MURA-209
Internal

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

ACCELERATION BY FIXED FREQUENCY CAVITIES INDEPENDENT OF THE ROTATIONAL FREQUENCY OF PARTICLES

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ABSTRACT: A Klystron-like acceleration method which was proposed in MURA TO-7 is examined in several cases. The method is superior to a pure stochastic method at low energies but the very low rate of acceleration at high energies and inherent large energy spread make it appear not practical.

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Two cavities having the same frequency are inserted in a circular accelerator; their accelerating gaps are perpendicular to the orbits to minimize R.F. knockout effects.

Let

$$V_1 = V_{10} \sin(\nu t + \varphi_{10}) \quad \text{be the voltage on cavity \#1}$$
$$\text{and} \quad V_2 = V_{20} \sin(\nu t + \varphi_{20}) \quad \text{be the voltage on cavity \#2.} \tag{1}$$

Particles crossing cavity #1 at $t = t_o$ receive an energy increment

$$\Delta E_1 = V_{10} \sin \psi_1 \tag{2}$$
$$\text{where} \quad \psi_1 = \nu t_o + \varphi_{10}$$

and will arrive at cavity #2 after a coasting time T.

$$T = \frac{\ell(E + \Delta E_1)}{c\ \beta(E + \Delta E_1)} \tag{3}$$

where ℓ is the distance between cavities and is a function of energy.

By using (2), (3) becomes

$$T = \frac{\ell(E)}{c\,\beta(E)} + \frac{d}{dE}\left(\frac{\ell}{c\beta}\right) V_{10} \sin \psi_1 \tag{4}$$

The phase at cavity #2 is given by

$$\nu(t_o + T) + \varphi_{20} = \psi_1 + \alpha + \nu\frac{\ell}{c\beta} + \frac{\nu}{c}\frac{d}{dE}\left(\frac{\ell}{\beta}\right) V_{10} \sin \psi_1 \tag{5}$$
$$\text{where} \quad \alpha = \varphi_{20} - \varphi_{10}$$

and the energy gain ΔE_2 at cavity #2 is

$$\Delta E_2 = V_{20} \sin\left\{ \psi_1 + \alpha + \nu\frac{\ell}{c\beta} + \frac{\nu}{c}\frac{d}{dE}\left(\frac{\ell}{\beta}\right) V_{10} \sin \psi_1 \right\} \tag{6}$$

From (2) and (6), the total energy gain as a function of ψ_1 is given by

$$\Delta E = \Delta E_1 + \Delta E_2$$
$$= V_{10} \sin \psi_1 + V_{20} \sin\left\{ \psi_1 + \alpha + \nu\frac{\ell}{c\beta} + \frac{\nu}{c}\frac{d}{dE}\left(\frac{\ell}{\beta}\right) V_{10} \sin \psi_1 \right\} \tag{7}$$

If we assume particles are uniformly distributed in phase before entering cavity #1, the acceleration on the average will be given by

(3)

$$
\begin{aligned}
\langle \Delta E \rangle_{\psi_1} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta E \, d\psi_1 \\
&= - V_{20} \sin\phi \;\; J_1(\xi) \\
&\sim - V_{20} \sin\phi \;\; \frac{\xi}{2} \qquad\qquad (\text{ if } \quad \xi \ll 1 \;)
\end{aligned}
\tag{8}
$$

where

$$\phi = \alpha + \frac{\nu \ell}{c\beta} \quad , \qquad \xi = \nu \frac{d}{dE}\left(\frac{\ell}{c\beta}\right) V_{10}$$

Particles are accelerated on the average by the amount given by (8) and at the same time receive an energy spread. The maximum energy gain or loss is given by

$$|\Delta E|^2_{max} = V_{10}^2 + V_{20}^2 + 2 V_{10} V_{20} \cos\phi \tag{9}$$

Since the average energy gain depends approximately on a product of two voltages, it may be close to optimum to choose both voltages equal. Then the average gain is proportional to the square of the voltage, while the energy spread is proportional to the voltage. Therefore it is obvious that the higher the voltage, the more efficient the system is.

To repeat this one turn acceleration, it is necessary that the particles forget their previous phases relative to the cavities after one turn.(1)

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(1) At first glance this assumption seems to be unnecessary because a particle will cross the cavities at different phases in a long time. However it has been pointed out by K. R.Symon that if W, Θ (see MURA KRS/AMS-1) are conjugate variables, $\frac{\partial \langle \dot{W} \rangle_\theta}{\partial E} = 0$ and therefore $\langle \dot{W} \rangle_\theta$ =const. In our case the calculated $\langle \dot{W} \rangle_\theta$ is not constant and W and Θ are not conjugate. Hence the above assumption is necessary to break the conjugate relation of W and

Θ by introducing another degree of freedom, i.e., the betatron oscillation. Presumably this means we are neglecting the effects of cavity #2 $\longrightarrow$ cavity #1 (particles modulated by cavity #2 and accelerated by #1), which might compensate the effects of cavity #1 $\longrightarrow$ cavity #2.

This assumption is rather reasonable at lower energy because the harmonic numbers are higher and the amplitude of the betatron oscillation is larger. But at higher energy, the betatron oscillation has been damped and the frequency of particles has become high. So the orbit length is rather well defined and the above assumption may not hold.

Since the magnetude of the acceleration depends on ϕ and ξ, we shall estimate them in several cases.

A) two cavities in a straight section (ℓ is constant)

$$\phi = \alpha + \frac{\nu \ell}{c} \frac{E}{\sqrt{E^2-1}}$$

$$\xi = -\frac{\nu \ell}{c} \frac{\nabla_{10}}{\sqrt{E^2-1}^{\,3}} \tag{10}$$

B) Two radial cavities

$$\ell(E) = R(E)\,\delta$$

$$\phi = \alpha + \frac{\nu}{\omega}\delta$$

$$\xi = -\frac{\nu\delta}{\omega} K \nabla_{10}$$

δ : azimuthul distance between cavities.

ω : angular frequency of particles

$$K = \frac{E}{\omega}\frac{d\omega}{dE}$$

In a high energy machine (high k machine) the quantity $\frac{\kappa}{\omega E}$ varies by a factor 10^4 while $\frac{1}{\omega}$ varies by a factor 10. The following tables show examples of these quantities and the average acceleration per turn in the machine in the MURA Proposal with two 1-Mev cavities.

A) Straight Section Cavities

E (total in units of rest mass)	ϕ	ξ	$\langle \Delta E \rangle$/turn
1.005	2.50	0.212	63.5KV
1.010	1.92	0.075	35.2KV
1.020	1.52	0.026	13.3KV
1.050	1.15	0.0064	2.93KV
1.100	0.98	0.0022	0.913KV
1.200	0.86	0.00073	0.273KV
1.500	0.77	0.00015	53V
2000	0.73	0.000041	14 V

B) Radial Cavities

E (total)	$\frac{\omega}{2\pi}$	κ	ϕ	ξ	$\langle \Delta E \rangle$/turn
1.005	66.3KC	98.8	2.50	0.197	60KV
1.010	93.9	49.4	1.91	0.069	33KV
1.020	132.4	24.7	1.50	0.024	12KV
1.050	204.8	9.58	1.14	0.0059	2.6KV
1.100	282.8	4.69	0.97	0.0020	0.83KV
1.200	377.1	2.23	0.85	0.00065	0.24KV
1.500	511.1	0.78	0.75	0.00013	46V
2.000	597.8	0.32	0.72	0.000035	12V

As shown above, both of the configurations give similar results for the machine.

In an electron machine the cavity voltage in units of total energy is larger than in a proton machine and this gives a larger acceleration voltage. However, the frequencies of particles are higher and it would be harder to use high harmonics.

Since this method has a stochastic feature, i.e. energy spread, we must consider the probability of finding particles at the out-put energy. Let us assume the distribution of energy after a certain number of turns receiving random voltages is given by a Gaussian distribution.

$$p(E)\,dE = \frac{2}{\sqrt{\pi}\sqrt{N}\,V}\, e^{-\frac{(E-E_0)^2}{N V^2}}\, dE \tag{12}$$

$p(E)dE$: probability of finding particles between E and $E + dE$

V : voltage on cavity, E_0 : initial energy.

Since there is an acceleration on the average per turn (12) becomes

$$p(E)\,dE \sim \frac{2}{\sqrt{\pi}\sqrt{N}\,V}\, e^{-\frac{(E-E_0-NV_a)^2}{N V^2}}\, dE \tag{13}$$

V_a : average acceleration per turn.

The probability of finding particles below the initial energy increases depending on the square root of N due to random voltage and decreases depending on N due to the average acceleration. The latter voltage is smaller and the probability will increase first and decrease after a certain number of turns as shown in Fig. 1.

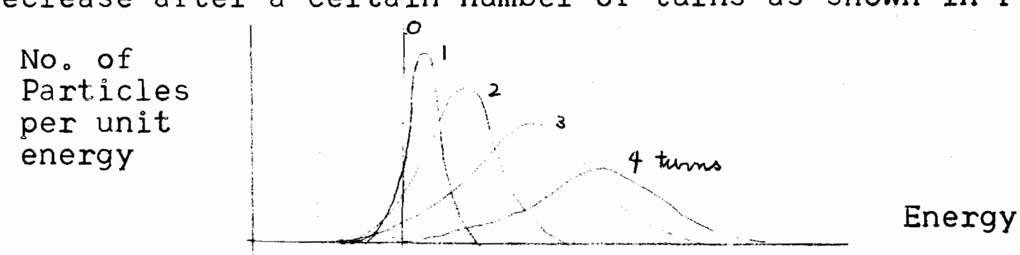

Figure 1.

Energy distribution of particles at successively later times following injection.

At higher energy the average acceleration becomes negligibly small and the system works as a stochastic acceleration system.[2] The probability of finding particles between E and E + dE has a maximum after Nmax turns, where Nmax is given by

$$N_{max} = \frac{2(E-E_0)^2}{V^2} \tag{14}$$

and the probability is given by

$$p_{max} = \sqrt{\frac{2}{\pi}}\,\frac{e^{-1/2}}{E-E_0} \tag{15}$$

It must be noted that in a stochastic system, there is a net acceleration after a certain <u>time interval</u>. Naturally, after a certain number of <u>turns</u> there is no net acceleration and the distribution is symmetric around the initial energy. However higher energy particles have larger frequency and take less time than lower energy particles to reach their energy.

To estimate approximately the distribution of energy after a certain time interval, it may be plausable to assume the distribution of W $\left(W = \int \frac{dE}{\omega(E)}\right)$ in a Gaussian after a time interval.

$$p(W)\,dW = \frac{2}{\sqrt{\pi}\,\sqrt{\frac{\omega_0}{2\pi}t}\,V}\; e^{-\frac{(W-W_0)^2}{V^2\frac{\omega_0}{2\pi}t}}\,dW \tag{16}$$

where W_0, ω_0 are initial values of W and ω.

Unit time is assumed as initial one turn period.

By putting

$$\omega(E) = \omega_0\,(1 + a(E-E_0)) \tag{17}$$

we get

$$W - W_0 = \frac{1}{a}\ln\{1 + a(E-E_0)\} \tag{18}$$

and (16) becomes

$$p(E)\,dE = \frac{2}{\sqrt{\pi}}\,\frac{1}{\mu}\,e^{-\frac{\ln^2\{1+a(E-E_0)\}}{a^2\mu^2}}\;\frac{dE}{1+a(E-E_0)} \tag{19}$$

where

$$\mu = V\sqrt{\frac{\omega_0}{2\pi}t}$$