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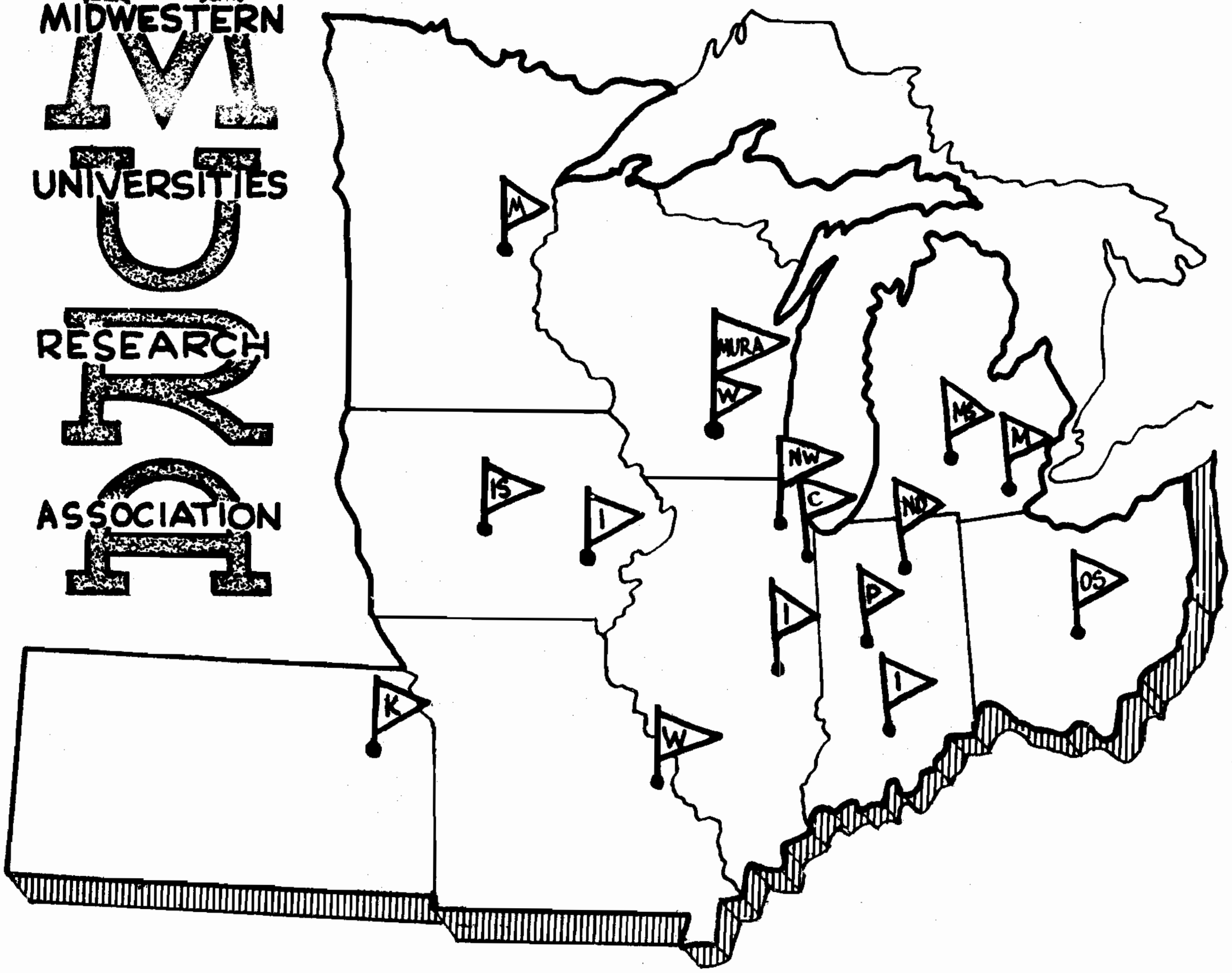


PROPERTIES OF NEUTRALIZED RELATIVISTIC ELECTRON BEAMS

J. Enoch

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2203 University Avenue, Madison, Wisconsin


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## ABSTRACT

The equilibrium properties of an intense, radiation-pinched, electron beam, which is neutralized by positive ions in the laboratory system, are studied. The motion of a test electron in such a beam is treated statistically, taking into account radiation damping and collisions with positive ions. By requiring that the calculated distribution of electrons as a function of distance from the center of the beam be equal to the distribution assumed in writing down the equations of motion, an expression for the equilibrium current is obtained. An expression for the equilibrium minor radius of the beam in terms of the acceleration per turn available is also obtained. An apparent discrepancy between the value of the equilibrium current as obtained in the present report and the value given by Budker[1] is explained in terms of assumptions about the energy distribution of the electrons.



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## INTRODUCTION

Budker has recently[1, 2], proposed an accelerator in which the interior of a neutralized relativistic electron beam circulating in an exterior magnetic field serves as a guide field for the acceleration of high energy particles. The advantage of such a machine is the fact that a relatively small magnet (of radius 100 cm, say, and $H = 2 \times 10^3$ gaus ) is needed in order to maintain a circulating electron beam capable of providing focussing forces for protons of energies in the Bev. region.

In the present report the equilibrium properties of a circulating electron beam which has been neutralized by residual gas ions are investigated. The method differs considerably from that of Budker although the basic idea of the "radiation pinch" is retained. An approximate stochastic equation of motion for the electrons is written down and a density distribution of electrons as a function of radial distance from the center of the beam is obtained. This permits the calculation of the equilibrium value of $\gamma i$, where $i$ is the electron current and $\gamma = \frac{E_{lab.}}{mc^2}$. Subsequently the energy losses of the electrons by collision with positive ions and by radiation are calculated and an expression for the minor radius of the beam as a function of the radiation loss per turn is obtained.

At first sight it seems that there is a discrepancy between the results of the present report and Budker's results. A closer look at Budker's work, however, reveals the reason for the discrepancy.

A. Qualitative description of the motion.

It will be assumed that in the laboratory system the electron beam has been completely neutralized by ionization of the residual gas molecules. Furthermore, the net charge density within the beam will be supposed to vanish, or, equivalently, the distribution of ions as a function of distance from the center of the beam is equal to the distribution of the electrons. Under these circumstances the electrons will be subject to three types of forces.

1) A strong magnetic field, $H_B$ , resulting from the intense electron current will be present. It will be assumed that $H_B \gg$ Hext. , where Hext. is the external guide field.

2) The reaction of the intense electromagnetic radiation provides a dissipative force causing, among other things, a damping of the transverse oscillations of the electrons.

3) The electrons will be continually scattered from the positive ions. The small angle scattering tends to increase the amplitude of oscillations through a random walk process. Large angle scattering can cause ejection of electrons from the beam. The latter will not be included in the final equation of motion but will be treated in section (H) from the point of view of its limitation on the lifetime of the beam.

B. Motion in the absence of radiation and scattering.

Since $H_B \gg$ Hext. , and since the major radius of the beam, R, is much larger than its minor radius, r*, it is natural to treat the motion of the electrons in the beam as if it were a linear beam. We choose the direction of motion of the electrons as the z-axis and introduce cartesian coordinates

$(x, y, z)$. Cylindrical coordinates $(r, \varphi, z)$ related to $(x, y, z)$ in the usual fashion will be occasionally used.

The electron density will be assumed to be uniform in the $z$ direction, and its dependence on distance from the center of the beam is assumed to have the form

$$\begin{aligned} f(r) &= \frac{N}{2\pi^2 R r^{*2}} , \quad r < r^* \\ f(r) &= 0 , \quad r > r^* , \end{aligned} \tag{1}$$

where $N$ is the total number of electrons in the beam. The magnetic field is then given by

$$H_\varphi = -\frac{e}{c}\bar{\dot{z}}\frac{Nr}{\pi R r^{*2}} \tag{2}$$

where $\bar{\dot{z}}$ is the average velocity of the electrons in the direction of the beam. It is easilty shown that the equations of motion are

$$\gamma m \ddot{x} + \left(\frac{e^2 \bar{\dot{z}} N}{c^2 \pi R r^{*2}}\right)\dot{z} x = 0 \tag{3}$$

$$\gamma m \ddot{y} + \left(\frac{e^2 \bar{\dot{z}} N}{c^2 \pi R r^{*2}}\right)\dot{z} x = 0 \tag{4}$$

and

$$\frac{d}{dt}\left[\gamma m \dot{z} - \left(\frac{e^2 \bar{\dot{z}} N}{c^2 \pi R r^{*2}}\right)(x^2 + y^2)\right] = 0 \tag{5}$$

eq. (5) can be integrated immediately to give (x, y, z). Cylindrical coordinates $(r, \varphi, z)$ related to (x, y, z) in the usual fashion will be occasionally used.

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$$

eq. (5) can be integrated immediately to give

$$\dot{z} = \frac{e^2 \bar{\dot{z}} N}{\gamma m c^2 \pi R r^{*2}} (x^2 + y^2) + \dot{z}(r=0) \tag{6}$$

Define $\nu = \dfrac{r_0 N}{2\pi R}$ . With this definition.

$\dfrac{e^2 N}{\gamma m c^2 \pi R} = 2\dfrac{(\gamma\nu)}{\gamma^2}$ . We shall show later on that

$\gamma\nu \approx$ 10. We choose $\gamma$ = 100 and note that

$$\frac{x^2 + y^2}{r^{*2}} \lesssim 1 .$$

With the aid of equation (6) we obtain the approximate relationship

$$\dot{z} \simeq 2 \times 10^{-3} \bar{\dot{z}} + \dot{z}(r=0) \tag{7}$$

For relativistic $\dot{z}(r=0)$ and $\bar{\dot{z}}$ , $\dot{z}$ is constant for all practical purposes. We shall approximate $\dot{z}(r=0) \approx \bar{\dot{z}} \approx \dot{z} \approx c$ . The x and y dependence of $\dot{z}$ contributes, therefore, only a small flutter to equations (1) and (2). By neglecting this flutter we can rewrite equation (1) as follows:

$$\ddot{x} + \omega^2 x = 0 , \tag{1'}$$

where

$$\omega^2 = \frac{e^2 N}{\gamma m \pi R r^{*2}} = \frac{2c^2(\gamma\nu)}{\gamma^2 r^{*2}} ,$$

$$\omega = \frac{\sqrt{2}\, c\, (\gamma\nu)^{\frac{1}{2}}}{\gamma r^*} .$$

By setting $r^* = 5 \times 10^{-3}$ cm, which we shall show later on is a reasonable value, we get

$$\omega = 3 \times 10^{11} .$$

We can also get an idea of the kind of transverse velocities that will be present:

$$\dot{x} \approx \dot{y} \approx \omega r^{*} = 1.2 \times 10^{9} \frac{cm}{sec.}$$

C. Radiation Reaction.

The relativistically invariant form of the radiation reaction force has the form[3]

$$f_i = \frac{dp_i}{ds} = \frac{2e^2}{3c}\left(\frac{d^2u_i}{ds^2} + u_i u_k \frac{d^2u_k}{ds^2}\right)$$

( $u_i$ is the four velocity with normalization $u_i^2 = -1$, $ds^2 = \frac{c^2 dt^2}{\gamma^2}$ . )

Under the conditions of the present problem the, unfortunately non-linear, second term on the right is dominant. Dropping the first term gives, in three dimensional form,[3]

$$\frac{d\vec{p}}{dt} = \frac{\vec{v}}{c} W$$

where W is the total power radiated. Equation (1') is now modified to read as follows;

$$\ddot{x} + \sigma \dot{x} + \omega^2 x = 0 \tag{8}$$

where

$$\sigma = \frac{W}{\gamma m c^2} .$$

W of course depends on x and y, but it is easily shown that

$$\langle \omega^2 x^2 \rangle_{Av.} \gg \left\langle \frac{\sigma \dot{x}^2}{\omega} \right\rangle_{Av.} ,$$

that is, the energy dissipated per oscillation is small compared to the total energy of oscillation. This suggests using an appropriate average value, $\overline{W}$ , instead of W. With this simplification the linearity of the problem, which is essential for the ensuing calculations, is preserved. Thus

$$\overline{W} = \frac{2e^4}{3m^2c^3}\gamma^2\overline{H^2} = \frac{2e^4}{3m^2c^3}\gamma^2\frac{e^2N^2\overline{r^2}}{2\pi^2 r^{*4}} .$$

By letting $\overline{r^2} = r^{*2}$ (see section G), and substituting $\nu = \frac{r_0 N}{2\pi R}$ we get

$$\overline{\sigma} = \frac{8r_0(\gamma\nu)^2 c}{3\gamma r^{*2}} .$$

D. Collisions with positive ions.

In the collision of a highly relativistic electron with a practically stationary ion the momentum transfer is substantially transverse and given to a good degree of approximation by:[4]

$$\Delta p = \frac{2e^2}{bc} ,$$

where b is the impact parameter.

The complete equation of motion, including scattering, takes on the form

$$\ddot{x} + \overline{\sigma}\dot{x} + \omega^2 x = \frac{1}{\gamma m}\sum_i \Delta p_x^i \,\delta(t - t_i) , \tag{9}$$

with a similar equation for the y motion. $\Delta p_x^i$ is the transfer of momentum in the x-direction in the $i$'th collision. (9) can be written as

$$\ddot{x} + \overline{\sigma}\dot{x} + \omega^2 x = \frac{2e^2}{\gamma mc}\sum_i \frac{\cos\varphi_i}{b_i}\delta(t - t_i) , \tag{9'}$$

where $\varphi_i$ is the azimuthal direction of the $i$'th momentum "kick".

## E. Statistical treatment of the motion.

Equation (9') is not an equation of motion in the ordinary sense since our information about the right hand side of the equation is statistical in nature. This type of equation is quite similar to equations encountered in the study of the effect of quantum fluctuations on the motion of electrons in synchrotrons, and has been treated by a number of author's(5) on the basis of Campbell's theorem.(6) The same technique will be adopted here. The result is

$$\overline{x^2} = \frac{Nc}{2\pi R}\left(\frac{2e^2}{\gamma mc}\right)\int_{b_{min}}^{b_{max}} \frac{P(b)}{b^2}db \int_0^{2\pi} \cos^2\varphi_i\, d\varphi_i \int_{-\infty}^{\infty} |S(t)|^2 dt \tag{10}$$

$\frac{Nc}{2\pi R}$ is, of course, simply the rate at which collisions occur. $P(b)db$ is the probability of a collision with impact parameter in the range (b, b+db). The distribution (1) implies that

$$P(b)db = \frac{2db}{r^{*2}b}$$

The distribution of $\varphi_i$'s is assumed uniform. S (t) is the solution of

$$\ddot{x} + \overline{\sigma}\dot{x} + \omega^2 x = \delta(t)$$

For the detailed statistical theory and for the conditions under which equation (10) is valid the reader is referred to reference(6).

By making the above substitutions equation (10) can be written as

$$\frac{\overline{r^2}}{2} = \overline{y^2} = \overline{x^2} = \frac{Nc}{2\pi R}\left(\frac{2e^2}{\gamma mc}\right)^2 \frac{1}{r^{*2}}\int_{b_{min}}^{b_{max}} \frac{db}{b}\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi|\omega'^2 - i\overline{\sigma}\omega' - \omega^2|^2} \cdot$$

Integration yields

$$\overline{r^2} = \frac{Nc}{\pi R}\left(\frac{2e^2}{\gamma mc}\right)^2 \frac{1}{r^{*2}} \ln\left(\frac{b_{max}}{b_{min}}\right) \frac{1}{2\bar{\sigma}\,\omega^2} \quad . \tag{11}$$

For $b_{max}$ we choose r*, the minor radius of the beam. $b_{min}$ is chosen so as to correspond to an energy transfer of $\Delta E = 2\gamma mc^2$ , which is approximately the maximum energy transfer possible classically in this type of collision. This gives $b_{min} = \frac{r_0}{\gamma}$ .

After appropriate substitutions equation (11) reduces to

$$\overline{r^2} = 3\frac{r^{*2}}{(\gamma\nu)^2} \ln\left(\frac{\gamma r^*}{r_0}\right) \tag{12}$$

The probability distribution for r, $\phi(r)$ , is approximately given by[6]

$$\phi(r)\,r\,dr\,d\varphi = \frac{2}{\sqrt{\pi^2 \overline{r^2}}}\, e^{-\frac{r^2}{\overline{r^2}}}\, r\,dr\,d\varphi \quad . \tag{13}$$

F. Energy losses.

In the foregoing it was assumed that $\gamma$ was constant. This will only hold if the energy lost by collisions and by radiation is constantly compensated for by some external means.

The energy exchanged in a typical collision is (4)

$$\Delta E = \frac{4e^4}{2\gamma mc^2 b^2} \quad .$$

The energy lost per turn by collisions is

$$W_c = \frac{N 4e^4}{2\gamma mc^2}\int_{b_{min}}^{b_{max}} \frac{2\,db}{r^{*2} b}$$

$$= \frac{4Ne^4 \ln\left(\frac{\gamma r^*}{r_0}\right)}{\gamma mc^2 r^{*2}} = \frac{8\pi R r_0 (\gamma \nu)^3 mc^2 \overline{r^2}}{3\gamma^2 r^{*2}}$$

The energy lost per turn by radiation is given by

$$W_R = \frac{2\pi R}{c}\frac{2e^4\gamma^2}{3m^2c^3}\overline{H^2}$$

$$= \frac{8\pi R r_0 mc^2 (\gamma\nu)^2}{3r^{*2}} \approx \frac{\gamma^2}{\gamma\nu} W_c \tag{14}$$

Thus with $\gamma\nu \approx 10$, $\gamma = 100$ we have $W_R = 10^3 W_c$; so that radiation losses are much more important. Actually $W_R$ is to a very good approximation equal to the total energy lost. This is so because $W_c$ represents mostly energy converted into transverse oscillation energy of the electrons. The latter is all radiated away under equilibrium conditions (see next section), so that to the extent that heavy ion recoil can be neglected

$$W_{total} = W_R \quad .$$

## G. Equilibrium conditions.

To obtain equilibrium two conditions must be satisfied.

1) The distribution function $\phi(r)$ (eq. 13) must be the same as $f(r)$ (with

appropriate normalization) given in equation (1). To accomplish this (approximately) we require that

$$\overline{r^2} = r^{*2} \quad . \tag{15}$$

This gives, using eq. (12),

$$(\gamma\nu)^2 = 3\ln\left(\frac{\gamma r^*}{r_0}\right)$$

A reasonable range for $r^*$ is $10^{-3}_{cm} < r^* < 10^{-2}_{cm}$, giving

$$\gamma\nu \simeq 10 \tag{16}$$

or

$$\dot{i} = \frac{17 \times 10^4}{\gamma} \quad \text{amperes} \quad , \tag{17}$$

where $\dot{i}$ is the electron current.

2) Power equal to $W_R$ must be fed into the beam. Equation (14) can be rewritten as

$$r^{*2} = \frac{8\pi R r_0 mc^2 (\gamma\nu)^2}{3 W_R} \, cm^2 \quad ,$$

or

$$r^* = 3.2 \times 10^{-4} \sqrt{\frac{R}{W_{kev.}}} \tag{18}$$

where $W_{kev}$ is the energy in kilovolt radiated per turn. If $W_{kev}$ is supplied by means of a betatron type induction field eq. (18) can be rewritten as follows:

$$r^* = \frac{4 \times 10^{-3}}{\sqrt{E}} \, cm \quad , \tag{18'}$$

where E is the induction field in volts/cm.

Equations (17) and (18) expresses the equilibrium conditions of the beam in terms of actual machine parameters, and are in this sense one indication of the feasibility of constructing such a machine. Naturally, many problems must still be solved before the final word on feasibility can be said. In particular, difficult questions about the macroscopic stability of such a configuration must be answered. These questions will not be treated in the present report. Stability against fluctuations in $\gamma$, however, can be discussed qualitatively in terms of equations (12) and (14).

Suppose that $\gamma$ is slightly smaller than the equilibrium value; equation (12) implies that this will cause the beam to start blowing up. On the other hand increasing r* means that $W_R$ decreases and if $W_{kev}$ is maintained constant this will cause the electrons to be accelerated thus tending to restore $\gamma$ to its equilibrium value. This argument can be reversed for the case $\gamma$ slightly smaller than the equilibrium value.

Another limitation is the loss of electrons from the beam by large angle scattering. We proceed to calculate the life time of the beam against single scattering.

H. Single scattering life time.

An electron will be ejected from the beam if it acquires enough transverse energy to overcome the attractive force of the beam and, as a results, hits the wall of the chamber. The minimum energy required for this is given by

$$\Delta E_{min} = \frac{e}{c}\int_{r^*}^{r_{ap}} \vec{v}\times\vec{H}\, dr \approx e\int_{r^*}^{r_{ap}} H_\varphi\, dr \tag{19}$$

where $r_{ap}$ is the effective aperture.

Now

$$H_\varphi(r) = \frac{eN}{\pi R r} \quad , \; r \geq r^* \quad ,$$

substitution in (19) and integration yield

$$\Delta E_{min} = \frac{e^2 N}{\pi R} \ln\left(\frac{r_{ap}}{r^*}\right)$$

In general[4]

$$\Delta E = \frac{2e^4}{\gamma m c^2 b^2}$$

for a collision with impact parameter b. All collisions with $b \leq b_{max}$, where $b_{max}^2 = \frac{2e^4}{\gamma m c^2 \Delta E_{min}}$, will result in the ejection of an electron from the beam. The cross-section for ejection, $\sigma_e$, is then given approximately by

$$\sigma_e = \pi b_{max}^2 = \frac{2\pi^2 r_0 R}{\gamma N \ln\left(\frac{r_{ap}}{r^*}\right)} \quad . \tag{20}$$

The life time of the beam is equal to the mean free "ejection collision" time, $\tau$.

$$\tau = \frac{1}{cn\sigma_e} \quad , \quad \text{where } n = \frac{N}{2\pi^2 R r^{*2}} = \text{ion density} .$$

Thus

$$\tau = \frac{2\pi R^2 r^{*2}}{c N \sigma_e}$$

$$= \frac{\gamma r^{*2} \ln\left(\frac{r_{ap}}{r^*}\right)}{c r_0} \quad .$$

For

$r^* = 4 \times 10^{-3}$ cm , $r_{ap} = 10$cm , and $\gamma = 100$

we obtain $\tau$ = 1.6 seconds

## I. Comparison with Budker's results.

The results of Budker's calculation of the equilibrium conditions can also be summarized in the form of equations (17) and (18). Equation (18) agrees well with Budker's result, but our value of the electron current (eq. 17) seems to differ by a factor of 5 from his value. The reason for this discrepancy lies in the definition of $\gamma$ . The $\gamma$ employed in reference (1) (henceforth to be called $\gamma_0$ ) is that of the electron "rest frame". $\gamma^*$, the average $\gamma$ of the electrons in their "rest frame", is given by[1]

$$\gamma^* \approx 1 + \gamma_0 \nu .$$

If the electron is supposed to move primarily in the transverse direction its $\gamma$ in the laboratory frame is given by

$$\gamma = \gamma^* \gamma_0$$

But $1 + \gamma_0 \nu = 2.7$ (see ref. 1), therefore

$$i = \frac{17 \times 10^4}{\gamma} = \frac{6.3 \times 10^4}{\gamma_0}$$

which differs only by a factor of 2 from reference (1).

The question of whether $\gamma$ or $\gamma_0$ is more convenient for expressing the current depends on the assumptions made about the energy distribution of the electrons. If, as is assumed in reference (1), the electrons are in thermal equilibrium in some average rest frame moving with "velocity" $\gamma_0$ in the laboratory frame, then $\gamma_0$ seems to be the reasonable quantity to use. It is important to realize, however, that the assumption of equilibrium implies a spread of $\sim 2.7$ in $\gamma^*$ and hence a spread of $\sim 2.7\,\gamma_0$ in the laboratory frame. If for example $\gamma_0 mc^2 = 30$ Mev., the energy spread of the electrons in the laboratory frame will be 90 Mev. ! ! It is difficult to justify such an energy distribution, although admittedly, it depends on how the beam is created.

## References

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6) S. O. Rice, Bell System Tech. J. 23, 282 (1944).