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MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

THE REACTION OF A CAVITY ON THE BEAM CURRENT

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November 30, 1956

ABSTRACT: This report investigates the action of a cavity gap on the accelerator beam current. The point of view will be to consider the gap as part of a cavity fed from an input line, and not as a fixed voltage generator. The gap voltage is found to be the superposition of two voltages: one related directly to the line input voltage, the second induced by the passing particle beam. The mathematics used in the report are similar to those presented by Slater.[1] They lead, in a rigorous manner, to the representation of the loaded cavity in terms of current generators and resonant circuits, a representation which has often been used on intuitive grounds.

*Supported by Contract AEC #AT(11-1)-384

1. J. C. Slater: "Microwave Electronics." Ch. IV.

I. General form of the equivalent circuit of a cavity containing currents.

(a) the normal modes of the cavity.

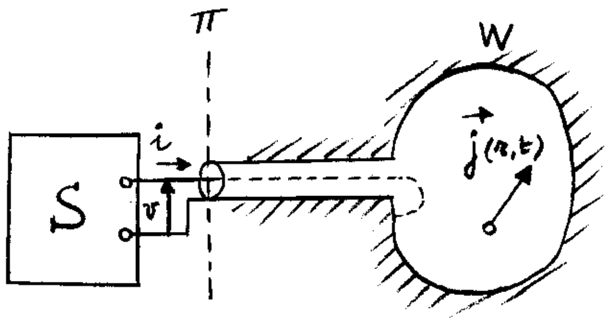

Fig. 1

The cavity, with perfectly conducting walls, is connected to an outside system S through a coaxial line. We shall assume that the frequency is sufficiently low for the lowest mode (the TEM mode of elementary transmission line theory) to be the only propagated wave along the line.[1] Let π be a plane where all higher modes have been attenuated. Voltage and current should be found in that particular plane. We take the following sign conventions: current positive when it goes into the cavity through the center conductor; voltage positive when the center conductor is positive with respect to the shield. The choice of a coaxial line input is a natural one at the frequencies involved in accelerator techniques, while the neglect of cavity wall losses will avoid mathematical difficulties.

The mathematical study of the cavity is based on the use of two sets of eigenvectors. The first one is generated by the vectorial differential equation:

$$-\text{curl curl } \vec{e}_a + \frac{w_a^2}{c^2} \cdot \vec{e}_a = 0 \tag{1.1}$$

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1. This condition is not very restrictive in the present accelerator problem. The second lowest mode is launched at a wavelength roughly equal to $\pi(R_o + R_i)$ where R_o and R_i are resp. the outer and inner radii of the coaxial line. This wavelength is of the order of a fraction of a meter or less for practical lines, i.e., the cut-off frequency is of the order of 1000 Mc/s or more.

with the following boundary conditions:[1] $\vec{e_a}$ normal to the cavity walls W, curl $\vec{e_a}$ normal to the π plane ($\vec{e_a}$ tangential to π). Let the various $\vec{e_a}$ be normalized in the space "coaxial line + cavity" (see Appendix 1 for the dimensions of the quantities appearing in this report). The real vectors $\vec{e}_a(r)$, all of which are solenoidal, form an orthonormal set, suitable to expand the solenoidal part of a vector field.

The second set of eigenvectors is $\vec{f}_b = \mathrm{grad}\, \varphi_b$, where φ_b is an eigenfunction of

$$\nabla^2 \varphi_b + \frac{\omega_b^2}{c^2} \cdot \varphi_b = 0 \qquad (1.2)$$

vanishing on the walls W and the terminal plane π . The normalized $\vec{f}_b$ form an orthonormal set of irrotational vectors, suitable to expand the irrotational part of a vector field. In accordance herewith, electrical field and current densities will be expanded as

$$\vec{e}(r,t) = \sum_a e_a(t) \cdot \vec{e}_a(r) + \sum_b f_b(t) \cdot \vec{f}_b(r) \qquad (1.3)$$

$$\vec{j}(r,t) = \sum_a j_a(t) \cdot \vec{e}_a(r) + \sum_b j_b(t) \cdot \vec{f}_b(r) \qquad (1.4)$$

The time-dependent coefficients can be given the name of "coupling coefficients" to the various "a" and "b" modes. The second part of the current expansion is irrotational, and is associated with charge effects in the gap.

Of interest for the sequel is the quantity V_a, computed at the terminal plane π, and representing in effect the average value of the "difference of

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1. The theory could also be developed with eigenvectors $\vec{e_a}$ normal to W and π , but in a less convenient form.

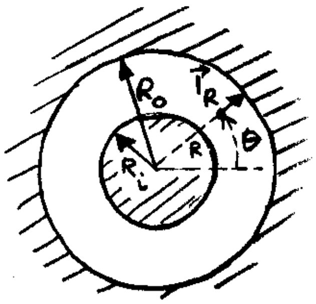

Fig. 2

potential" created by the "a"th mode between inner and outer conductor. The formula for $\mathcal{V}_a$ is:

$$\mathcal{V}_a = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_{R_o}^{R_i} \left(\vec{e}_a \cdot \vec{1}_R\right) dR \tag{1.5}$$

The actual value of $\mathcal{V}_a$ depends, among other factors, on the coupling between coaxial line and cavity.

(b) equivalent circuits

The problem to be solved can be expressed as follows: given j_a and j_ℓ, find e_a and e_ℓ. The differential equation for e_ℓ is very simple:

$$j_\ell + \varepsilon \frac{d f_\ell}{dt} = 0 \tag{1.6}$$

This equation, a consequence of the equation of continuity, will allow computation of that part of the electric field due to space-charge effects, namely

$$\Sigma\, f_\ell(t)\, \vec{f}_\ell(r)$$

The coefficient e_a is found to depend only on j_a and on the coaxial-line current i.

$$\varepsilon\mu \frac{d^2 e_a}{dt^2} + \frac{\omega_a^2}{c^2} e_a = -\mu \frac{d j_a}{dt} + \mu \mathcal{V}_a \frac{di}{dt} \tag{1.7}$$

This equation can be solved by operational methods for example. To take a simple situation, if all initial values are zero, i.e., if the system starts from rest, then $E_a(p)$ can be found from the simple equivalent circuit of Fig. 3, where capital letters indicate Laplace transforms

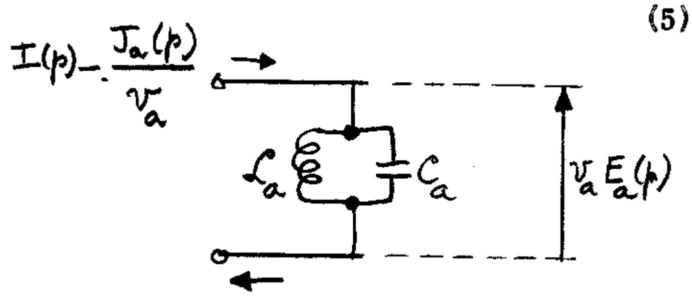

Fig. 3

with

$$\mathcal{L}_a = \frac{v_a^2}{\varepsilon \omega_a^2} \tag{1.8}$$

$$C_a = \frac{\varepsilon}{v_a^2} \tag{1.9}$$

and the indicial impedance of the resonance circuit is

$$Z(p) = \frac{\frac{p v_a^2}{\varepsilon}}{p^2 + \omega_a^2} \tag{1.10}$$

The equivalent circuit will give the amplitude of excitation of the "a"th mode (a radiative mode), provided the line current $\underline{i}$ is known. The latter, however, depends on the coupling to the external system $\mathcal{S}$. To find the line current, then, one considers the total electric field in plane π , i.e. $\sum_a e(t).(\vec{e}_a \cdot \vec{l}_R)$, expresses its integral between inner and outer conductor as being $\mathcal{V}(t)$, and obtains[1] the following equivalent circuit as seen from the input plane π (see Fig. 4). This circuit shows the cavity to behave as a current sink in parallel with an internal impedance composed of an infinite number of resonant circuits in series.

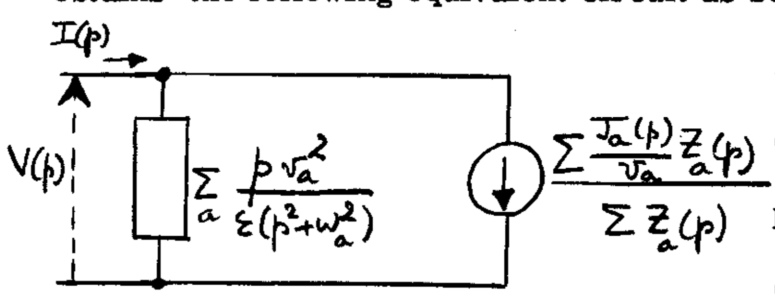

Fig. 4

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1. Under the restriction that the Fourier spectrum of the current density does not extend to higher frequencies than the cut-off of the second coaxial line mode. Generalizations to multi-mode propagation in the coaxial line are trivial, and will not be considered.

(c) Power delivered to the beam

The instantaneous power is

$$W(t) = \iiint \vec{j}(r,t)\,\vec{e}(r,t)\,dV = \sum_a e_a(t)\,j_a(t) + \sum_b f_b(t)\,j_b(t) \qquad (1.11)$$

The second term can be written as $-\sum_b f_b \frac{d f_b}{dt} = -\frac{\varepsilon}{2}\frac{d}{dt}\sum f_b^2(t)$ (1.12)

Clearly, it is the result of the interaction of the irrotational part of $\vec{j}$ with the field created by that part. It averages to zero over a time interval after which $\vec{j}$ (r, t) reverts to its original value. This condition prevails, for example, when $\vec{j}$ (r, t) is a periodic current.

(d) Sinusoidal phenomena

Equivalent circuits can be obtained by replacing p by $j\omega$ in the preceding considerations. Of particular interest is the study of the phenomena around one of the resonant frequencies of the cavity, say ω_p. Let us express the frequency ω as $\omega_p(1+\Delta)$ where Δ, the relative frequency excursion, is a small number. The cavity internal impedance takes the form:

$$Z(j\omega) = -\frac{j\,v_p^2}{2\varepsilon\omega_p\Delta} - \frac{j\,v_p^2}{4\varepsilon\omega_p} + \sum_{a\neq p}\frac{-j\,\omega_p v_a}{\varepsilon(\omega_p^2-\omega_a^2)} + \cdots \qquad (1.13)$$

and goes to infinity at resonance. If the exciting currents keep a constant amplitude around resonance, then the expression for the current sink can be written as:

$$\frac{\sum \frac{J_a}{v_a} Z_a(j\omega)}{\sum Z_a(j\omega)} = \frac{J_p}{v_p} + \Delta \cdot 2\sum_{a\neq p}\left(\frac{J_p}{v_p} - \frac{J_a}{v_a}\right)\left(\frac{v_a}{v_p}\right)^2 \frac{1}{\left(\frac{\omega_a}{\omega_p}\right)^2 - 1} + \cdots \qquad (1.14)$$

Clearly, at resonance, the cavity behaves as a current sink of value $\frac{J_p}{p}$, but this current is not a local extremum at all.

II. Application to an idealized accelerator beam current.

(a) model for the beam current.

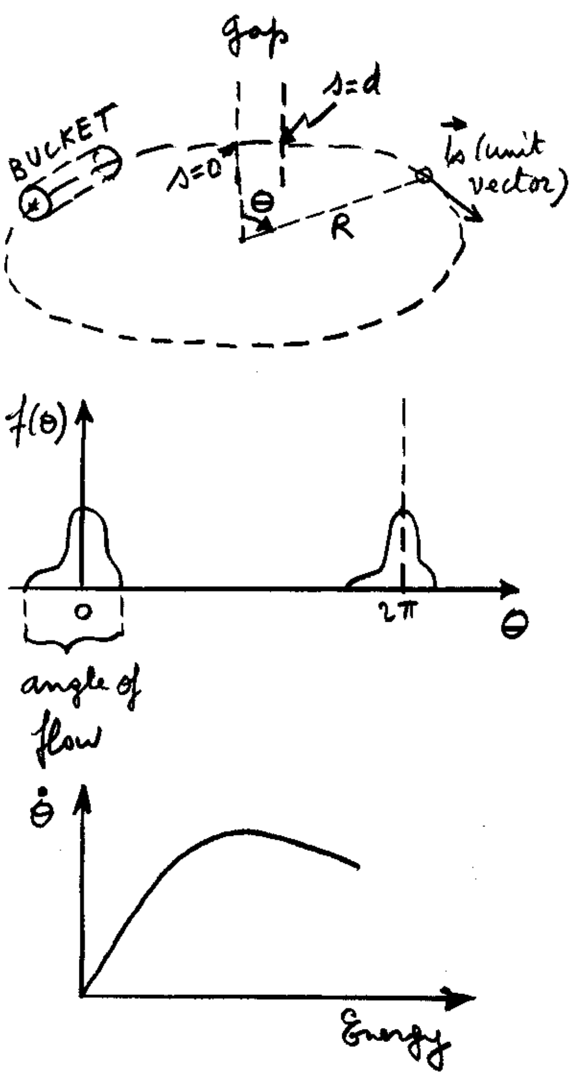


Fig. 5

It will be assumed that the bucket of particles moves as a rigid distribution of charges, of cylindrical shape, and of constant density throughout a cross-section of the cylinder. The assumption implies that oscillations within the bucket average out as far as charge density is concerned, and that the bucket shape does not vary appreciably over the small range of kinetic energies which will be considered here. The current density then becomes a function of the angular position of the bucket, namely:

$$\vec{j}(s,t) = \frac{I_{max}}{S} f\left(\int_0^t \dot{\Theta}\, dt - \frac{s}{R}\right) \vec{1}_s \qquad (2.1)$$

The origin of time coincides with a passage of the maximum current at the gap ($s = o$). The current density is a quasi-periodic quantity, the period $\frac{2\pi}{\dot{\Theta}}$ being considered as a very slowly varying function of time.

Expanded in angle harmonies, the current distribution can be written as

$$\vec{j}(s,t) = \frac{I_{max}}{S} \vec{1}_s \left[A_1 \cos\left(\int \dot{\Theta}\, dt - \frac{s}{R} + \varphi_1\right) + A_2 \cos\left(2\int \dot{\Theta}\, dt - \frac{2s}{R} + \varphi_2\right) + \cdots\right] \qquad (2.2)$$

The argument of each of the cosine functions is not precisely a periodic function of time so that, to be rigorous, the evaluation of the response of the cavity to a current like $\cos\left[\int_0^t \dot{\theta}\,dt - \frac{\Delta}{R} + \varphi_1\right]$ should involve transient analysis of the Laplace transform type. In practice, however, $\dot{\Theta}$ is a very slowly varying function of time, and we shall make the approximation that the response to

$$\cos\left(\int_0^t \dot{\theta}\,dt + \varphi_1\right) = \cos\left[\dot{\theta}t + \int_0^t \Delta\dot{\theta}\,dt - t\Delta\dot{\theta}(t) + \varphi_1\right] \tag{2.3}$$

where $\dot{\theta}_0$ is the angular velocity at time $t = 0$, and $\Delta\dot{\theta}(t) = \dot{\theta}(t) - \dot{\theta}_0$, is the static response to a sinusoidal function of pulsation $\dot{\theta}$ and phase angle $\phi_1(t) = \varphi_1 + \int_0^t \Delta\dot{\theta}\,dt - \Delta\dot{\theta}\,t$. The error involved could eventually be calculated, and has been in similar situations of circuit theory.[1] The adiabatic hypothesis implies also that quantities like $\int \cos\left[2\dot{\theta}t + 2\varphi_1(t)\right]dt$ average practically to zero if taken over sufficiently long periods. Such integrals will be found in beam power computations, and will systematically be discarded whenever averages are computed.

As a further simplification, involving only lighter algebraic notation without introducing any error in principle, it will be assumed that $\vec{e}_a$ is constant over the cross-sectional area of the bucket. Generalizations are trivial, and will involve replacing $\vec{e}_a$ by a cross-sectional weighted average. As a final remark, it will be noticed the I_{max} is proportional to the velocity of the bunch, and is, consequently, a function of the beam energy.

1. See e.g. G. Hok, J. Appl. Phys., 19, 242, 1948 for the response of a resonant circuit to a voltage with linearly varying frequency, and the difference of this response from the static, adiabatic one.

articular form of the equivalent circuit

The couplings of the nth harmonic to the "a"th and "b"th modes are respectively

$$J_a^n = I_{max} A_n e^{j\varphi_n(t)} \cdot \int_0^d e^{-j\frac{ns}{R}} e_{as}(s)\,ds = I^n_{beam}\; v_{gap}^{an} \tag{2.4}$$

$$J_b^n = I_{max} A_n e^{j\varphi_n(t)} \int_0^d e^{-j\frac{ns}{R}} f_{bs}(s)\,ds = I^n_{beam}\; v_{gap}^{bn} \tag{2.5}$$

It is useful, at this point, to introduce the notion of complex Q of a mode relative to the nth harmonic

$$Q_{an} = \frac{v_{gap}^{an}}{V_a} = \frac{1}{V_a}\left(\int_0^d \cos\frac{ns}{R}\, e_{as}(s)\,ds - j\int_0^d \sin\frac{ns}{R}\, e_{as}(s)\,ds \right) \tag{2.6}$$

The interpretation of Q_{an} will become clearer later on, but it appears from (2.6) that Q_{an} measures, roughly speaking, the ratio of voltage across the gap to input voltage for the ath mode. This ratio, which depends on many factors, such as coaxial line coupling, should be high for efficient beam acceleration. The equivalent circuit, as seen from the coaxial input, can be represented with the help of current transformers as:

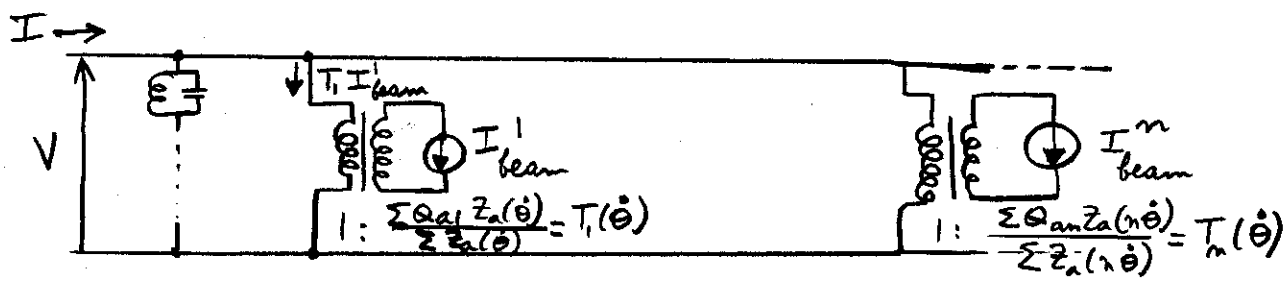


Fig. 6

The complex character of the turn ratios $T_1\ T_2 \ldots T_n \ldots$ is due to the finite transit angle of the gap, which can make phase-shifts along the gap non negligible. The current transformation is seen to involve an amplitude multiplication and a phase shift. The transformation ratio is, in addition, frequency dependent through Z_a. At the resonant frequency of the "p"th mode it is simply Q_{pm}.

(c) Gap voltage.

It is $\int_{gap} \vec{e}(s,t)\cdot d\vec{s} = \int_{gap}\left[\sum e_a(t)\vec{e}_a(r) + \sum f_b(t)\vec{f}_b(r)\right]\cdot d\vec{s} = \sum_a e_a(t)\, v_a^{gap} + \sum f_b(t)\, v_b^{gap}$ (2.7)

This expression is simplified if one introduces the real q of the ath mode, namely

$$q_a = \frac{v^a_{gap}}{V_a} = \frac{1}{V_a}\int_0^d e_{as}(s)\,ds \tag{2.8}$$

It will be noted that qa, a real quantity, can be positive or negative. There are, indeed, sign conventions for V_a and v^a gap, namely: positive from inner to outer conductor in the coaxial line, and from entrance to exit grid in the gap. Equation (2.7) leads to an equivalent circuit for the calculation of the gap voltage.

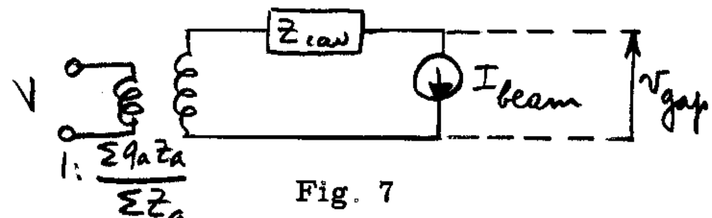


Fig. 7

The turn ratio $\frac{\sum q_a Z_a}{\sum Z_a}$ is frequency dependent, and should be computed for each of the frequencies contained in the input voltage V. The voltage drop in the impedance indicated as Z_{cav} represents the voltage induced by the passing beam.

$$Z_{cav} = \sum_a Z_a q_a Q_a - \frac{\sum_a q_a Z_a \sum_a Q_a Z_a}{\sum_a Z_a} + \frac{j}{\omega\varepsilon}\sum_b v^b_{gap} V^{bn}_{gap} \tag{2.9}$$

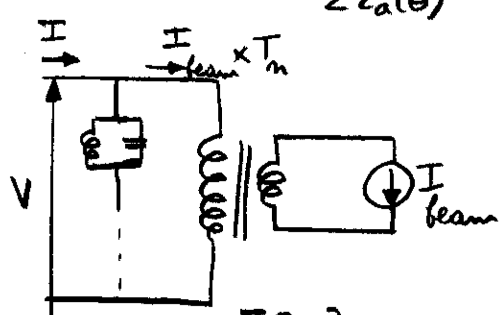
and has to be computed, for each harmonic contained in the beam current, by using the corresponding values of frequency and complex Q. Around a resonant frequency ω_p of the cavity, the first two terms reduce to

$$q_p Q_p \sum_{a\neq p} Z_a + \sum_{a\neq p} q_a Q_a Z_a - Q_p \sum_{a\neq p} q_a Z_a - q_p \sum_{a\neq p} Q_a Z_a$$

(d) Effective gap voltage.

The instantaneous power delivered to the beam is not, because of the finite gap width, the product of gap voltage and beam current. Let us try to define an instantaneous effective gap voltage which, multiplied by the beam current, will give the instantaneous power. We start with the expression for the current absorbed by the cavity:

$$i(t) = \overbrace{I_1 \frac{\sum Q_{a1} Z_a(\dot{\theta})}{\sum Z_a(\dot{\theta})} + I_2 \frac{\sum Q_{a2} Z_a(2\dot{\theta})}{\sum Z_a(2\dot{\theta})} + \cdots}^{\text{transformed cavity current}} + \overbrace{\underbrace{\frac{V_1}{\sum Z_a(\dot{\theta})} + \frac{V_2}{\sum Z_a(2\dot{\theta})}}_{\text{beam frequencies}} + \cdots + \underbrace{\frac{V'}{\sum Z_a(\omega')}}_{\text{other frequencies.}}}^{\text{current due to } V} \tag{2.10}$$



$1 : \underbrace{\dfrac{\sum Q_{an} Z_a}{\sum Z_a}}_{T_n}$

Fig. 8

From this knowledge of i(t), the terms $e_a(t)$ and $j_a(t)$, necessary for the evaluation of the instantaneous power, can be computed as

$$j_a(t) = V_a Q_a^1 I_1 + V_a Q_a^2 I_2 + \cdots \tag{2.11}$$

$$e_a(t) = \frac{1}{V_a}\left[\overbrace{-Z_a Q_{a1} I_1 + Z_a I_1 \frac{\sum Q_{a1} Z_a(\dot{\theta})}{\sum Z_a(\dot{\theta})}}^{A} + \cdots + \overbrace{\frac{Z_a(\dot{\theta})}{\sum Z_a(\dot{\theta})} V_1 + \cdots \frac{Z_a(\omega')}{\sum Z_a(\omega')} V'}^{B}\right] \tag{2.12}$$

The power is $\sum_a e_a(t)\, j_a(t)$, neglecting the "b" modes, which average fast to zero energy. It is a simple matter to see that part A will interact with ja(t) to give zero average power. The effective interaction results from the effect of part B, the various terms of which depend on the external system connected to the cavity. A term like $\frac{Z_a(\dot{\theta})}{\sum Z_a(\dot{\theta})} V_1$ will interact only with $V_a Q_{a1} I_1$, and not with the harmonics. Its average power will be:

$$\frac{1}{2} \mathrm{Re} \sum \left[\frac{Z_a(\dot{\theta})}{\sum Z_a(\dot{\theta})} V_1 Q_{a1}^* I_1^*\right] = \frac{1}{2} \mathrm{Re}\left[V_1 T_1^* I_1^*\right] = \frac{1}{2} \mathrm{Re}\left[V_1 T_1^* I_1\right] \text{ if } I_1 \text{ is taken as origin of phases} \tag{2.13}$$

Terms of the form $V' \frac{Z_a(\omega')}{\sum Z_a(\omega')}$ will interact with the various beam current components to give power contributions

$$\sum e_a(t)\, j_a(t) = \sum_a |Q_{a1}|\,|I_1| \cos\left(\int \dot{\theta}\, dt + \varphi_1 + \beta_{a1}\right) V' \cos(\omega' t + \psi') \frac{Z_a(\omega')}{\sum Z_a(\omega')} \quad (2.14)$$

the phase angle β_{a1} being the argument of Q_{a1}.

The product of the two cosine terms will be transformed into the sum of two cosines, one of them averaging rapidly to zero, the second one susceptible of giving birth to a slow variation of power, i.e. to a useful effect on the beam. This second term can be rewritten as:

$$\sum_a I_1 \cos\left(\int \dot{\theta}\, dt + \varphi_1\right) V' \cos(\omega' t + \psi' - \beta_{a1}) \frac{|Q_{a1}|\, Z_a(\omega')}{\sum Z_a(\omega')} \quad (2.15)$$

i.e. the interaction of the beam current with $V'(T')^*$. For purposes of "effective" gap voltage then, one can use the following circuit for each of the frequencies contained in the input voltage

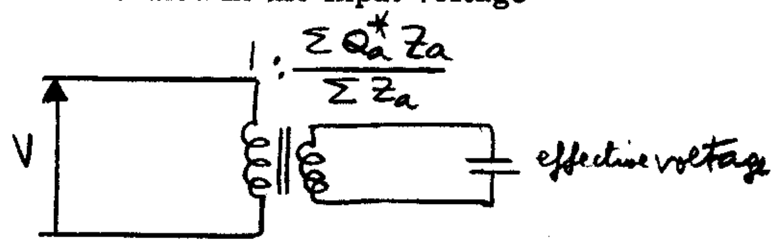


Fig. 9

provided the turn ratio is computed carefully. One must notice, in particular, that Z_a must be evaluated at the frequency of the voltage component being studied, and that Q_a refers to the harmonic component of the beam current with which interaction is being considered.

(e) Narrow gap.

This is a gap for which phase effects are negligible, even for the highest non-vanishing harmonics of the beam current. Let n be the order of the highest term which should be retained. The criterion for "narrowness"

1. This is not really the instantaneous power any more, but its short time average, its quasi-instantaneous value.

is then $\frac{nd}{R} \ll \frac{\pi}{2}$

A narrow gap introduces considerable simplification. Real and complex Q become equal, and so do gap voltage and effective gap voltage. One overall equivalent circuit can be used.

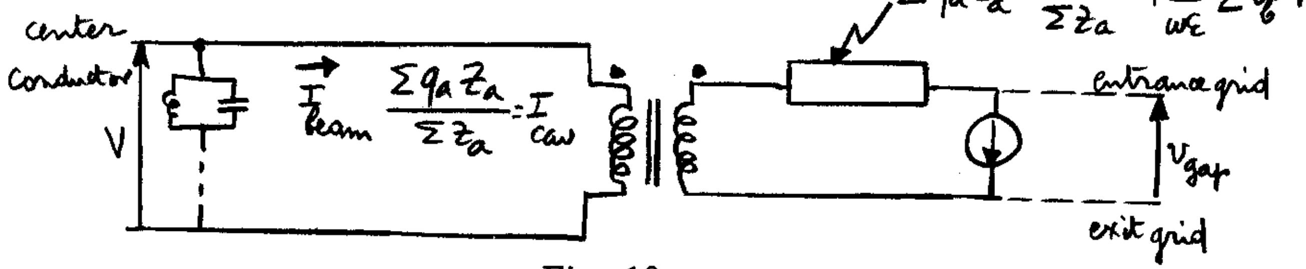


Fig. 10

III. A few examples of beam acceleration.

(a) Deceleration by a passive dissipative load.

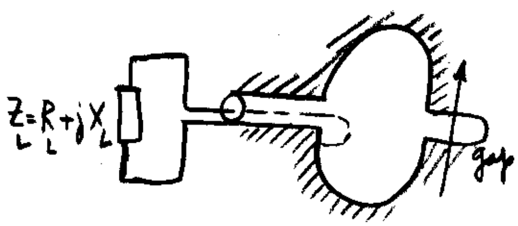


Fig. 11

The power dissipated in the load is

$$W = \frac{1}{2} \sum_n R_L \frac{|I_{cav}|^2 |X_{cav}|^2}{R_L^2 + (X_L + X_{cav})^2} \tag{3.1}$$

The various terms are, in general, functions of the instantaneous angular velocity $\dot{\theta}$. Assuming the relation between beam energy and $\dot{\theta}$ to have been computed from a knowledge of the action of focussing devices, and representing this relation by a local linear law $\Delta\dot{\theta} = k\,\Delta E$, valid except around the transition energy, then, assuming no losses except in the cavity:

$$\frac{d\dot{\theta}}{dt} = k(\dot{\theta}) \cdot W(\dot{\theta}) \tag{3.2}$$

is the law of deceleration. In the vicinity of a resonant frequency, and referring to (1.14) and (2.4)

$$\frac{d\Delta}{dt} = -\frac{1}{2\omega_p} R_L(\omega_p)\, k(\omega_p)\, |Q_{p1}|^2\, |I_{max}(\omega_p)|^2\, |A_1|^2 \tag{3.3}$$

(b) Narrow gap cavity with constant voltage source.

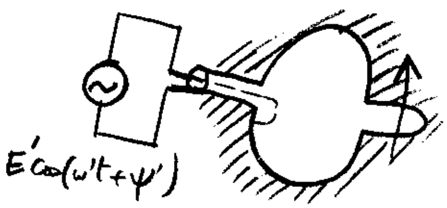


Fig. 12

The equivalent circuit of Fig. 10 can be applied, with the simplifying feature that V contains only the frequency ω'. Let us assume that ω' is close to the beam angular velocity $\dot{\theta}$. Then the law of acceleration is:

$$\frac{d\dot{\theta}}{dt} = \frac{k}{2} \underbrace{E' \frac{\Sigma q_a z_a(\omega')}{\Sigma z_a(\omega')}}_{\text{gap voltage}} \cdot I_1 \cos\Big[\underbrace{\int_0^t \dot{\theta}\, dt + \varphi_1 - \omega' t - \psi'}_{\psi}\Big] \qquad (3.4)$$

where ψ is the instantaneous "phase angle" of voltage and first harmonic of beam current. When the generator keeps constant frequency ω', the differential equation for ψ is

$$\frac{d^2\psi}{dt^2} = \text{constant} \times \cos\psi \qquad (3.5)$$

As a result, the bunch of particles will be subjected to pendulum oscillations. These have been investigated thoroughly in the literature,[1] in particular with respect to the stability of the oscillations. When the generator frequency increases slowly, in an adiabatic manner, (3.4) will still describe the phenomena, but now:

$$\frac{d^2\psi}{dt^2} = \frac{k}{2} \frac{E' \Sigma q_a z_a(\omega')}{\Sigma z_a(\omega')} I_1 \cos\psi - \frac{d^2}{dt^2}(\omega' t) \qquad (3.6)$$

In particular, if ω' varies linearly as $\omega' = \omega'_0 + \varepsilon t$

$$\frac{d^2\psi}{dt^2} = \frac{kE'}{2} \frac{\Sigma q_a z_a(\omega')}{\Sigma z_a(\omega')} I_1 \cos\psi - 2\varepsilon \qquad (3.7)$$

Equation of a biased pendulum with time-dependent torque.

Let us assume that the particle bunch has been "captured" by a generator of fixed frequency ω'. Then $\dot{\theta}$ will oscillate around ω'. For one particular

1. See e.g. Kaiser: Proc. Phys. Soc., LXIII, p. 52, 1950

phase relation, however, no oscillations will occur, namely if, when the particle reaches a velocity ω', beam current and gap voltage are 90° out of phase ($\varphi_1 - \phi' = 90°$, so that $\frac{d\dot{\theta}}{dt} = 0$). This is the equilibrium position. The bunch will follow a voltage of linearly increasing frequency

$\omega' = \omega'_0 + \varepsilon t$ without oscillations if

$$\cos\psi = \frac{\varepsilon}{\frac{k}{2} E' \frac{\Sigma q_a Z_a(\omega')}{\Sigma Z_a} I_1} \tag{3.8}$$

provided the gap voltage is maintained constant, which, in view of resonance properties, will require programming of E' or servo-control of the gap voltage.

(c) Narrow gap cavity with source having internal impedance.

The preceding paragraph was a summary of well known results. The effect of Z_i will perhaps be more interesting to investigate. Voltage V now will include terms in $\dot{\theta}$, $2\dot{\theta}$ ... etc., in addition to the applied voltage frequency ω'.

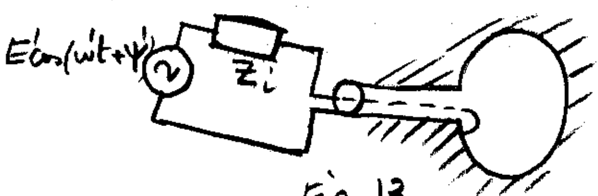


Fig. 13

In evaluating the gap voltage, the following amplitudes and arguments of complex quantities are of interest:

$$A' e^{j\alpha'} = \frac{\Sigma q_a Z_a(\omega')}{Z_i(\omega') + \Sigma Z_a(\omega')} \qquad B_1 e^{j\beta_1} = \frac{Z_i(\dot{\theta}) \, \Sigma Z_a(\dot{\theta})}{Z_i(\dot{\theta}) + \Sigma Z_a(\dot{\theta})} \quad \cdots \tag{3.9}$$

The gap voltage is then

$$A' E' \cos(\omega' t + \phi' + \alpha') - [T_1(\dot{\theta})]^2 \cdot B_1 I_1 \cos(\dot{\theta} t + \phi_1 + \beta_1) - \left[\Sigma q_a^2 X_a + \frac{(\Sigma q_a X_a)^2}{\Sigma X_a}\right] I_1 \cos(\dot{\theta} t + \phi_1 + 90°) \tag{3.10}$$

+ similar harmonic terms.

The instantaneous power is obtained by multiplying this expression by the beam current $I_1 \cos(\dot{\theta} t + \phi_1) + I_2 \cos(2\dot{\theta} t + \phi_2) + \cdots$

If we assume that ω' is close to $\dot{\theta}$, then the only terms which do not rapidly average out to zero are

$$W(t) = \frac{A'E'I_1}{2}\cos\Big[\underbrace{(\dot{\theta}-\omega')t+\phi_1-\psi'-\alpha'}_{\psi}\Big] - \frac{B_1 I_1^2}{2}\left|\frac{\Sigma q_a Z_a(\dot{\theta})}{\Sigma Z_a(\dot{\theta})}\right|^2 \cos\beta_1 - \cdots \quad (3.11)$$

If Z_i is purely reactive, then all terms in cos β disappear, $\alpha' = 0$ and one falls back on equation (3.4)

If Z_i has a resistive part, α' is a shift which can be varied with the value of Z_i, and so are the various cos terms. It then appears that ψ again satisfies a biased pendulum equation. The situation, however, is complicated by the fact that the biasing term is a function of the oscillating variable $\dot{\theta}$.

The equilibrium angle

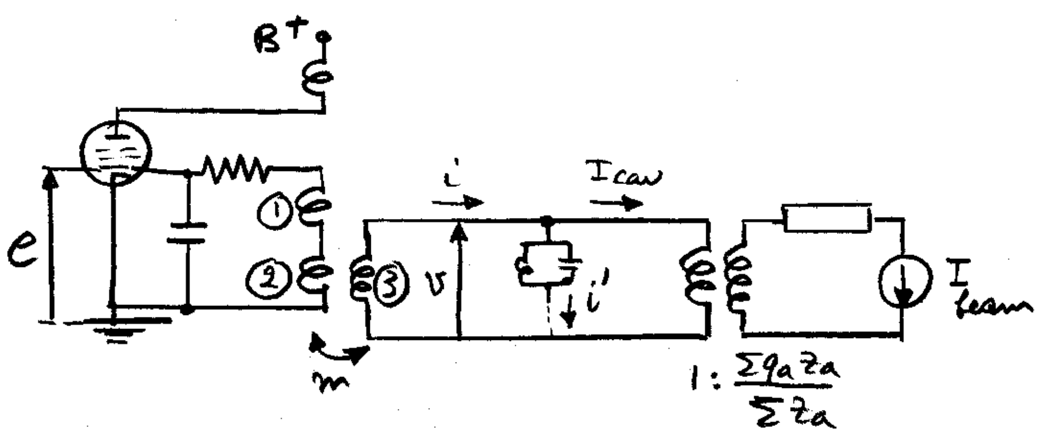
$$\cos\psi_{eq} = \frac{2}{A'(\omega')\,E'I_1}\left[B_1(\omega')\,\frac{I_1^2}{2}\left|\frac{\Sigma q_a Z_a(\omega')}{\Sigma Z_a(\omega')}\right|^2 \cos\beta_1(\omega') + \cdots\right] \quad (3.12)$$

is a function of Z_i as well as ω'. When the source frequency coincides with a cavity resonant frequency ω_p

$$\cos\psi_{eq} = \frac{2}{\underset{\text{gap voltage}}{\underbrace{q_p}}\,E'I_1}\left[q_p^2\cdot|Z_i(\omega_p)|\cdot\frac{I_1^2}{2}\cos\gamma + \underset{\text{terms}}{\text{harmonic}}\cdots\right] \quad (3.13)$$

where γ is the argument of the internal impedance Z_i at the frequency ω_p.

(d) Frequency pulling of an oscillator by the beam.



By assuming, as Minorski does, that the pentode characteristics are of the form $i_a = S e_g \left(1 - \frac{e_g^2}{3V_s^2}\right)$, the equation for the tube behavior can be written as:

$$\ddot{e} - \alpha \dot{e} + \gamma (\dot{e})^3 + \omega_0^2 e = -\beta . m . \frac{di}{dt} \tag{3.14}$$

The second member represents the injected voltage (containing e.g. all the harmonics of the beam voltage), the influence of which might eventually "lock in" the oscillator frequency. The parameters $\alpha \; \beta \; \gamma \; \omega_0$ depend only on tube and circuit characteristics. In the solution of 7.1 presented in textbooks on non-linear theory, the synchronizing voltage was simply $E_0 \sin \omega_1 t$, a fixed affair. In the present situation, however, i is a much more complicated function

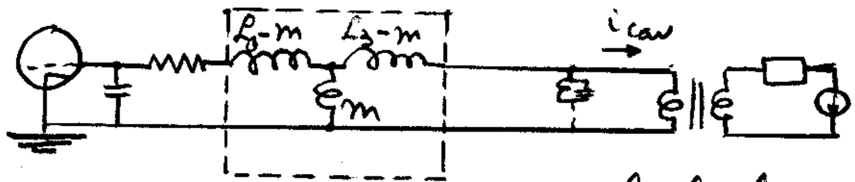
$$i = i_{cav} + i' = i' + I_1 . T_1 \cos(\dot{\theta} t + \phi_1) + \cdots \tag{3.15}$$

i', on the other hand, is the current flowing into the series of resonant circuits under the influence of v. But $v = \delta m \ddot{e} + \mathcal{L}_3 \frac{di'}{dt}$, where δ depends on tube characteristics, so that $\frac{di}{dt}$ will contain terms in $\dddot{e}$, $\ddot{e}$ -- etc. To make matters still more complicated, $\dot{\theta}$ is not constant, but

$$\frac{d\dot{\theta}}{dt} = k W(t) = k \, v_{gap} \, i_{beam}(t) \tag{3.16}$$

where v_{gap} depends on v. If v contains a discrete series of sinusoidal terms, then v_{gap} is easy to calculate. If v has a continuous Fourier spectrum $V(j\omega)$, then the Fourier spectrum of v_{gap} will be $V(j\omega) \frac{\Sigma q_a Z_a(j\omega)}{\Sigma Z_a(j\omega)}$

The study of the whole structure might be facilitated by the replacement of the transformer by an equivalent circuit



$$\mathcal{L}_0 = \mathcal{L}_1 + \mathcal{L}_2$$
$$m = k\sqrt{\mathcal{L}_2 \mathcal{L}_3}$$

One may expect that, in the study of pulling conditions, the harmonics of current and oscillator waveforms will play a non negligible part by their mutual interactions.

APPENDIX: Dimensions of certain quantities

$\vec{e}_a : m^{-\frac{3}{2}}$	$E_a : V\, m^{\frac{1}{2}}$	C_a : Farad
$J_a : A\, m^{-\frac{1}{2}}$	$F_b : V\, m^{\frac{1}{2}}$	$V_a^{gap} : m^{-\frac{1}{2}}$
$\vec{f}_b : m^{-\frac{3}{2}}$	$v_a : m^{-\frac{1}{2}}$	
$J_b : A\, m^{-\frac{1}{2}}$	$\mathcal{L}_a$: Henry	