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USE OF A SCALAR POTENTIAL IN TWO-DIMENSIONAL MAGNETOSTATIC COMPUTATIONS WITH DISTRIBUTED CURRENTS

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ABSTRACT: Relaxation computations to provide the solution to two-dimensional magnetostatic problems in the presence of simple current distributions may be performed by the aid of scalar potential functions. The standard algorisms for determining an improved value of the "potential" at certain points need only be supplemented by the addition of prescribed constants to allow for the current. Some incomplete comments are appended concerning the possibility of applying similar methods to the "scaling" field of a spiral-sector accelerator.

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1. The Derivation of Magnetic Fields from Scalar Potentials:

We indicate below methods whereby a two-dimensional magnetic field may be derived from scalar "potential" functions in cases such that, where currents are present, the current-density is a function of x only or of y only. For each of the methods attention should also be directed to the continuity conditions at the boundaries of the current-carrying region and to the need for a "cut" to avoid encountering multiple valued functions.

In each case it is required that div $\vec{B} = 0$ everywhere and that curl $\vec{H} = 4\pi \vec{j}$ at points where current is present (unrationalized emu). We consider the permeability to be unity at all points occupied by current.

(a) $\vec{j} = j(y)\,\hat{e}_z$:

(i) If one writes

$$H_x = -\frac{\partial \Psi}{\partial x} \quad \text{and} \quad H_y = -\frac{\partial \Psi}{\partial y} + 4\pi (x + c)\, j(y),$$

the curl condition on $\vec{H}$ is satisfied identically and the divergence condition is satisfied by requiring

$$\nabla \cdot (\mu \nabla \Psi) = 4\pi\mu (x + c)\frac{\partial j}{\partial y}.$$

(ii) Alternatively, if one writes

$$H_x = -\frac{\partial \Psi}{\partial x} - 4\pi \int^y j(y)\, dy \quad \text{and} \quad H_y = -\frac{\partial \Psi}{\partial y},$$

the curl condition is again satisfied and the divergence condition requires

$$\nabla \cdot (\mu \nabla \Psi) = 0.$$

(b) $\vec{j} = j(x)\,\hat{e}_z$:

(i) If one writes

$$H_x = -\frac{\partial \Psi}{\partial x} - 4\pi (y + c)\, j(x) \quad \text{and} \quad H_y = -\frac{\partial \Psi}{\partial y},$$

the curl condition is automatically satisfied and the divergence condition requires

$$\nabla \cdot (\mu \nabla \Psi) = -4\pi\mu\,(y + c)\frac{\partial j}{\partial x} \quad .$$

(ii) Alternatively, if one writes

$$H_x = -\frac{\partial \Psi}{\partial x} \quad \text{and} \quad H_y = -\frac{\partial \Psi}{\partial y} + 4\pi\int^{x} j(x)\,dx,$$

the curl condition is again satisfied and the divergence condition requires

$$\nabla \cdot (\mu \nabla \Psi) = 0.$$

As mentioned above, boundary conditions would have to be examined for any particular method adopted. Additional methods of employing a scalar potential might also be contrived -- as by superposition. We consider below the boundary conditions for the specific case a - ii.

2. The Boundary Conditions:

We consider here in some detail the nature of the boundary condition for a specific case. Outside the region occupied by current, the magnetic field may be expressed simply as the negative gradient of a scalar potential function V, save for the necessary introduction of a cut (extending to the current-carrying region) across which there must be a discontinuity of V given by 4π I or 4π j (Area): *

$$H_x = -\frac{\partial V}{\partial x}, \qquad H_y = -\frac{\partial V}{\partial y}, \qquad \text{outside.}$$

Within the current-carrying region we write

$$H_x = -\frac{\partial \Psi}{\partial x} - 4\pi\int_{y_{ref}}^{y} j(y)\,dy, \qquad H_y = -\frac{\partial \Psi}{\partial y},$$

with (cf. Sect. a-ii)

$$\nabla \cdot (\mu \nabla \Psi) = 0.$$

Since, with distributed currents, $\vec{H}$ is continuous (away from the boundary of a magnetic medium, where only H_t and B_n are continuous), it follows that

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* It is seen that I represents the number of abampere-turns.

$$\frac{\partial V}{\partial x} = \frac{\partial \Psi}{\partial x} + 4\pi \int_{y_{ref}}^{y} j(y)\, dy$$

$$\frac{\partial V}{\partial y} = \frac{\partial \Psi}{\partial y} \qquad \text{at a boundary of a current region;}$$

if j is independent of y as well as being independent of x,

$$\frac{\partial V}{\partial x} = \frac{\partial \Psi}{\partial x} + 4\pi\, j\, (y - y_{ref})$$

$$\frac{\partial V}{\partial y} = \frac{\partial \Psi}{\partial y} \qquad \text{at a boundary.}$$

From the above connections between the derivatives of V and the derivatives of Ψ it is clear that at a boundary (between a current-carrying region and a region free of current) there will be a discontinuity between Ψ and V which augments by $4\pi I$ as one progresses around the boundary. Specifically, continuing for simplicity with the case j = const., if we have a discontinuity $V - \Psi = K$ at the cut, the discontinuity elsewhere on the boundary is given by

$$V(P) = \Psi(P) + K + 4\pi j \int_{cut}^{P} (y - y_{ref})dx;$$

upon progressing counter-clockwise completely around the boundary to the other side of the cut the difference $V - \Psi$ grows to attain the value $K - 4\pi\, j \cdot (\text{area})$ or $K - 4\pi I$.

With respect to higher order derivatives, the relationships between the first partial derivatives of V and Ψ are identities in y along a vertical boundary, permitting differentiation with respect to y, and are identities in x along a horizontal boundary, permitting differentiation with respect to x. One concludes in this manner that

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 \Psi}{\partial x^2}$$

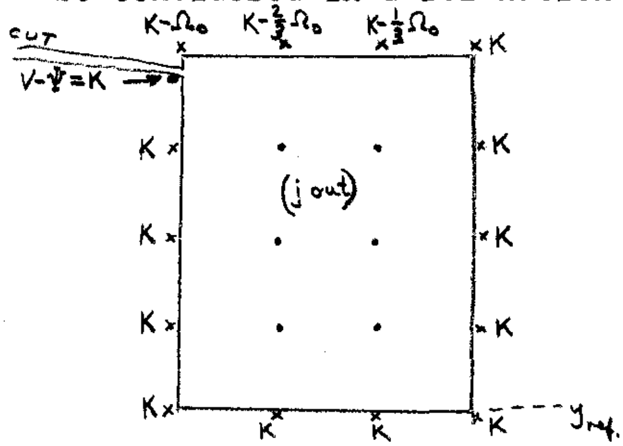
$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 \Psi}{\partial y^2}$$

$$\frac{\partial^2 V}{\partial x \partial y} = \begin{cases} \dfrac{\partial^2 \Psi}{\partial x \partial y} + 4\pi j & \text{at a vertical boundary} \\[2ex] \dfrac{\partial^2 \Psi}{\partial x \partial y} & \text{at a horizontal boundary.} \end{cases}$$

(It will be noted from this last relation that there is an ambiguity concerning the cross derivative at a corner, which is directly connected to the fact that (despite the continuity of H_x, in particular, on a vertical boundary and of H_y on a horizontal boundary) it is inconsistent to assert that $\frac{\partial H_x}{\partial y}$ and $\frac{\partial H_y}{\partial x}$ exist as continuous quantities at a corner where curl H is discontinuous. This difficulty is not of great importance in what follows, but if one wishes to employ the cross derivative in a formal way at a corner it may be considered least objectionable to take $\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 \Psi}{\partial x \partial y} + 2\pi j$.)

Example:

We give here an illustration of the discontinuity between V & Ψ for a rectangular coil of width 3h and height 4ℓ. We denote $4\pi I = 4\pi j \cdot (\text{area}) = 4\pi j\,(12h\ell)$ by Ω_o. The points along the boundary at which the "potential" might conveniently be considered in a relaxation problem are denoted by x in Fig. 1.



The quantities affixed to these points denote the values of $V - \Psi$ if $y_{ref.}$ is taken as the ordinate of the lower boundary of the coil.

Fig. 1. Illustration of Progressive Discontinuity of $V - \Psi$.

3. Modification of Algorisms for the Potential:

For a harmonic potential one of the following standard algorisms is normally employed:

4-Point Algorism:

$$V'_{00} = \left[\begin{array}{ccc} & \frac{1}{2(1+r^2)} V_{01} & \\ + \frac{r^2}{2(1+r^2)} V_{-10} & & + \frac{r^2}{2(1+r^2)} V_{10} \\ & + \frac{1}{2(1+r^2)} V_{0-1} & \end{array} \right]$$

8-Point Algorism:

$$\left[\begin{array}{ccc} \frac{1}{20} V_{-11} & + \frac{5-r^2}{10(1+r^2)} V_{01} & + \frac{1}{20} V_{11} \\ + \frac{5r^2-1}{10(1+r^2)} V_{-10} & & + \frac{5r^2-1}{10(1+r^2)} V_{10} \\ + \frac{1}{20} V_{-1-1} & + \frac{5-r^2}{10(1+r^2)} V_{0-1} & + \frac{1}{20} V_{1-1} \end{array} \right],$$

where $r \equiv \ell/h$ with ℓ denoting the vertical interval and h the horizontal interval between adjacent mesh points.

In the present problem such relations must be re-examined for those cases in which both functions, V and Ψ, are involved. As an illustrative case we may consider the algorism for the potential (V) at the point in Fig. 1 which lies on the top boundary one unit (h) to the right of the cut. By Maclaurin expansions and use of the continuity relations one obtains the following equations in which the derivatives are evaluated at this point:

$$V_{10} \cong V_{00} + h V_x + \tfrac{1}{2} h^2 V_{xx}$$

$$V_{-10} \cong V_{00} - h V_x + \tfrac{1}{2} h^2 V_{xx}$$

$$V_{01} \cong V_{00} + \ell V_y + \tfrac{1}{2} \ell^2 V_{yy}$$

$$\Psi_{0-1} \cong \Psi_{00} - \ell \Psi_y + \tfrac{1}{2} \ell^2 \Psi_{yy}$$

$$= V_{00} - K + \tfrac{2}{3} \Omega_0 - \ell V_y + \tfrac{1}{2} \ell^2 V_{yy}$$

$$V_{11} \cong V_{00} + h V_x + \ell V_y + \tfrac{1}{2} h^2 V_{xx} + h \ell V_{xy} + \tfrac{1}{2} \ell^2 V_{yy}$$

$$V_{-11} \cong V_{00} - h V_x + \ell V_y + \tfrac{1}{2} h^2 V_{xx} - h \ell V_{xy} + \tfrac{1}{2} \ell^2 V_{yy}$$

$$\Psi_{1-1} \cong \Psi_{00} + h\Psi_x - l\Psi_y + \tfrac{1}{2}h^2\Psi_{xx} - hl\Psi_{xy} + \tfrac{1}{2}l^2\Psi_{yy}$$
$$= V_{00} - K - \tfrac{2}{3}\Omega_0 + hV_x - lV_y + \tfrac{1}{2}h^2V_{xx} - hlV_{xy} + \tfrac{1}{2}l^2V_{yy}$$

$$\Psi_{-1-1} \cong \Psi_{00} - h\Psi_x - l\Psi_y + \tfrac{1}{2}h^2\Psi_{xx} + hl\Psi_{xy} + \tfrac{1}{2}l^2\Psi_{yy}$$
$$V_{-1-1} \cong V_{00} - \tfrac{2}{3}\Omega_0 - hV_x - lV_y + \tfrac{1}{2}h^2V_{xx} + hlV_{xy} + \tfrac{1}{2}l^2V_{yy}.$$

By multiplying the above expressions for V_{10}, V_{-10}, V_{01}, Ψ_{0-1} or for V_{10}, V_{-10}, V_{01}, Ψ_{0-1}, V_{11}, V_{-11}, Ψ_{1-1}, V_{-1-1} by the appropriate weights appearing in the corresponding algorism, one finds

$$V_{00} = (\text{std. 4-point algorism}) + \frac{K - \frac{2}{3}\Omega_0}{2(1+r^2)} \qquad \text{or}$$

$$V_{00} = (\text{std. 8-point algorism}) + \frac{\frac{1}{20}\left((11-r^2)K - \frac{2}{5}\Omega_0\right)}{1+r^2}.$$

The constant term which must be appended to the algorims for other points may be obtained similarly, with attention given to possible discontinuities in the cross derivative entering into derivation of the 8-point algorism.

Upon further examination it appears that a relatively simple receipt may be given for obtaining the supplemental term, denoted (CV), for any particular point. We write this formula below for the case j = constant:

$$[CV] = \sum_{P_i} w_i \left\{ \frac{\Omega_0}{Area}\left(\bar{y} - y_{ref}\right)(\Delta x) - \sum_{\text{along path}} (\text{jumps of potential}) \right\}.$$

In this equation w_i represents the weight which the algorism in question attaches to the point P_i, $\bar{y}$ denotes the average value of y and Δx the x-displacement along each of those straight lines which traverse the current region in going from the central point (0,0) to P_i, and the sum of sudden jumps of potential representing discontinuities between V and Ψ is to be formed along the same paths and should also include any effect of the cut in the exterior region. The expression $\frac{\Omega_0}{Area}$ represents $4\pi j$ and should carry the sign of the current (positive if j is directed out of the paper).

Example:

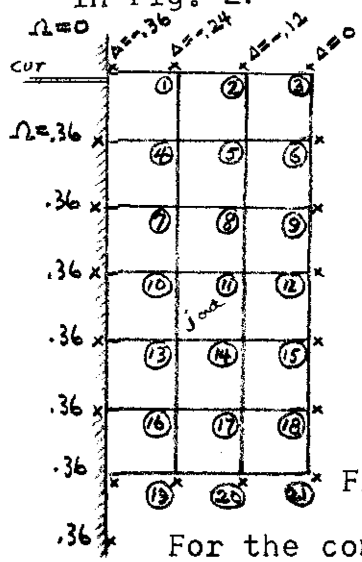
By way of illustration, this formula would involve, in the example for which the Maclaurin expansions were made, the points (-1,-1), (0,-1), and (1,-1) in the 8-point algorism; thus

$$
\begin{aligned}
[CV] &= \frac{1}{20}\left\{\frac{\Omega_0}{12hl}\left(\frac{7}{2}l\right)(-h) - \left[\left(-K+\frac{2}{3}\Omega_0\right)+K\right]\right\} \\
&\quad + \frac{5-r^2}{10(1+r^2)}\left\{\frac{\Omega_0}{12hl}\left(\frac{7}{2}l\right)(0) - \left[-K+\frac{2}{3}\Omega_0\right]\right\} \\
&\quad + \frac{1}{20}\left\{\frac{\Omega_0}{12hl}\left(\frac{7}{2}l\right)(h) - \left[-K+\frac{2}{3}\Omega_0\right]\right\} \\
&= \frac{3}{5(1+r^2)}\left[K-\frac{2}{3}\Omega_0\right] - \frac{1}{20}K \\
&= \frac{\frac{1}{20}(11-r^2)K - \frac{2}{5}\Omega_0}{1+r^2}, \text{ as before.}
\end{aligned}
$$

4. A Detailed Example:

In the preceding paragraphs we have emphasised the method of Sect. 1 a-ii in the belief that this method may be a good procedure on which to standardize for the introduction of currents (in a two-dimensional approximation) into the FOROCYL computational program. The program, as written by Dr. J. N. Snyder, is prepared to accept current values in the form of prescribed constants whereby the standard algorism is modified at specified points. It may, therefore, be of interest to illustrate this method below in some detail for the case of a 3 x 6 coil ($r = 1$) intended to provide the magnetomotance for a magnet pole against which it is located.

Since the IBM computational program is designed to work with "potentials" Ω less than 1/2, we arbitrarily take the magnetomotance to be 0.36 units, corresponding to 0.02 units per each basic square mesh cell of the coil. The discontinunities of potential, $\Delta \equiv V - \Psi$, are indicated along the top of the coil in Fig. 2.



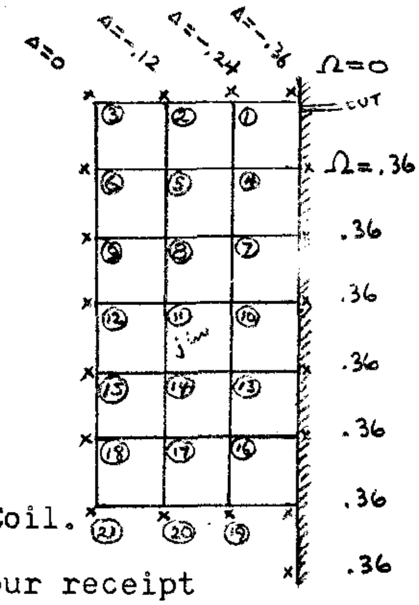


Fig. 2. Location of Current-Carrying Coil.

For the configuration of Fig. 2 application of our receipt gives the following "current values", which should supplement the standard algorism, for points associated with each of the coils:

Point	CV	
	For 4-Point Algorims	For 8-Point Algorims
1	-.06	-.072
2	-.03	-.036
3	0	-.0055
4	+.06	+ .072
5	+.03	+.036
6	-.025	-.024
7	0	0
8	0	0
9	-.02	-.024
10	0	0
11	0	0
12	-.015	-.018
13	0	0
14	0	0

15	-.01	-.012
16	0	0
17	0	0
18	-.005	-.006
19	0	0
20	0	0
21	0	-.0005

For a similar coil providing magnetomotance for a "negative" magnet (field directed up into the iron), the current values would, of course, be of opposite sign.

Some trial FOROCYL computations were made for coils of the type illustrated, with current values derived from the table. By use of an extensive (90 x 52) mesh the case of a coil on an infinitely long pole could be closely simulated and the interference of one coil with the field produced by the other could be regarded as small. (For convenience in preparing the computations, the fields and consequently the current values were reversed from the case illustrated in Fig. 2 ($V_2 = .36 - V_1$, $CV_2 = -CV_1$, $j_2 = -j_1$). The coil extended from j = 23 to j = 29. The value of Ω entered at j = 51 was shaded off somewhat in the expected way from 0.36 to 0.2696 as one approched the center of the space between the poles (i = 45) and values extending from 0 to 0.0904 were similarly entered at j = 1. Along the vertical pole-surfaces Ω had the value 0 for $1 \leq j \leq 28$ and the value 0.36 for $29 \leq j \leq 51$. In FOROCYL run 104 only the pure Laplace phase, with the 4-point algorism was used; in run 105 the main phase was used but in an almost Laplacian manner, since the constants $1/w = 0$, $k' = 0.5$ and N = 3850 were employed. In citing the results of these computations we shall, for convenience, reconvert the potentials to those appropriate to the more conventional field-polarity illustrated in Fig. 2.)

The results of the FOROCYL computations may be compared with the image calculations of Mr. Weinberg (Elliot Weinberg, MURA notes dtd. 6/30/56 and revised graph subsequently distributed) for a single coil situated on an infinitely long pole. For this case the exterior potential is found to vary very nearly linearly as one proceeds around the exposed periphery of the coil and, in particular, to conform exactly to this linear relationship at the points denoted 3, 12, and 21 in Fig. 2. The results are summarized below:

Point	Nominal Value	Image Calculation	Computational Results: With 4-Point Algorism	Computational Results: With 8-Point Algorims
Pole, above cut	0	0	(0)	(0)
1	.03	.03279168	.0326	.0326
2	.06	.06373808	.063 4	.0634
3	.09	.09	.089 7	.0896
6	.12	.11626192	.116 2	.1159
9	.15	.14720832	.147 1	.1469
12	.18	.18	.179 9	.1798
15	.21	.21279168	.212 7	.2126
18	.24	.24373808	.243 7	.2436
21	.27	.27	.270 1	.2700
20	.30	.29626192	.296 5	.2963
19	.33	.32720832	.327 3	.3272
Pole Below Cut	.36	.36	(.36)	(.36)

Similarly, if the configuration of Fig. 2 is represented by a mesh with r = 2/3, so that a single coil would be 3h wide by 9ℓ high, the appropriate current values would be

Point	CV (8-Point)	Point	CV (8-Point)	Point	CV (8-Point)
1	-.099 692 3077	11	0	21	-.0073846154
2	-.049 846 1538	12	-.014 769 2308	22	0
3	-.005 666 6667	13	0	23	0
4	+.099 692 3077	14	0	24	-.0049230769
5	+.049 846 1538	15	-.012 307 6923	25	0
6	-.013 692 3077	16	0	26	0
7	0	17	0	27	-.0024615385
8	0	18	-.009 846 1538	28	0
9	-.017 230 7692	19	0	29	0
10	0	20	0	30	-.0003333333

The results of such a run are summarized below:

Point	Nominal Value	Image Calc.	Comput. Result	Point	Nominal Value	Imag Calc.	Comput. Result
Pole, above cut	0	0	(0)	18	.19	.19101585	.1920
1	.03	.03279168	.0332	21	.21	.21279168	.2138
2	.06	.06373808	.0646	24	.23	.23376174	.2347
3	.09	.09	.0915	27	.25	.25323033	.2540
6	.11	.10676967	.1081	30	.27	.27	.2706
9	.13	.12623826	.1274	29	.30	.29626192	.2968
12	.15	.14720832	.1483	28	.33	.32720832	.3275
15	.17	.16898415	.1700	Pole, Below Cut	.36	.36	(.36)

(In the computational examples, convergence of the iteration process was not necessarily complete. Thus, in the last example (with r = 2/3 and a "90 x 77" mesh) the potential for point "6" would be .1092 after 393 main iterations and .1081 after 781 main iterations (requiring about 6 hrs.). The agreement with the expected value appeared sufficiently good for the present purpose, however, that, in the interest of saving computer time, the problem was not carried further.)

5. Conclusion Concerning Application of Two-Dimensional Analysis:

From the analysis and examples given, the method described in the preceding sections appears to provide an adequate and convenient means for including spatially-extended currents, of uniform current density, in substantially two-dimensional problems for the determination of magnetic fields by relaxation methods. The possible application of similar techniques to the scaling field of FFAG accelerators (cf. MURA-LJL-8, Revised) remains to be investigated. Because of the relatively minor rôle commonly played by the current-distribution (save to determine the magnetomotive potential developed in the poles) and the comparitively simple form of the results for a strictly two-dimensional configuration, one may in any case be inclined to exploit the close similarity between FFAG and two-dimensional problems to retain the procedure described here in formulating FOROCYL agenda.

6. Extension to a Scaling Three-Dimensional Field:

It appears possible again to employ a scalar "potential" function in certain problems involving three-dimensional magnetic fields which scale in the sense of MURA-LJL-8 (Rev.). The problem is not uniquely defined, however, unless information is available concerning the direction of the currents which are introduced to augment the main magnetizing current as one proceeds to larger and larger radii.

If one supposes that the magnetic field is given in the usual way from the gradient of a scalar-potential plus supplementing terms which are only required in the presence of currents, one writes

$$H_z = -(1+x)^k \frac{\sqrt{1+(\omega N)^2}}{2\pi\omega} \frac{\partial\Omega}{\partial\eta} + (1+x)^k f(\xi,\eta)$$

$$H_r = -(1+x)^k \left\{(k+1)\Omega + \frac{1}{2\pi\omega}\frac{\partial\Omega}{\partial\xi} - \eta\frac{\partial\Omega}{\partial\eta}\right\} + (1+x)^k g(\xi,\eta)$$

$$H_\theta = (1+x)^k \frac{N}{2\pi}\frac{\partial\Omega}{\partial\xi} + (1+x)^k h(\xi,\eta).$$

The current-density is derived from H by taking the curl and will, of course, be divergence-free since div curl vanishes identically.

$$\frac{4\pi r_1}{(1+x)^{k-1}} J_z = (k+1)h + \frac{1}{2\pi\omega}\frac{\partial h}{\partial\xi} - \eta\frac{\partial h}{\partial\eta} + \frac{N}{2\pi}\frac{\partial g}{\partial\xi}$$

$$\frac{4\pi r_1}{(1+x)^{k-1}} J_r = -\frac{N}{2\pi}\frac{\partial f}{\partial\xi} - \frac{\sqrt{1+(\omega N)^2}}{2\pi\omega}\frac{\partial h}{\partial\eta}$$

$$\frac{4\pi r_1}{(1+x)^{k-1}} J_\theta = \frac{\sqrt{1+(\omega N)^2}}{2\pi\omega}\frac{\partial g}{\partial\eta} - \frac{1}{2\pi\omega}\frac{\partial f}{\partial\xi} + \eta\frac{\partial f}{\partial\eta} - kf(\xi,\eta).$$

It may be noted that the vector line-element associated with changes in the coordinates x, ξ, and η is

$$\vec{ds} = r_1 \left\{\left[\hat{e}_r + \frac{2\pi\omega}{\sqrt{1+(\omega N)^2}}\eta\,\hat{e}_z + \frac{1}{\omega N}\hat{e}_\theta\right]dx \right.$$

$$\left. -\left[(1+x)\frac{2\pi}{N}\hat{e}_\theta\right]d\xi + \left[(1+x)\frac{2\pi\omega}{\sqrt{1+(\omega N)^2}}\hat{e}_z\right]d\eta;\right.$$

hence the unit vector in such a direction that ξ and η remain constant is

$$\hat{N} = \frac{\hat{e}_r + \frac{2\pi\omega}{\sqrt{1+(\omega N)^2}}\eta\,\hat{e}_z + \frac{1}{\omega N}\hat{e}_\theta}{\frac{\sqrt{1+(\omega N)^2}}{\omega N}\left[1 + \frac{4\pi^2\omega^4 N^2\eta^2}{[1+(\omega N)^2]^2}\right]^{\frac{1}{2}}}.$$

In determining the functions f, g and h it would be desirable (i) to arrange for the contribution to div H which arises from these supplemental terms to be zero, in order that Ω satisfy the differential equation which prevails in the absence of currents,

$$(k+1)g + \frac{1}{2\pi w}\frac{\partial g}{\partial \xi} - \eta\frac{\partial g}{\partial \eta} - \frac{N}{2\pi}\frac{\partial h}{\partial \xi} + \frac{\sqrt{1+(wN)^2}}{2\pi w}\frac{\partial f}{\partial \eta} = 0$$

and (ii) to choose functions such that $\vec{J}\cdot\hat{N}$ is substantially independent of ξ and η, with $\vec{J}$ itself directed primarily in the direction $\hat{N}$.

An example of the type of functions which might be considered to meet these conditions is

$$f = -\left\{\left[\frac{k-2}{2k+1} - \frac{(wN)^2}{k}\right]\frac{\sqrt{1+(wN)^2}}{2\pi(wN)^2} + \frac{\pi k w^2\eta^2}{\sqrt{1+(wN)^2}}\right\}c$$

$$g = c w \eta$$

$$h = -\frac{k(k-2)}{2k+1}\frac{c\eta}{N},$$

where the constant "c" is related to the spatial density of the current in the magnetizing windings and hence to the magnetomotance which these windings develop. With the foregoing form for the functions f, g, and h, the condition div $\vec{H} = 0$ leads to the customary differential equation for Ω, namely that which prevails in the absence of currents. The current-densities are of the form

$$\frac{4\pi r_1}{(1+x)^{k-1}} J_z = -\frac{k^2(k-2)}{2k+1}\frac{c\eta}{N}$$

$$\frac{4\pi r_1}{(1+x)^{k-1}} J_r = +\frac{k(k-2)}{2k+1}\frac{\sqrt{1+(wN)^2}}{2\pi w N}c$$

$$\frac{4\pi r_1}{(1+x)^{k-1}} J_\theta = +\left[\frac{k(k-2)}{2k+1}\frac{\sqrt{1+(wN)^2}}{2\pi(wN)^2} + k(k-2)\frac{\pi w^2\eta^2}{\sqrt{1+(wN)^2}}\right]c.$$

This current is divergence free, with the strongest components J_r and the η-independent part of J_θ in the ratio $1:\frac{1}{wN}$. The component of current-density $\vec{J} \cdot \hat{N}$ is independent of η through terms in η^2.

If this or some similar current configuration is regarded as representative of the current distribution of interest, the coil geometry must then be specifically considered in order to associate the scale-factor "c" with the total magnetomotance developed by the coil, to determine the boundary relationships, and to develop suitable algorisms for the potential function which are of the standard form save for an additive term. These detailed steps have not, however, as yet been carried through for any particular case and may involve some annoying complexity.