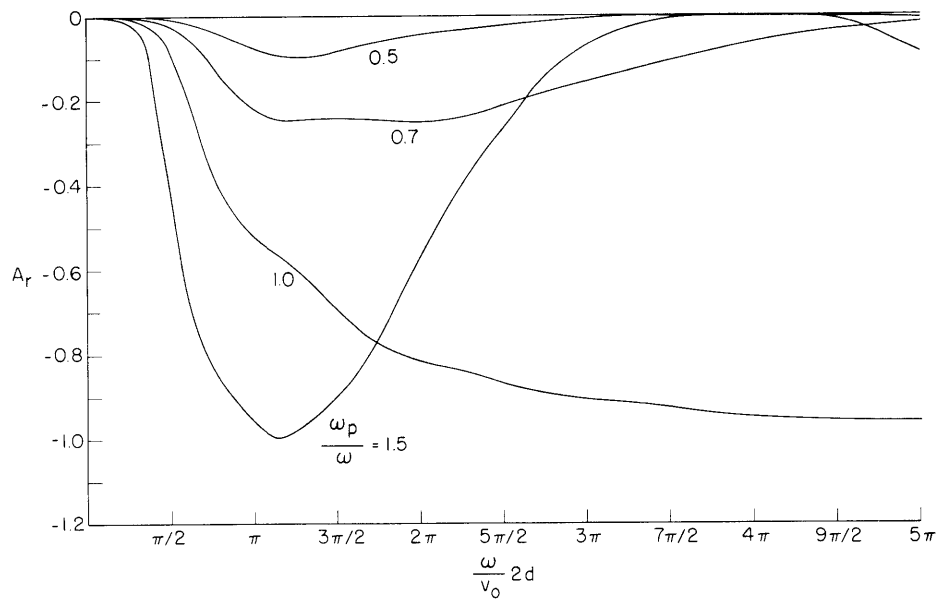
Fig. VI-4. Gap parameters of the interaction matrix.



$$Y_{el} = G_{el} + jB_{el}$$

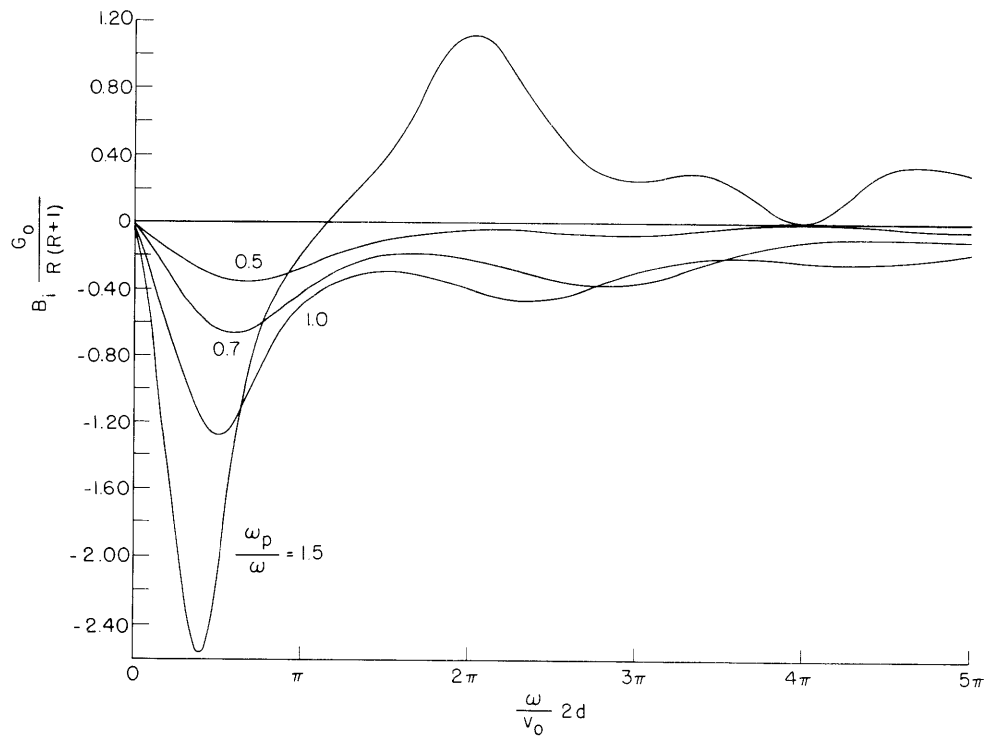
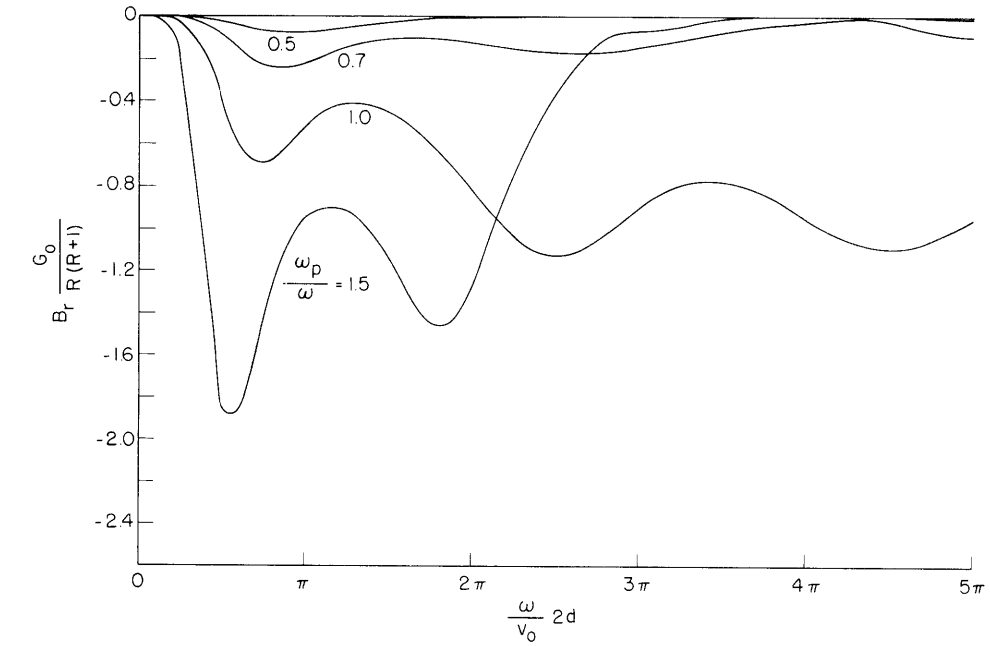


Fig. VI-4. Gap parameters of the interaction matrix.



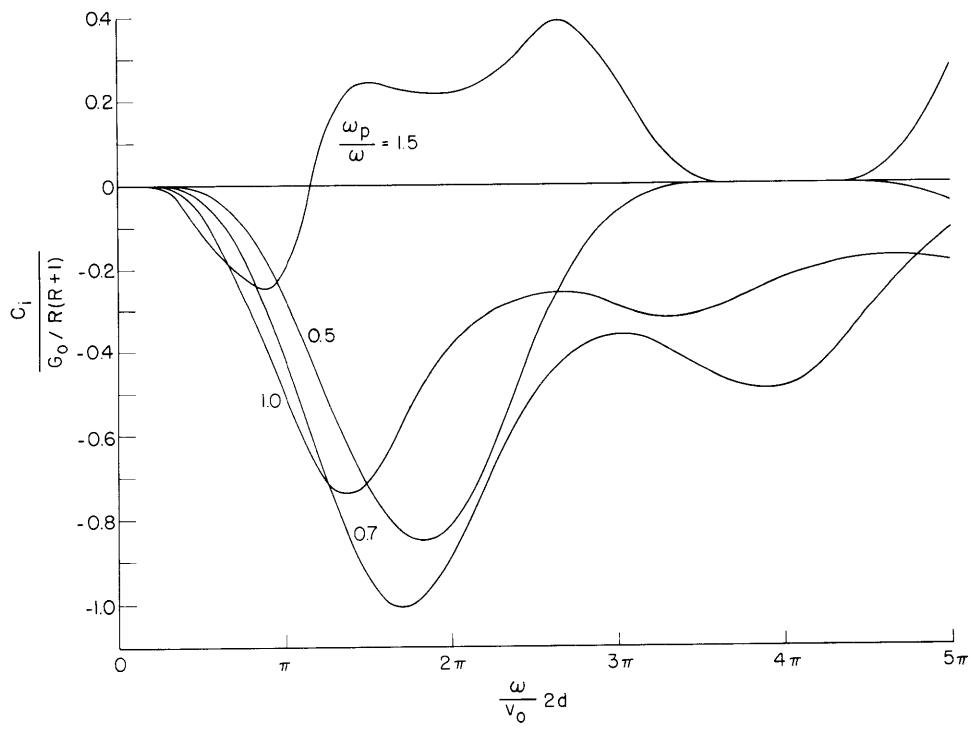
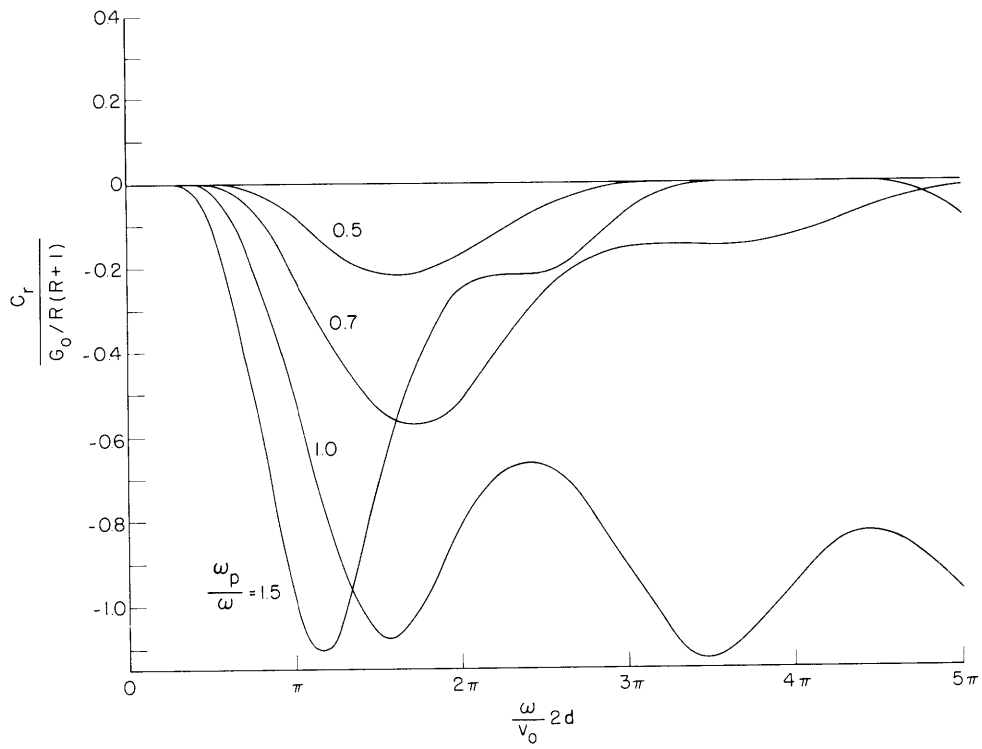
$$A_c = A_r + jA_i$$

Fig. VI-4. Gap parameters of the interaction matrix.



$$B_c = B_r + jB_i$$

Fig. VI-4. Gap parameters of the interaction matrix.



$$C_c = C_r + jC_i$$

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be calculated. These parameters are shown in Fig. VI-4, with the following notation

$$\underline{\underline{G}}_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} A_c & B_c \\ C_c & A_c \end{bmatrix} \quad (31)$$

$$\underline{\underline{K}}_c = \begin{bmatrix} M_c \\ Y_o N_c \end{bmatrix} \quad (32)$$

$$\underline{\underline{\Gamma}}_c = [Y_o N_c \quad M_c] \quad (33)$$

$$Y_{el} = G_{el} + jB_{el}. \quad (34)$$

The solution for the gap problem obtained from a weak-coupling theory<sup>3</sup> can be derived from the exact solution as follows. Comparing Eqs. 2 and 3 with Eqs. 1 and 2 in our previous report,<sup>3</sup> we note that their solutions are identical if we can identify  $K/j\omega\epsilon_o$  with  $E_z^c$ , which corresponds to identifying  $K/j\omega c_o$  with  $V_c$ . Such identification is possible when  $j\omega c_o \rightarrow \infty$  (see Fig. VI-3), which is the limit in which the circuit is shorted out (see Fig. VI-2). In this limit coupling with the circuit can exist if at the same time we allow  $K \rightarrow \infty$  so that  $(K/j\omega c_o) \rightarrow V_c$ . Hence, in the weak-coupling approximation the solution for the gap interaction is provided by Eq. 17 with  $(K/j\omega c_o) = V_c$ , or equivalently by Eqs. 28-30 with  $y = 0$ . This is precisely the result of our previous report.<sup>3</sup> (The notation there is only slightly different:  $I_g = K_i\sigma$ ;  $V_g = V_c$ ;  $Y_{el} = Y\sigma$ , with  $\sigma$  a cross-section area of the system.)

A comparison of the gap parameters for the exact solution, Fig. VI-4, with those of the weak-coupling solution (see Figs. VI-11 to VI-14 in the previous report)<sup>3</sup> leads to the following conclusions:

(a) The weak-coupling theory gives a good approximate description of the gap interaction for  $(\beta_e 2d) < \pi$  and  $(\omega_p/\omega) < 0.5$ . Hence the one-dimensional weak-coupling theory is strictly speaking also a weak space-charge theory.

(b) For gaps with transit angle  $(\beta_e 2d) < 2.6$  radians, the electronic loading conductance,  $G_{el}$ , increases with  $(\omega_p/\omega)$ , reaches a maximum, and then decreases; the shorter the gap, the higher the value of  $(\omega_p/\omega)$  at which  $G_{el}$  reaches a maximum.

(c) For  $(\omega_p/\omega) > 0.5$ , the voltage coupling coefficient  $M_c$ , for  $(\beta_e 2d) < \pi/2$ , shows a marked increase over the value unity and, for  $(\beta_e 2d) > \pi/2$ , a sharp cutoff accompanied by a large out-of-phase component  $M_i$ .

(d) For  $(\omega_p/\omega) > 0.5$ , the center-gap remodulations of an electron beam ( $\underline{\underline{G}}_c$  matrix) when  $V_c = 0$  become of considerable magnitude and cannot be neglected, as they are in a weak-coupling theory.

Conclusions (b), (c), and (d) have important consequences for the proper design

and possibilities of klystron and other microwave amplifiers employing high-density electron beams.

A. Bers

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#### B. GAP INTERACTION IN THE PRESENCE OF A POTENTIAL DEPRESSION – THIN-BEAM TWO-DIMENSIONAL, SPACE-CHARGE THEORY

Consider an electron beam in a waveguide whose walls are at a dc potential  $V_0$ . If the space charge in the beam is not completely neutralized, the potential at the position of the beam will be less than  $V_0$ , and vary over the beam cross section. In the presence of a gap in the waveguide wall the potential of the beam in the vicinity of the gap will also vary with distance along the beam, as shown in Fig. VI-5.

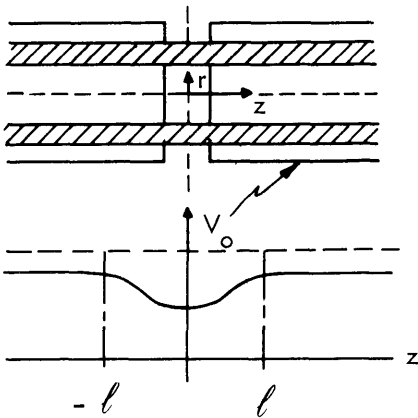


Fig. VI-5. D-C potential variation along the beam in the vicinity of a gap.

In the last report<sup>1</sup> we presented a kinematic theory for the gap interaction in such a system. Here we would like to present a theory that accounts approximately for the small-signal space-charge fields in the beam. We shall assume that the beam is thin so that the potential variation over its cross section is negligible, and consider the gap interaction in the presence of the longitudinal potential depression in the vicinity of the

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gap. The small-signal equations<sup>2</sup> describing a thin beam of weak space-charge density are:

$$\left(j\beta_e + \frac{\partial}{\partial z}\right) U = jZ_o\beta_q J \quad (1)$$

$$\left(j\beta_e + \frac{\partial}{\partial z}\right) J = jY_o\beta_q U. \quad (2)$$

All of the symbols have been previously defined.<sup>2</sup> Equations 1 and 2 are approximately valid in the presence of very gradual dc accelerations; here, we define a local plasma phase constant  $\beta_q$ . Under these conditions  $\beta_e$ ,  $\beta_q$ , and  $Z_o = 1/Y_o$  are all functions of  $z$ . Expressing the kinetic voltage and current density in terms of the fast- and slow-wave amplitudes, we have

$$U = \sqrt{2Z_o} (a_+ + a_-) \quad (3)$$

$$J = \sqrt{2Y_o} (a_+ - a_-). \quad (4)$$

Equations 1 and 2 become

$$\frac{\partial}{\partial z} a_+ = -j\beta_+ a_+ - \left(\frac{\partial}{\partial z} \ln Z_o\right) a_- \quad (5)$$

$$\frac{\partial}{\partial z} a_- = -\left(\frac{\partial}{\partial z} \ln Z_o\right) a_+ - j\beta_- a_- \quad (6)$$

where

$$\beta_{\pm} = \beta_e \mp \beta_q. \quad (7)$$

Equations 5 and 6 show that the slow and fast space-charge waves are coupled because of the longitudinal variation of the dc potential. These equations are analogous to the equations of an inhomogeneous transmission line.<sup>3</sup>

The natural solution of this system is obtained by assuming a  $z$ -dependence of

$$e^{-\gamma(z)} \quad (8)$$

where

$$\gamma(z) = \int_0^z \Gamma dz. \quad (9)$$

We find two independent solutions for

$$\Gamma_{1,2} = j \left[ \frac{\beta_+ + \beta_-}{2} \pm \sqrt{\left(\frac{\beta_+ - \beta_-}{2}\right)^2 - c_{12}^2} \right] \quad (10)$$

Here, we have designated the coupling coefficient of the space-charge waves as

$$c_{12} = \frac{\partial}{\partial z} (\ln Z_0). \quad (11)$$

The amplitudes of these solutions are related to the space-charge wave amplitudes by

$$A_{1,2} = (a_{\pm} + C_{\pm} a_{\mp}) \quad (12)$$

where

$$C_{\pm} = \frac{j\beta_{\pm} - \Gamma_{1,2}}{c_{12}}. \quad (13)$$

For small values of  $c_{12}$ ,  $C_{\pm} \sim c_{12}$ , so that in the absence of a dc potential variation  $c_{12}$  and  $C_{\pm}$  are zero and  $\Gamma_{1,2} = j\beta_{\pm}$ ,  $A_{1,2} = a_{\pm}$ , which are the usual space-charge wave propagation constants and wave amplitudes.

The potential depression resulting from a gap opening in the waveguide will practically extend over a limited region  $|z| < \ell$ ; see Fig. VI-5. Hence for  $|z| > \ell$  the solutions are described by space-charge waves, while for  $|z| < \ell$  the solutions are given by Eqs. 8-13.

Consider now the weak coupling between the electromagnetic fields in the guide caused by an excitation at the gap and the free solutions of the beam. We assume that the empty waveguide is below cutoff and, consistently with the weak-coupling and weak space-charge assumptions, we neglect electromagnetic power flow as compared with kinetic power flow. Furthermore, we assume that the electron flow is confined to the  $z$ -direction so that only the  $z$ -component of the electric field,  $E_z^C(z, b)$ , of the circuit at the position of the beam is of importance. Under these conditions, we first determine the excitations of the space-charge waves at  $z = \ell$  by an impulse-circuit electric field at  $z = z'$ ,  $|z'| < \ell$ . Then, using the superposition integral, we determine these space-charge wave excitations for an arbitrary spacial distribution of the circuit field. We find

$$a_{\pm}(\ell) = \frac{1}{2} m_{\pm} e^{-\gamma_{1,2}(\ell)} \quad (13)$$

where

$$m_{\pm} = \int_{-\infty}^{\infty} \frac{(1 + C_{\pm})}{\sqrt{2Z_0}} E_z(z) e^{\gamma_{1,2}(z)} dz \quad (14)$$

$$\gamma_{1,2}(\ell) = \int_0^{\ell} \Gamma_{1,2} dz \quad (15)$$

$$\gamma_{1,2}(z) = \int_0^z \Gamma_{1,2} dz. \quad (16)$$

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The evaluation of Eq. 14 by approximate integration techniques is now in progress.

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