VIII. STATISTICAL COMMUNICATION THEORY

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A. PHASE MODULATION IN NONLINEAR FILTERING

Consider the sketch in Fig. VIII-1 of a nonlinear filter. The linear networks $h_1, \ldots, h_n$ are chosen to provide a satisfactory representation of the input signal x and its past history. For the purpose of this report we are interested in the nonlinear, memoryless
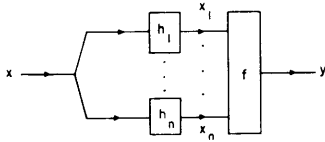

Fig. VIII-1. Nonlinear filter.

part of the filter, shown as f in Fig. VIII-1. Mathematically, $y = f(x_1, \ldots, x_n)$; that is, y is a function of the linear network outputs. Wiener (1), Barrett (2), and others have represented f by a series of orthogonal polynomials. This representation corresponds physically to a system containing only multipliers as nonlinear elements.

The suggestion offered in this report is to represent f by an orthogonal trigonometric series that corresponds physically to a system containing phase modulators.

To simplify matters, let us first assume that the signals $x_1, \ldots, x_n$ (Fig. VIII-1) are distributed uniformly and independently over the interval $(-\pi, \pi)$. Then the trigonometric series for f is

$$f \approx \sum_{k_1, \ldots, k_n} c_{k_1 \ldots k_n} \cos\left(k_1 x_1 + \ldots + k_n x_n + \phi_{k_1 \ldots k_n}\right) \tag{1}$$

In Eq. 1 the sum does not contain redundant terms (e.g., the pairs $k_1, \ldots, k_n$ and $-k_1, \ldots, -k_n$ are clearly equivalent). With the assumed joint distribution of $x_1, \ldots, x_n$, the coefficients $c_{k_1 \ldots k_n}$ are given by

$$c_{k_1 \ldots k_n} = \frac{2}{(2\pi)^n} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} f(x_1, \ldots, x_n) \cos\left(k_1 x_1 + \ldots + k_n x_n + \phi_{k_1 \ldots k_n}\right) dx_1 \ldots dx_n \tag{2}$$

Equation 2 can be derived with the aid of the orthogonality of each term in Eq. 1. The phase angle $\phi_{k_1 \ldots k_n}$ is chosen to maximize the corresponding $c_{k_1 \ldots k_n}$, and $c_{o \ldots o}$ is one-half the value given by Eq. 2, with $\phi_{o \ldots o} = 0$. (This is equivalent to the standard multidimensional sine-cosine series.) Now, let us briefly examine the physical
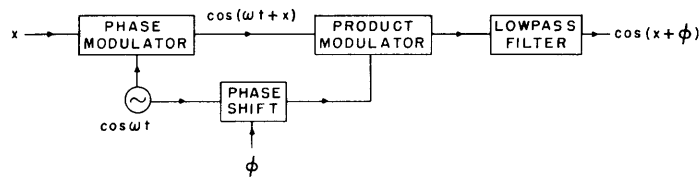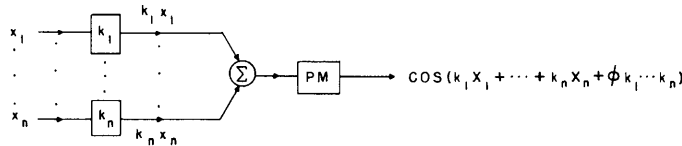
Fig. VIII-2. Phase-modulation system.

Fig. VIII-3. Series construction.

realization of the foregoing mathematics.

In the block diagram of Fig. VIII-2, the cosine of a signal is produced by carrier-demodulating the output of a phase modulator. The phase ϕ is determined by adjusting the phase-shift network. The major engineering problem is to obtain a sufficiently wide linear phase deviation in the phase modulator. This problem may be solved by the use of frequency multipliers, although care would be required to prevent loss of carrier phase synchronism. At any rate, the physical problems do not appear insurmountable.

In Fig. VIII-3, a system analogous to one of the terms in the series of Eq. 1 is outlined. The box PM represents all of Fig. VIII-2, with x replaced by $k_1 x_1 + \ldots + k_n x_n$, and ϕ by $\phi_{k_1 \ldots k_n}$. Hence, f can be constructed from a parallel combination of the systems of Fig. VIII-3 with outputs adjusted according to the coefficients $c_{k_1 \ldots k_n}$. These coefficients might be found experimentally, as illustrated in Fig. VIII-4. The meter M is assumed to multiply the two incoming signals and average the result. By adjusting $\phi_{k_1 \ldots k_n}$ for a maximum reading on the meter, we obtain the correct phase angle, and this maximum reading gives $c_{k_1 \ldots k_n}$. (For a more accurate phase-angle indication it is possible to adjust the phase angle for a null on the meter. The correct value of $\phi_{k_1 \ldots k_n}$ differs from the null value by 90°.)

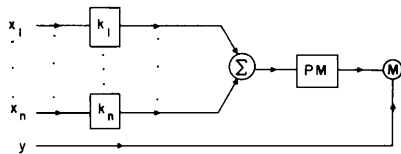Note that the signals $x_1, \ldots, x_n$ in Fig. VIII-1 need not be distributed uniformly because a suitable nonlinear, memoryless, monotonic transformation can correct any such non-uniformity. If these signals are not independent,

Fig. VIII-4. Coefficient and phase analysis.

two choices are open. Either the size (number of terms) of the series of Eq. 1 is determined in advance, and all coefficients are solved for simultaneously or orthogonal trigonometric terms are constructed, and the series is built up from these orthogonal terms. In the latter case, the better choice would be to orthogonalize all trigonometric terms of a given "degree" to all those of lower "degree" ("degree" is defined here as $|k_1| + \ldots + |k_n|$), just as Wiener (3) does with polynomials. Then it would just be necessary to solve for all coefficients of terms having the same degree simultaneously.

Alternatively, we could orthogonalize all of the terms. The first term might be the constant, 1. Then we have $g_0 = 1$. The second term might be $g_1 = \cos(x_1+\phi_1) + c_1(\phi_1)$ with $c_1(\phi_1)$ chosen so that g_1 is orthogonal to g_0 for all ϕ_1. Then we obtain

$$\int_{-\infty}^{\infty} \left[\cos(x_1+\phi_1) + c_1(\phi_1)\right] \cdot 1 \cdot dP(x_1) = 0 \tag{3}$$

where $P(x_1)$ is the distribution function of x_1. Next, we have

$$g_2 = \cos(x_2+\phi_2) + c_2(\phi_1, \phi_2) \cos(x_1+\phi_1) + c_3(\phi_1, \phi_2) \tag{4}$$

with $c_2(\phi_1, \phi_2)$ and $c_3(\phi_1, \phi_2)$ chosen to make g_2 orthogonal to g_1 and g_0. Thus we might proceed through the first-degree terms, thence through the second-degree, and so on. The complexity of this construction is obvious, but there is the partially compensating advantage of having complete pairwise orthogonality of the trigonometric terms.

Finally, it should be pointed out that the analytical work suggested here is not difficult to perform when the signals $x_1, \ldots, x_n$ have a joint Gaussian distribution. For, it is not difficult to see that a product of trigonometric terms can be reduced by various identities to a weighted sum of terms of the form $\cos z$, where z is a weighted sum of $x_1, \ldots, x_n$, plus a constant such that $\bar{z} = 0$. Here the horizontal bar means "average value of." But z is itself a Gaussian variable, and it follows (4) that

$$\overline{\cos z} = e^{-\overline{z^2}/2} \tag{5}$$

The $\overline{z^2}$ average is simple to compute for Gaussian variables, therefore the analytical work is considerably simplified in this case.

A. D. Hause

References

1. N. Wiener, Nonlinear Problems in Random Theory (The Technology Press of Massachusetts Institute of Technology, and John Wiley and Sons, Inc., New York, 1958).

2. J. F. Barrett, The use of functionals in the analysis of non-linear physical systems, Statistical Advisory Unit Report 1/57, Ministry of Supply, Great Britain, 1957.

3. N. Wiener, op. cit., pp. 28-38.

4. The basis of this calculation is found in N. Wiener, op. cit., pp. 53-54.

B. STATISTICAL MODEL OF COUPLED OSCILLATORS

Many systems of coupled oscillators can be viewed as systems in which the frequency of each oscillator is affected by the sum of the outputs of all of the other oscillators in the system — for example, a city's electric power supply system that consists of many coupled generators. In this report, we shall propose a model of such systems from which certain statistical characteristics can be determined. For our model, we shall assume that the system consists of an infinite number of oscillators that are not phase-locked. The practical question of how many oscillators must be coupled together in order for our solution for the infinite system to be a good approximation is being investigated.

In describing our statistical model, let us focus our attention on one oscillator of the system. It is acted upon by the sum of the outputs of an infinite number of coupled oscillators that are not phase-locked. It has been shown that this sum, $y(t)$, has a Gaussian distribution, irrespective of the coupling or the periodic waveform involved (1); it can also be shown that $y(t)$ is a Gaussian wave. If we represent the output of the oscillator that we are observing as $\exp[j\theta(t)]$, then according to our model, the phase angle, $\theta(t)$, is some function of the Gaussian wave, $y(t)$. If $\theta(t)$ is a linear function of $y(t)$ — that is,

$$\theta(t) = \int_{-\infty}^{\infty} h(t-\sigma) \, y(\sigma) \, d\sigma$$

we shall say that the coupling is linear. In general, the coupling is nonlinear, and $\theta(t)$ is some nonlinear function of $y(t)$.

A virtue of this model is that by means of it we can examine the general effects of various types of coupling by studying the spectrum of

$$f(t) = \exp(jF[y(t)]) \tag{1}$$

for various nonlinear functions, F. It is interesting to observe that $f(t)$ is merely the output of a phase-modulated oscillator, as depicted in Fig. VIII-5. Thus we have an electrical analog that is useful as an analog computer for determining the general effects of various types of coupling. Without techniques of analysis, however, an experimental analog is cumbersome to use because we then have neither a measure of how fine our measurements must be nor any indication of what regions we should investigate for the desired effects. The analytical technique has been presented by Wiener (2).
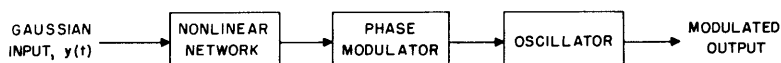

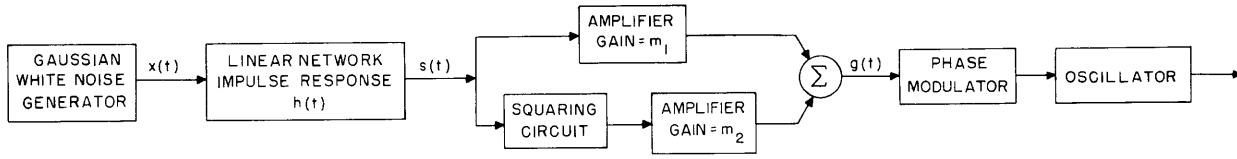Fig. VIII-5. General electrical analog.

Fig. VIII-6.   Analog of coupling with second-order nonlinearity.

To illustrate this analytical technique, we shall determine the effect on the spectrum of adding some second-order nonlinearity to the coupling of a linearly coupled system of oscillators.   Since any Gaussian wave can be considered to be the Gaussian white-noise response of a linear filter, our general electrical analog of Fig. VIII-5 becomes the system that is shown in Fig. VIII-6, and the general equation, Eq. 1, becomes

$$f(t) = \exp(j[\omega_o t + g(t)]) \tag{2}$$

in which

$$g(t) = m_1 s(t) + m_2 s^2(t) \tag{3}$$

and

$$s(t) = \int_{-\infty}^{\infty} h(t-z) \, x(z) \, dz \tag{4}$$

Here, $h(t)$ is the impulse response of the linear filter, $x(t)$ is a Gaussian white-noise signal, and ω_o is the unmodulated oscillator frequency.   The spectrum of $f(t)$ is that of $\exp[jg(t)]$ centered about the frequency ω_o.   We thus note that no generality is lost if we assume that $\omega_o = 0$.   We are then interested in the spectrum of

$$f(t) = \exp[jg(t)] \tag{5}$$

Let us normalize $h(t)$ by letting

$$h(t) = K\phi(t) \tag{6}$$

in which

$$K^2 = \int_{-\infty}^{\infty} h^2(t) \, dt \tag{7}$$

so that

$$\int_{-\infty}^{\infty} \phi^2(t) \, dt = 1 \tag{8}$$

We shall also use the definitions:

$$a_1 = jm_1 K$$

$$a_2 = jm_2 K^2 \tag{9}$$

Then from Eqs. 3, 4, 6, and 9 we can write

$$jg(t) = a_1 \int_{-\infty}^{\infty} \phi(t-z) \, x(z) \, dz + a_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(t-z_1) \, \phi(t-z_2) \, x(z_1) \, x(z_2) \, dz_1 dz_2 \tag{10}$$

In order to determine the spectrum, we shall first obtain the development of f(t) in terms of Wiener's fundamental orthogonal functionals:

$$f(t) = \sum_{n=0}^{\infty} C_n G_n [\phi^{(n)}(t)] \tag{11}$$

The autocorrelation of f(t) is

$$R(\tau) = \overline{f(t) \, f^*(t+\tau)}$$

$$= \sum_{n=0}^{\infty} n! \, |C_n|^2 \left[\int_{-\infty}^{\infty} \phi(t) \, \phi(t+\tau) \, dt \right]^n \tag{12}$$

The desired spectrum is the Fourier transform of $R(\tau)$. The coefficients, C_n, of Eq. 11 will be determined by projecting f(t) upon the function space generated by a linearly phase-modulated wave, v(t):

$$v(t) = \exp\left[b \int_{-\infty}^{\infty} \phi(t-z) \, x(z) \, dz \right] \tag{13}$$

Wiener has shown that the fundamental orthogonal functional expansion of v(t) is

$$v(t) = \exp\left(\frac{1}{2} b^2 \right) \sum_{p=0}^{\infty} \frac{b^p}{p!} \, G_p [\phi^{(p)}(t)] \tag{14}$$

If we define

$$y(t) = \int_{-\infty}^{\infty} \phi(t-z) \, x(z) \, dz \tag{15}$$

then we note from Eq. 8 that y is a Gaussian random variable with a mean of zero and a variance of one. The probability density distribution of y is thus

$$p(y) = \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{1}{2} y^2 \right) \tag{16}$$

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The projection of $f(t)$ is obtained by taking the average product of $f(t)$ and $v(t)$. From Eqs. 11 and 14 this average product is

$$\overline{f(t)\ v(t)} = \exp\left(\frac{1}{2}b^2\right) \sum_{p=0}^{\infty} b^p C_p \tag{17}$$

and from Eqs. 5, 13, and 16 we obtain

$$\overline{f(t)\ v(t)} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp\left(a_1 y + a_2 y^2\right) e^{by} \exp\left(-\frac{1}{2}y^2\right) dy$$

$$= \frac{1}{(1-2a_2)^{1/2}} \exp\left[\frac{(a_1+b)^2}{2(1-2a_2)}\right] \tag{18}$$

Thus we obtain from Eqs. 17 and 18

$$\exp\left(\frac{1}{2}b^2\right) \sum_{p=0}^{\infty} b^p C_p = \frac{1}{(1-2a_2)^{1/2}} \exp\left[\frac{(a_1+b)^2}{2(1-2a_2)}\right] \tag{19}$$

which can be written in the form

$$\sum_{p=0}^{\infty} b^p C_p = \frac{1}{(1-2a_2)^{1/2}} \exp\left[\frac{a_1^2}{2(1-2a_2)}\right] \exp\left[\frac{a_2 b^2 + a_1 b}{1 - 2a_2}\right] \tag{20}$$

The coefficients C_p can now be determined by expanding the exponential in a power series and equating like powers of b. Thus

$$\sum_{p=0}^{\infty} b^p C_p = \frac{1}{(1-2a_2)^{1/2}} \exp\left[\frac{a_1^2}{2(1-2a_2)}\right] \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_1^m a_2^n b^{m+2n}}{m!\, n!\, (1-2a_2)^{m+n}} \tag{21}$$

The autocorrelation of $f(t)$ is then obtained by use of Eq. 12 and noting from Eq. 9 that a_1 and a_2 are imaginary numbers. Thus, for example,

$$|C_0|^2 = \frac{1}{\left(1+4|a_2|^2\right)^{1/2}} \exp\left(-\frac{|a_1|^2}{1 + 4|a_2|^2}\right)$$

$$|C_1|^2 = \frac{|a_1|^2}{1 + 4|a_2|^2} |C_0|^2 \tag{22}$$

The general expression for $|C_p|^2$ is complicated. The case in which we are interested is that for small nonlinearities; that is, a_2/a_1 is small. For this case we shall obtain an approximate expression for the spectrum. From Eqs. 20 and 21 we can write

$$\sum_{p=0}^{\infty} b^p C_p = C_o \exp\left(\frac{a_1 b}{1 - 2a_2}\left[1 + \frac{a_2}{a_1}b\right]\right)$$

$$= C_o \sum_{n=0}^{\infty} \left(\frac{a_1}{1 - 2a_2}\right)^n \frac{b^n}{n!}\left[1 + \frac{a_2}{a_1}b\right]^n \tag{23}$$

With the assumption that a_2/a_1 is small, we shall say that

$$\sum_{p=0}^{\infty} b^p C_p = C_o \sum_{n=0}^{\infty} \left[\frac{a_1}{1 - 2a_2}\right]^n \frac{b^n}{n!}\left[1 + n\frac{a_2}{a_1}b + \frac{(n)(n-1)}{2}\left(\frac{a_2}{a_1}\right)^2 b^2\right] \tag{24}$$

By equating like powers of b, this expression yields the exact equation for C_p through C_5. For higher orders of C_p, the basic approximation is found to be

$$a_1^2 \gg \frac{2}{3}\frac{a_2}{\left(1+4a_2^2\right)^{1/2}} \tag{25}$$

By solving Eq. 24 for C_p, substituting this result in Eq. 12, and summing the resultant series, we obtain a closed-form expression for $R(\tau)$ after a considerable amount of algebraic manipulation. The result is

$$R(\tau) = |C_o|^2 e^{\rho(\tau)}\left\{1 - 4\left(\frac{a_2}{a_1}\right)^2\rho^2(\tau) + \left(\frac{a_2}{a_1}\right)^2\frac{1 + 4a_2^2}{a_1^2}\left[2+4\rho(\tau)+\rho^2(\tau)\right]\rho^2(\tau)\right.$$

$$-\left(\frac{a_2}{a_1}\right)^2\frac{1 - 4a_2^2}{a_1^2}\rho^4(\tau) - 2\left(\frac{a_2}{a_1}\right)^4\frac{1 + 4a_2^2}{a_1^2}\left[12+8\rho(\tau)+\rho^2(\tau)\right]\rho^4(\tau)$$

$$\left. + \frac{1}{4}\left(\frac{a_2}{a_1}\right)^4\left(\frac{1 + 4a_2^2}{a_1^2}\right)^2\left[24+96\rho(\tau)+72\rho^2(\tau)+16\rho^3(\tau)+\rho^4(\tau)\right]\rho^4(\tau)\right\} \tag{26}$$

in which

$$\rho(\tau) = \frac{a_1^2}{1 + 4a_2^2}\int_{-\infty}^{\infty} \phi(t)\,\phi(t+\tau)\,dt \tag{27}$$

and the magnitude signs on a_1 and a_2 have been omitted.

In order to study the spectrum, the Fourier transform of Eq. 26 is required. It is evident that the total transform of $R(\tau)$ is a complicated function of frequency and would be difficult to interpret. This difficulty may be circumvented by first expanding Eq. 26 in a power series of $\rho(\tau)$:

$$R(\tau) = |C_o|^2 \sum_{n=0}^{\infty} B_n \rho^n(\tau) \tag{28}$$

When this is done, the coefficients B_n are found to be

$$
\left.
\begin{aligned}
B_o &= 1 \\[2mm]
B_1 &= 1 \\[2mm]
B_2 &= \frac{1}{2!} + 2\left(\frac{a_2}{a_1}\right)^2 \left[\frac{1 + 4a_2^2}{a_1^2} - 2\right] \\[4mm]
B_3 &= \frac{1}{3!} + 2\left(\frac{a_2}{a_1}\right)^2 \left[3 \, \frac{1 + 4a_2^2}{a_1^2} - 2\right]
\end{aligned}
\right\}
\tag{29}
$$

Now, the Fourier transform, $F_n(\omega)$, of $\rho^n(\tau)$ is a positive function of ω. Thus, from the central limit theorem, the bandwidth of $F_n(\omega)$ increases with increasing n as it approaches its limiting form, which is a Gaussian curve. Hence the spectrum of $R(\tau)$ becomes more narrow as the coefficients B_n are reduced. We observe from Eq. 29 that both B_2 and B_3 are reduced for a_2 in the range

$$a_2^2 < \frac{2a_1^2 - 3}{12} \tag{30}$$

which is within the range of the approximation given by Eq. 25. This implies that the spectral bandwidth of a linearly coupled system for which this model is applicable may be reduced somewhat by the addition of a quadratic nonlinearity!

The solution for this example was obtained in a straightforward manner by the use of techniques developed by Wiener. For many types of nonlinear coupling, the computation is involved because the solution involves integrals that are not readily evaluated. An example of such a case is one in which there is some cubic non-linearity. This case is of special interest because many investigators claim that oscillators tend to synchronize as a result of the presence of a cubic nonlinearity (3). However, even though the computation for this case is involved, the effect of the presence of varying amounts of cubic nonlinearity in the coupling can be experimentally

investigated by the use of the electrical analog depicted in Fig. VIII-5.

Our model was derived to represent systems of oscillators that are not phase-locked; but in certain special cases, it can be used to represent systems of phase-locked oscillators. An example is our initial one of an electric power supply system for a city (2). Although the generators are phase-locked, it can be argued that as people in the city turn electrical appliances on and off, the load presented to any one generator of the system will fluctuate approximately in a Gaussian manner. Thus our model is applicable, and it can be used to study the effects of various nonlinearities upon the spectrum of the generators en masse.

<div style="text-align: right">M. Schetzen</div>

References

1. M. Schetzen, Statistical behavior of coupled oscillators, Quarterly Progress Report No. 58, Research Laboratory of Electronics, M.I.T., July 15, 1960, pp. 160-165.

2. N. Wiener, Nonlinear Problems in Random Theory (The Technology Press, M.I.T., and John Wiley and Sons, Inc., New York, 1958).

3. F. E. Terman, Electronics and Radio Engineering (McGraw-Hill Book Company, Inc., New York, 4th ed., 1955), p. 496.

C. AN OPTIMUM METHOD FOR SIGNAL TRANSMISSION

It was brought to my attention after the publication of my report (1) that the solution for the optimum combination of linear pre-emphasis network and linear filter for transmission through a noisy channel was published by Costas (2) in 1952.

<div style="text-align: right">M. Schetzen</div>

References

1. M. Schetzen, An optimum method for signal transmission, Quarterly Progress Report No. 59, Research Laboratory of Electronics, M.I.T., Oct. 15, 1960, pp. 109-115.

2. J. P. Costas, Coding with Linear Systems, Technical Report 226, Research Laboratory of Electronics, M.I.T., Feb. 20, 1952.

D. PROPERTIES OF THE TRANSFORMS OF THE KERNELS OF A NONLINEAR SYSTEM

The input-output relation of a nonlinear system can frequently be expressed by the Volterra type of functional power series

$$y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} K_n(t-\tau_1, \ldots, t-\tau_n) \, x(\tau_1) \cdots x(\tau_n) \, d\tau_1 \cdots d\tau_n \tag{1}$$

in which $x(t)$ is the input, $y(t)$ is the output, and K_n are the kernels. The multidimensional Fourier transforms of the kernels are $K_1(j\omega_1)$, ..., $K_n(j\omega_1, \ldots, j\omega_n)$, ... We shall develop the following properties of the transforms.

(1) The form of transform necessary for the corresponding impulse response to be a real function.

(2) The nonlinear analog to real-part sufficiency and a generalized Hilbert transform relation.

(3) The constraint between the gain and minimum-phase relations in each kernel.

(4) A criterion that provides a sufficiency test for the realizability of an nth-order gain function.

(5) Necessary and sufficient conditions for the stability of a rational n^{th}-order kernel. (The Routh criterion is generalized to provide a straightforward method of testing stability of rational transforms.)

As one would expect, the approach to the problem is a generalization of the approach to the properties of linear transforms described by Lee (1) and Mason (2). The transform pair that we shall use is

$$K(S_1, S_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(t_1, t_2) \exp(-S_1 t_1) \exp(-S_2 t_2) \, dt_1 dt_2 \tag{2}$$

$$k(t_1, t_2) = \left(\frac{1}{2\pi}\right)^2 \int_{-j\infty}^{+j\infty} \int_{-j\infty}^{+j\infty} K(S_1, S_2) \, e^{S_1 t_1} e^{S_2 t_2} \, dS_1 dS_2 \tag{3}$$

where $S_1 = \sigma_1 + j\omega_1$, $S_2 = \sigma_2 + j\omega_2$. We are concerned with functions whose region of convergence includes the imaginary axis in each variable so that the Fourier transform is included in the definition of Eqs. 2 and 3. All properties will be illustrated with the two-dimensional case, but they hold for the nth-dimensional case.

The first property follows trivially from Eq. 2. If we consider S_1, S_2 to be real variables, the integrand is a real function for all real impulse responses. This implies that $K(S_1, S_2)$ has the form of a real function of S_1 and S_2. Thus, the most general rational transform corresponding to a real second-order impulse response is

$$K(S_1, S_2) = \frac{\displaystyle\sum_{i=0}^{N} \sum_{j=0}^{N} a_{ij} S_1^i S_2^j}{\displaystyle\sum_{i=0}^{M} \sum_{j=0}^{M} b_{ij} S_1^i S_2^j} \tag{4}$$

where a_{ij} and b_{ij} are real coefficients. Since any kernel can be made symmetrical, $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$.

1. Realizable impulse responses

We shall now consider realizable impulse responses and their properties. Let $k(t_1, t_2) = 0$ for either t_1 or $t_2 < 0$. A typical response is shown in Fig. VIII-7. We can write

$$k_2(t_1, t_2) = k_2^i\left(t_1^e, t_2^e\right) + k_2^{ii}\left(t_1^e, t_2^o\right) + k_2^{iii}\left(t_1^o, t_2^o\right) + k_2^{iv}\left(t_1^o, t_2^e\right) \tag{5}$$

where $k_2^i\left(t_1^e, t_2^e\right)$ denotes a function that is even with respect to the variables t_1 and t_2 as shown in Fig. VIII-8. We exclude the singularity function $\mu_o(t_1)\,\mu_o(t_2)$ and its derivatives from the class of $k_2(t_1, t_2)$ to be considered. For a realizable system, it follows that

$$k_2^i\left(t_1^e, t_2^e\right) = k_2^{ii}\left(t_1^e, t_2^o\right) = k_2^{iii}\left(t_1^o, t_2^o\right) = k_2^{iv}\left(t_1^o, t_2^e\right) = \frac{1}{4}k_2(t_1, t_2) \qquad t_1, t_2 \geqslant 0 \tag{6}$$

Substituting Eq. 5 in the direct Fourier transform equation gives

$$K_2(j\omega_1, j\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[k_2^i\left(t_1^e, t_2^e\right) + k_2^{iii}\left(t_1^o, t_2^o\right) + k_2^{iv}\left(t_1^o, t_2^e\right) + k^{ii}\left(t_1^e, t_2^o\right)\right]$$

$$\times\left[\cos\omega_1 t_1 \cos\omega_2 t_2 - \sin\omega_1 t_1 \sin\omega_2 t_2 - j\sin\omega_1 t_1 \cos\omega_2 t_2 - j\cos\omega_1 t_1 \sin\omega_2 t_2\right]$$

$$\times dt_1 dt_2 \tag{7}$$

Taking advantage of the oddness and evenness of the various terms in the integrand, we have

$$K_2(j\omega_1, j\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_2^i\left(t_1^e, t_2^e\right)\cos\omega_1 t_1 \cos\omega_2 t_2\, dt_1 dt_2$$

$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_2^{iii}\left(t_1^o, t_2^o\right)\sin\omega_1 t_1 \sin\omega_2 t_2\, dt_1 dt_2$$

$$- j\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_2^{iv}\left(t_1^o, t_2^e\right)\sin\omega_1 t_1 \cos\omega_2 t_2\, dt_1 dt_2$$

$$- j\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_2^{ii}\left(t_1^o, t_2^e\right)\sin\omega_1 t_1 \cos\omega_2 t_2\, dt_1 dt_2 \tag{8}$$

By dividing $K_2(j\omega_1, j\omega_2)$ into its real and imaginary parts and then into even and odd parts, we have

$$K_2(j\omega_1, j\omega_2) = K_R\left(j\omega_1^e, j\omega_2^e\right) + K_R\left(j\omega_1^o, j\omega_2^o\right) + jK_I\left(j\omega_1^o, j\omega_2^e\right) + jK_I\left(j\omega_1^e, j\omega_2^o\right) \tag{9}$$

where

$$K_R\left(j\omega_1^e, j\omega_2^e\right) = \int_0^\infty \int_0^\infty k_2(t_1, t_2) \cos \omega_1 t_1 \cos \omega_2 t_2 \, dt_1 dt_2 \qquad (10)$$

$$K_R\left(j\omega_1^o, j\omega_2^o\right) = -\int_0^\infty \int_0^\infty k_2(t_1, t_2) \sin \omega_1 t_1 \sin \omega_2 t_2 \, dt_1 dt_2 \qquad (11)$$

and

$$K_I\left(j\omega_1^o, j\omega_2^e\right) = -\int_0^\infty \int_0^\infty k_2(t_1, t_2) \sin \omega_1 t_1 \cos \omega_2 t_2 \, dt_1 dt_2 \qquad (12)$$

Here, symmetry has been used to change the limits of integration.

Clearly then, the inverse transform corresponding to Eq. 10, Eq. 11, or Eq. 12, can be used to find $k_2(t_1, t_2)$.

$$k_2(t_1, t_2) = \left(\frac{1}{2\pi}\right)^2 \int_0^\infty \int_0^\infty K_R\left(j\omega_1^e, j\omega_2^e\right) \cos \omega_1 t_1 \cos \omega_2 t_2 \, d\omega_1 d\omega_2 \qquad (13)$$

The conclusion is that any one of the four parts of a second-order transform is sufficient to specify the corresponding realizable impulse response. Similarly, for n = 3, the transform can be divided into eight parts:

$$K_3(j\omega_1, j\omega_2, j\omega_3) = K_R\left(\omega_1^e, \omega_2^e, \omega_3^e\right) + K_R\left(\omega_1^e, \omega_2^o, \omega_3^o\right) + K_R\left(\omega_1^o, \omega_2^e, \omega_3^o\right) + K_R\left(\omega_1^o, \omega_2^o, \omega_3^e\right)$$

$$+ j\left[K_I\left(\omega_1^o, \omega_2^o, \omega_3^o\right) + K_I\left(\omega_1^o, \omega_2^e, \omega_3^e\right) + K_I\left(\omega_1^e, \omega_2^o, \omega_3^e\right) + K_I\left(\omega_1^e, \omega_2^e, \omega_3^o\right)\right]$$

$$(14)$$

and any one of the eight parts is sufficient to specify the corresponding impulse response.

Since $K_R\left(j\omega_1^e, j\omega_2^e\right)$ completely determines the transform, one expects that there
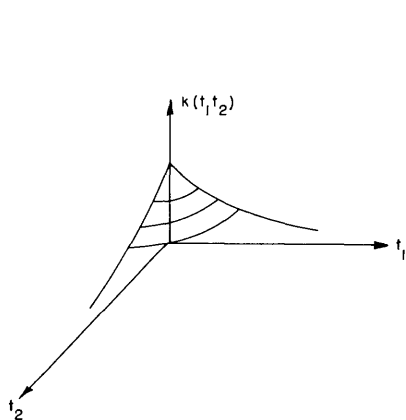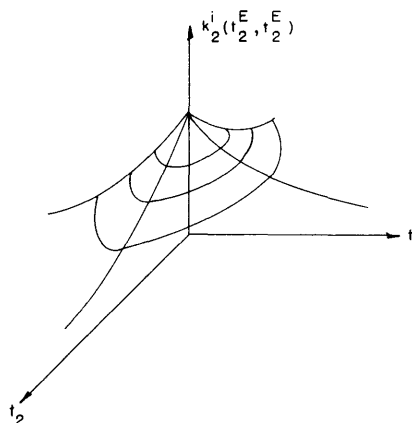

Fig. VIII-7. Typical realizable impulse response.

Fig. VIII-8. Even part of impulse response.

exists a generalized Hilbert transform relation that expresses $K_R\left(j\omega_1^o, j\omega_2^o\right)$, $K_I\left(j\omega_1^o, j\omega_2^e\right)$, and $K_I\left(j\omega_1^e, j\omega_2^o\right)$ as explicit functions of $K_R\left(j\omega_1^e, j\omega_2^o\right)$ and vice versa. A typical relation will now be derived.

2. Generalized Hilbert transform

Let $k(t_1, t_2)$ be an arbitrary, realizable kernel that is Fourier-transformable. Let

$$k(t_1, t_2) = (2\pi)^2 f(t_1, t_2) d(t_1, t_2) \qquad \text{for all } t_1, t_2 \tag{15}$$

We can choose $f(t_1, t_2)$ to be any realizable function as long as it is not zero at any point at which $k(t_1, t_2)$ is nonzero. This will specify $d(t_1, t_2)$ in the range $t_1, t_2 > 0$. The relation corresponding to Eq. 15 in the transform domain is

$$K(j\omega_3, j\omega_4) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(j\omega_3 - j\omega_1, j\omega_4 - j\omega_2) D(j\omega_1, j\omega_2) \, d\omega_1 d\omega_2 \tag{16}$$

Let

$$f(t_1, t_2) = \left(\frac{1}{\pi}\right)^2 \mu_{-1}(t_1) \exp(-a_1 t_1) \mu_{-1}(t_2) \exp(-a_2 t_2) \tag{17}$$

then

$$F(j\omega_1, j\omega_2) = \left(\frac{1}{\pi}\right)^2 \frac{a_1 - j\omega_1}{a_1^2 + \omega_1^2} \cdot \frac{a_2 - j\omega_2}{a_2^2 + \omega_2^2} = \left(\frac{1}{\pi}\right)^2 \frac{a_1 a_2 - \omega_1 \omega_2 - j(a_1\omega_2 + a_2\omega_1)}{\left(a_1^2 + \omega_1^2\right)\left(a_2^2 + \omega_2^2\right)}$$

We note that if $a_1 = a_2$, then, since $k(t_1, t_2)$ is symmetrical in its arguments, $d(t_1, t_2)$ will be symmetrical. Since $d(t_1, t_2)$ is arbitrary for either $t_1, t_2 < 0$, we can make it an even function in both arguments. This implies that $D(j\omega_2, j\omega_2)$ is real and even in both arguments.

$$K(j\omega_3, j\omega_4) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{\pi}\right)^2 \frac{a_1 - j(\omega_3 - \omega_1)}{a_1^2 + (\omega_3 - \omega_1)^2} \cdot \frac{a_2 - j(\omega_4 - \omega_2)}{a_2^2 + (\omega_4 - \omega_2)^2} \cdot D(j\omega_1, j\omega_2) \, d\omega_1 d\omega_2 \tag{19}$$

Since $D(j\omega_1, j\omega_2)$ is real, we have

$$K_R(j\omega_3, j\omega_4) = \left(\frac{1}{\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{a_1 a_2 - (\omega_3 - \omega_1)(\omega_4 - \omega_2)}{\left(a_1^2 + (\omega_3 - \omega_1)^2\right)\left(a_2^2 + (\omega_4 - \omega_2)^2\right)} D(j\omega_1, j\omega_2) \, d\omega_1 d\omega_2 \tag{20}$$

Since $D(j\omega_1, j\omega_2)$ is even in both arguments, the even part of $F_R(j\omega_1, j\omega_2)$ gives the even part of $K_R(j\omega_1, j\omega_2)$

$$K_R\left(j\omega_3^e, j\omega_4^e\right) = \left(\frac{1}{\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{a_1 a_2}{\left[a_1^2+(\omega_3-\omega_1)^2\right]\left[a_2^2+(\omega_4-\omega_2)^2\right]} D(j\omega_1, j\omega_2)\, d\omega_1 d\omega_2 \qquad (21)$$

Similarly,

$$K_I\left(j\omega_3^o, j\omega_4^e\right) = \left(\frac{1}{\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-a_2[\omega_3-\omega_1]}{\left[a_1^2+(\omega_3-\omega_1)^2\right]\left[a_2^2+(\omega_4-\omega_2)^2\right]} D(j\omega_1, j\omega_2)\, d\omega_1 d\omega_2 \qquad (22)$$

Let $a_1 = a_2 \to 0$. Then

$$\frac{a_1}{a_1^2 + (\omega_3-\omega_1)^2} \to \pi\mu_o(\omega_3-\omega_1) \qquad (23)$$

$$\frac{a_2}{a_2^2 + (\omega_4-\omega_2)^2} \to \pi\mu_o(\omega_4-\omega_2) \qquad (24)$$

and Eq. 21 becomes

$$K_R\left(j\omega_3^e, j\omega_4^e\right) = D(j\omega_3, j\omega_4) \qquad (25)$$

In Eq. 22,

$$\frac{\omega_3 - \omega_1}{a_1^2 + (\omega_3-\omega_1)^2} \to \frac{1}{\omega_3 - \omega_1} \qquad (26)$$

and this yields

$$K_I\left(j\omega_3^o, j\omega_4^e\right) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\omega_3 - \omega_1}\, D(j\omega_1, j\omega_4)\, d\omega_1 \qquad (27)$$

or

$$K_I\left(j\omega_3^o, j\omega_4^e\right) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\omega_3 - \omega_1}\, K_R\left(j\omega_1^e, j\omega_4^e\right)\, d\omega_1 \qquad (28)$$

The complete imaginary part is

$$K_I(j\omega_3, j\omega_4) = K_I\left(j\omega_3^o, j\omega_4^e\right) + K_I\left(j\omega_3^e, j\omega_4^o\right) \qquad (29)$$

where the second term follows directly from symmetry.

In retrospect, one notes that Eq. 28 could have been obtained by equating the inverse transforms of Eqs. 10 and 12:

$$\int_0^{\infty} \int_0^{\infty} K_R\left(j\omega_1^e, j\omega_2^e\right) \cos \omega_1 t_1 \cos \omega_2 t_2\, d\omega_1 d\omega_2$$

$$= -\int_0^{\infty} \int_0^{\infty} K_I\left(j\omega_1^o, j\omega_2^e\right) \sin \omega_1 t_1 \cos \omega_2 t_2\, dt_1 dt_2 \qquad (30)$$

or

$$\int_0^\infty \cos \omega_2 t_2 \, d\omega_2 \int_0^\infty K_R\left(j\omega_1^e, j\omega_2^e\right) \cos \omega_1 t_1 \, d\omega_1$$

$$= -\int_0^\infty \cos \omega_2 t_2 \, d\omega_2 \int_0^\infty K_I\left(j\omega_1^o, j\omega_2^e\right) \sin \omega_1 t_1 \, d\omega_1 \tag{31}$$

But since the outer transform relation is unique, we have

$$\int_0^\infty K_R\left(j\omega_1^e, j\omega_2^e\right) \cos \omega_1 t_1 \, d\omega_1 = -\int_0^\infty K_I\left(j\omega_1^o, j\omega_2^e\right) \sin \omega_1 t_1 \, d\omega_1 \tag{32}$$

This is merely the linear relation with an arbitrary ω_2. Thus, one can write Eq. 29 directly.

The corresponding relation follows similarly:

$$K_R\left(j\omega_3^e, j\omega_4^e\right) = +\frac{1}{\pi} \int_{-\infty}^\infty \frac{1}{\omega_3 - \omega_1} K_I\left(j\omega_1^o, j\omega_4^e\right) d\omega_1 \tag{33}$$

Similar reasoning leads to a useful constraint relation between the odd and even parts of the real part of $K_2(j\omega_1, j\omega_2)$

$$K_R\left(j\omega_1^o, j\omega_2^o\right) = \frac{1}{\pi^2} \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{\omega_1 - \omega_3} \cdot \frac{1}{\omega_2 - \omega_4} K_R\left(j\omega_3^e, j\omega_4^e\right) d\omega_3 d\omega_4 \tag{34}$$

Similar relations follow for higher-order kernels. Starting with any one of the eight terms in Eq. 14, we can find the other terms by a series of integrations that are similar to Eqs. 28 and 33.

The next step is to generalize the idea of gain-phase relations in linear networks to higher-order kernels.

3. Gain-phase relations

Let

$$\log K(j\omega_1, j\omega_2) = \log \left| K(j\omega_1, j\omega_2) \right| + j\theta(j\omega_1, j\omega_2)$$

$$= G(j\omega_1, j\omega_2) + j\theta(j\omega_1, j\omega_2) \tag{35}$$

where $G(j\omega_1, j\omega_2)$ is the gain function. Now if we can show that $G(j\omega_1, j\omega_2)$ has all of the properties of a real part of a realizable, stable system function, then $\theta(\omega_1, \omega_2)$ can be determined by using the Hilbert transforms of the previous subsection.

Clearly, if $\log K(j\omega_1, j\omega_2)$ is a realizable, stable system function, then $K(j\omega_1, j\omega_2)$ and $1/K(j\omega_1, j\omega_2)$ are realizable and stable.

First, write $G(S_1, S_2)$ and consider S_1, S_2 to be real variables. Then $G(S_1, S_2)$ must be a real function. Assuming that this condition is satisfied, we can write

$$G(j\omega_1, j\omega_2) = G\left(j\omega_1^e, j\omega_2^e\right) + G\left(j\omega_1^o, j\omega_2^o\right) \tag{36}$$

If the even and odd parts of $G(j\omega_1, j\omega_2)$ satisfy the constraints of Eq. 34, then $G(j\omega_1, j\omega_2)$ is the gain of a realizable system. This condition is sufficient but does not appear to be necessary.

To prove the minimum-phase property, we have to show that any other realizable system with the same gain function has zeros in the half-planes belonging to $\sigma_1 > 0$ and $\sigma_2 > 0$. Then we must show that this implies that the second function accumulates phase shift faster for any path in the ω_1, ω_2 plane.

Let the original minimum phase kernel be $K_M(j\omega_1, j\omega_2)$. Consider a new function

$$H(j\omega_1, j\omega_2) = K_M(j\omega_1, j\omega_2) \frac{A(j\omega_1, j\omega_2)}{B(j\omega_1, j\omega_2)} \tag{37}$$

with the constraint

$$\left| H(j\omega_1, j\omega_2) \right| = \left| K_M(j\omega_1, j\omega_2) \right| \tag{38}$$

This implies

$$\left| \frac{A(j\omega_1, j\omega_2)}{B(j\omega_1, j\omega_2)} \right| = \frac{A(j\omega_1, j\omega_2)}{B(j\omega_1, j\omega_2)} \times \frac{A(-j\omega_1, -j\omega_2)}{B(-j\omega_1, -j\omega_2)} = 1 \tag{39}$$

and yields

$$\frac{A(j\omega_1, j\omega_2)}{B(j\omega_1, j\omega_2)} = \frac{B(-j\omega_1, -j\omega_2)}{A(-j\omega_1, -j\omega_2)}. \tag{40}$$

Making the substitutions $S_1 = +j\omega_1$, $S_2 = +j\omega_2$ we obtain

$$\frac{A^2(S_1, S_2)}{B^2(S_1, S_2)} = \frac{A(S_1, S_2)}{B(S_1, S_2)} \frac{A(S_1, S_2)}{B(S_1, S_2)} = \frac{A(S_1, S_2)}{B(S_1, S_2)} \frac{B(-S_1, -S_2)}{A(-S_1, -S_2)} \tag{41}$$

For this class of rational functions, we can write

$$A(S_1, S_2) = a_{00} + a_{10}S_1 + a_{01}S_2 + a_{20}S_1^2 + a_{11}S_1 S_2 + a_{02}S_2^2 + a_{30}S_1^3$$

$$+ a_{21}S_1^2 S_2 + a_{12}S_1 S_2^2 + a_{03}S_2^3 + \cdots a_{0n}S_2^n \tag{42}$$

and

$$B(S_1, S_2) = b_{00} + b_{10}S_1 + b_{01}S_2 + b_{20}S_1^2 + b_{11}S_1S_2 + b_{02}S_2^2 + b_{30}S_1^3$$

$$+ b_{21}S_1^2S_2 + b_{12}S_1S_2^2 + b_{03}S_2^3 + \ldots b_{0n}S_2^n \qquad (43)$$

These conditions imply

$$A(S_1, S_2) = B(-S_1, -S_2) \qquad (44)$$

The phase function can be written as

$$\theta(\omega_1, \omega_2) = \tan^{-1} \left(\frac{-b_{10}\omega_1 - b_{01}\omega_2 + b_{30}\omega_1^3 + b_{21}\omega_1^2\omega_2 + \ldots}{b_{00} - b_{20}\omega_1^2 - b_{11}\omega_1\omega_2 - b_{02}\omega_2^2 + \ldots} \right)$$

$$- \tan^{-1} \left(\frac{+b_{10}\omega_1 + b_{01}\omega_2 - b_{30}\omega_1^3 - b_{21}\omega_1^2\omega_2 + \ldots}{b_{00} - b_{20}\omega_1^2 - b_{11}\omega_1\omega_2 - b_{02}\omega_2^2 + \ldots} \right) \qquad (45)$$

We want to show that $\theta(\omega_1, \omega_2)$ is a decreasing function of ω_1 and ω_2. Consider the $\omega_1 - \omega_2$ plane shown in Fig. VIII-9; we want to study an arbitrary path that is subject to the restriction that both ω_1 and ω_2 are either increasing or constant along the path. For any arbitrary path we can write $\omega_2 = g(\omega_1)$. (The only exceptions are segments on which $\omega_2 = K_o$, which can be treated separately.) Therefore, we can write the all-pass part of our transform as

$$\frac{A(S_1, S_2)}{B(S_1, S_2)} = \frac{A(S_1, jg(\omega_1))}{B(S_1, jg(\omega_2))} \qquad (46)$$

which is just a one-dimensional transform. If we can show that the system is stable for any $g(\omega_1)$, then the pole-z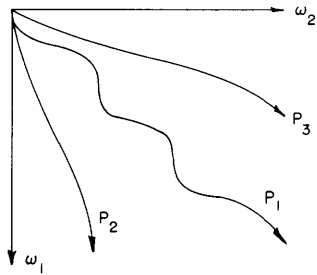ero plot in the S_1 plane is symmetric about the $j\omega_1$ axis and the desired minimum-phase property follows directly. The necessary and sufficient conditions for a multidimensional transform to be stable are shown in (d). It will be seen that stability of $\dfrac{A(S_1, S_2)}{B(S_1, S_2)}$ implies stability of $\dfrac{A(S_1, jg(\omega_1))}{B(S_1, jg(\omega_1))}$. Clearly, the all-pass term must be stable for $H(S_1, S_2)$ to be stable. Hence the desired minimum-phase property is proved.

Fig. VIII-9. Paths in ω_1, ω_2 plane.

Therefore, the even part of any allowable gain function specifies a phase

function through the use of Eqs. 28 and 29, so that the resulting system is realizable. This system is not unique, but it has the minimum-phase property demonstrated above.

The extension of this discussion to a kernel of any order follows directly.

4. Stability of nth-order kernels

We would like to investigate the stability of rational transforms defined in Eq. 2. For $k(t_1, t_2)$ to be a realizable, stable impulse response, it is necessary and sufficient for the integral to converge for all $\sigma_1 : 0 \le \sigma_1 < \infty$ and $\sigma_2 : 0 \le \sigma_2 < \infty$. For rational transfer functions, a generalized Routh criterion may be used.

Consider the denominator of Eq. 4. Instead of writing it as a double sum, we shall construct a simple array of the coefficients.

$$
\begin{array}{ccccc}
 & S_2 & S_2^2 & S_2^3 \\
a_{00} & a_{01} & a_{02} & a_{03} \\
S_1 & a_{10} & a_{11} & a_{12} \\
S_1^2 & a_{20} & a_{21} \\
S_1^3 & a_{30}
\end{array}
$$

Initially, we can apply the Routh test to a polynomial in S_1:

$$P(S_1) = A + BS_1 + CS_1^2 + DS_1^3 \tag{47}$$

where

$$A = \sum_{i=0}^{3} a_{0i} S_2^i, \quad B = \sum_{i=0}^{2} a_{1i} S_2^i, \quad C = \sum_{i=0}^{1} a_{2i} S_2^i, \quad D = a_{30}$$

The requirements for stability are

$$A, B, C > 0 \qquad D \ge 0 \qquad BC - DA > 0 \tag{48}$$

Since A, B, and C are functions of S_2, we have to satisfy the inequalities in expression 48 for $S_2 : 0 \le \sigma_2 < \infty$. Assume $a_{00} > 0$. Then the requirement, A > 0 for all $S_2 : 0 \le \sigma_2 < \infty$, implies that

$$A = a_{00} + a_{01} S_2 + a_{02} S_2^2 + a_{03} S_2^3 = 0 \tag{49}$$

can have no roots with positive real parts. Using the Routh criterion gives the requirements

$$a_{03} \geq 0; \quad a_{02}, \ a_{01}, \ a_{00} > 0; \quad a_{02}a_{00} > a_{03}a_{01} \tag{50}$$

Similarly, $B > 0$ requires

$$a_{10}, \ a_{11} > 0; \quad a_{12} \geq 0 \tag{51}$$

and $C > 0$, $D \geq 0$ require

$$a_{20} > 0; \quad a_{21} \geq 0; \quad a_{30} \geq 0 \tag{52}$$

The inequality $BC-DA > 0$ requires that

$$\left(a_{10}+a_{11}S_2+a_{12}S_2^2\right)(a_{20}+a_{21}S_2) - a_{30}\left(a_{00}+a_{01}S_2+a_{02}S_2^2+a_{03}S_2^3\right) = 0 \tag{53}$$

shall have no roots with positive real parts. This implies the following inequalities:

$$a_{12}a_{21} - a_{03}a_{03} \geq 0; \quad a_{11}a_{21} + a_{20}a_{12} - a_{30}a_{02} > 0; \quad a_{10}a_{21} + a_{11}a_{20} - a_{30}a_{01} > 0;$$

$$a_{10}a_{20} - a_{30}a_{00} > 0 \tag{54}$$

and

$$(a_{11}a_{21}+a_{20}a_{12}-a_{30}a_{02})(a_{10}a_{21}+a_{11}a_{20}-a_{30}a_{01}) - (a_{12}a_{21}-a_{30}a_{03})(a_{10}a_{20}-a_{30}a_{00}) > 0$$

Inequalities 49-54 must be satisfied for the transform to represent a realizable and stable impulse response. If the region of convergence includes the $j\omega_1$ and $j\omega_2$ axes, then failure to satisfy one of the inequalities implies unrealizability. If the region of convergence lies to the right of the locus of singularities in the S_1 and S_2 planes, then the transform represents a realizable, but unstable system.

The extension to n dimensions requires increasing the dimension of the array. The Routh criterion is applied in succession to each dimension.

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