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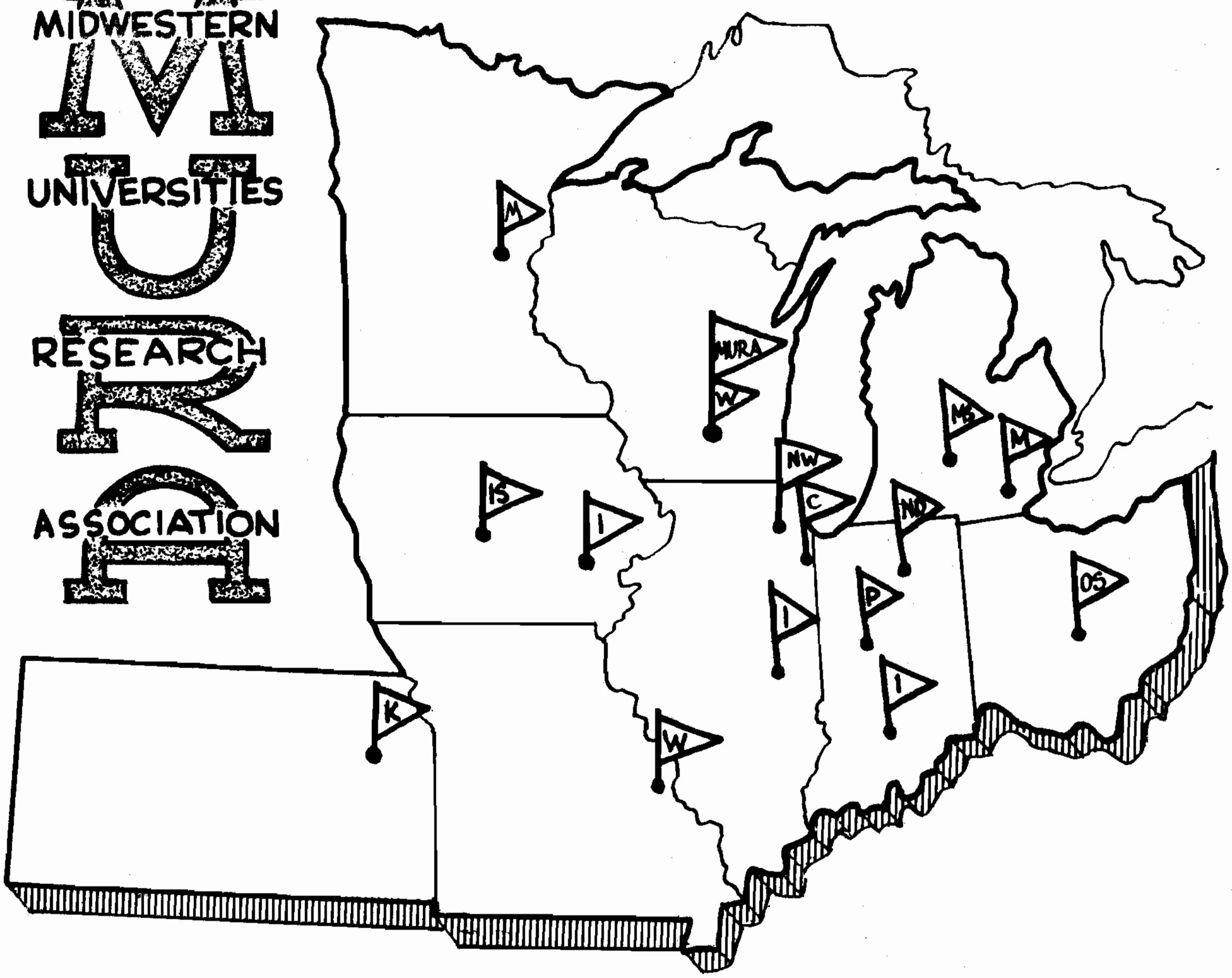

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ABSTRACT

The magentostatic problem in the presence of distributed currents can be formulated in terms of a scalar boundary value problem in which solutions of Laplace's equation are found that conform to prescribed single and double layer distributions at the copper-air and the copper-iron interfaces. It is shown that the prescribed discontinuities are not unique and may be modified to yield a variety of solutions to the potential problem. These alternative formulations have no effect on the magnetic fields but do permit in some cases a simplification of the potential problem. Application is made to the case of a scaling spiral sector FFAG guide field.

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I. MAGNETOSTATIC PROBLEM

The magnetic field within a current carrying region may be expressed as[1]

$$\vec{H} = 4\pi\vec{\mathcal{L}}\tau + \nabla\times\vec{\mathcal{L}}U \tag{1}$$

where $\vec{\mathcal{L}}$ is the operator

$$\vec{\mathcal{L}} = \nabla q\times\nabla. \tag{2}$$

The function q is chosen by considerations of convenience in any particular problem. Since cylindrical coordinates will be of interest in the example selected, q is chosen so that $\nabla q = \vec{k}$, the unit vector along the cylinder axis. Thus Eq. (1) becomes

$$\vec{H} = 4\pi\vec{\mathcal{L}}\tau - 4\pi\vec{k}\sigma - \nabla\frac{\partial U}{\partial z}, \tag{3}$$

where

$$\nabla^2 U = -4\pi\sigma. \tag{4}$$

It is seen that

$$W \equiv \frac{\partial U}{\partial z} \tag{5}$$

may be taken as the magnetostatic potential since, in the absence of the current source,

$$\vec{J} = \vec{\mathcal{L}}\sigma + \nabla\times\vec{\mathcal{L}}\tau, \tag{6}$$

the magnetic field is the negative gradient of the scalar W. In many problems sufficient flexibility in the expression for the current density is obtained even though σ does not contain the longitudinal coordinate z. Thus Eq. (4) may be replaced by

$$\nabla^2 W = 0. \tag{7}$$

Equation (7), subject to prescribed potential and gradient discontinuities at the boundary interfaces, yields the desired solution of the magnetostatic problem.[1] Since the gradient of W must be supplemented with terms depending on the source densities, σ and τ, in order to obtain the magnetic field, it is possible to formulate alternative potential problems by removing prescribed functions from W to form a new potential U. Suppose C is a given function in the region of the current sources and is zero outside. Let

$$U \equiv W + C, \tag{8}$$

where U has no direct relation to the function used in Eq. (4). Equation (3) becomes

$$\vec{H} = 4\pi \vec{\ell}\tau - 4\pi \vec{k}\sigma + \nabla C - \nabla U, \tag{9}$$

and Eq. (6) remains unaltered. Now, however, instead of Eq. (7) the potential U satisfies

$$\nabla^2 U = \nabla^2 C, \tag{10}$$

where the inhomogeous term is known. The function C can be chosen to alter the prescription of the required discontinuities at the copper-air and copper-iron interfaces.

II. SCALING MAGNETOSTATIC PROBLEM

In the scaling case where the potential W in the copper is given by

$$W = \rho^{k+1}\,\Psi(\xi\,\eta), \tag{11}$$

and the potential in the air is given by

$$V = \rho^{k+1}\Omega(\xi\eta), \tag{12}$$

Eq. (7) becomes

$$\begin{aligned} \mathcal{M}\Psi \equiv (k+1)^2\Psi + \frac{2(k+1)}{w}\frac{\partial\Psi}{\partial\xi} - (2k+1)\eta\frac{\partial\Psi}{\partial\eta} - \frac{2\eta}{w}\frac{\partial^2\Psi}{\partial\xi\partial\eta} \\ + \left(\frac{1}{w^2}+N^2\right)\frac{\partial^2\Psi}{\partial\xi^2} + (1+\eta^2)\frac{\partial^2\Psi}{\partial\eta^2} = 0 , \end{aligned} \tag{13}$$

a similar equition holding for Ω .

An examination of the boundary conditions and potential discontinuities[2] reveals that it may be convenient to remove a part, $c(\xi\eta)$, of the potential discontinuities. Three cases are of interest. First, it may be desirable to remove the potentials on the iron-copper surfaces. In this case

$$c(\xi\eta) = \begin{cases} \dfrac{\Omega_3}{(\xi_1-\xi_0)(\eta_2-\eta_1)}\left[\dfrac{\eta}{k}\left(\xi_1-\xi+\dfrac{1}{kw}\right)-(\eta_2-\eta)(\xi_1-\xi)\right] & \text{for } \xi_0<\xi<\xi_1 \\ \dfrac{\Omega_2}{(\xi_1-\xi_0)(\eta_2-\eta_1)}\left[\dfrac{\eta}{k}\left(\xi_1+\xi-\dfrac{1}{kw}\right)-(\eta_2-\eta)(\xi_1+\xi)\right] & \text{for } -\xi_1<\xi<-\xi_0 , \end{cases} \tag{14}$$

where

$$\Omega_2 \equiv \Omega_0\, e^{\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}\xi_0}, \tag{15}$$

and

$$\Omega_3 \equiv \Omega_0\, e^{-\frac{k+1}{\frac{1}{w}+\frac{1}{w'}}\xi_0}. \tag{16}$$

The new potential, Φ , in the copper is related to the previous potential Ψ through

$$\Phi(\xi\eta) = \Psi(\xi\eta) + c(\xi\eta). \tag{17}$$

If it is desired to remove the potential discontinuity at the copper-air surface the function $c(\xi\eta)$ in Eq. (17) may be taken as

$$c(\xi\eta) = \begin{cases} \dfrac{\Omega_3}{(\xi_1-\xi_0)(\eta_2-\eta_1)}\left[\dfrac{\eta}{k}\left(\xi_1-\xi+\frac{1}{kw}\right)+(\eta-\eta_1)(\xi_1-\xi)\right] & \text{for } \xi_0<\xi<\xi_1 \\ \dfrac{\Omega_2}{(\xi_1-\xi_0)(\eta_2-\eta_1)}\left[\dfrac{\eta}{k}\left(\xi_1+\xi-\frac{1}{kw}\right)+(\eta-\eta_1)(\xi_1+\xi)\right] & \text{for } -\xi_1<\xi<-\xi_0 , \end{cases} \tag{18}$$

or, alternatively,

$$c(\xi\eta) = \begin{cases} \dfrac{\Omega_3}{(\xi_1-\xi_0)(\eta_2-\eta_1)}\cdot\dfrac{\eta}{k}\left(\xi_1-\xi+\frac{1}{kw}\right) & \text{for } \xi_0<\xi<\xi_1 \\ \dfrac{\Omega_2}{(\xi_1-\xi_0)(\eta_2-\eta_1)}\cdot\dfrac{\eta}{k}\left(\xi_1+\xi-\frac{1}{kw}\right) & \text{for } -\xi_1<\xi<-\xi_0 . \end{cases} \tag{19}$$

The case of Eq. (18) gives a linear variation of the potential across the current slot at $\eta=\eta_2$ and that of Eq. (19) gives a linear variation on the surface, $\xi=\xi_0$. In every case the differential equation for the new potential Φ is

$$\mathcal{M}\,\Phi = \mathcal{M}\,c , \tag{20}$$

where the inhomogeneous term is calculated from the various prescribed functions of Eqs. (14), (18), and (19).

Since Eq. (19) leads to the simplest choice for the inhomogeneous term in Eq. (20), this case is treated in detail. The boundary condition[2,3] for the new potential become

$$\Phi(\xi_0\eta) = \Omega_3\,\frac{\eta_2-\eta}{\eta_2-\eta_1} , \tag{21}$$

$$\Phi(-\xi_0\eta) = \Omega_2\,\frac{\eta_2-\eta}{\eta_2-\eta_1} , \tag{22}$$

$$\Phi(\pm\xi_1, \eta) = 0\,, \tag{23}$$

$$\Phi(\xi\, \eta_2) = 0\,, \tag{24}$$

$$\Omega(\xi\, \eta_1) - \Phi(\xi\, \eta_1) = 0\,, \tag{25}$$

and

$$\Omega_\eta(\xi\,\eta_1) - \Phi_\eta(\xi\,\eta_1) = \begin{cases} \dfrac{\Omega_3}{(\xi_1-\xi_0)(\eta_2-\eta_1)}(\xi_1-\xi) & \text{for } \xi_0<\xi<\xi_1 \\[2ex] \dfrac{\Omega_2}{(\xi_1-\xi_0)(\eta_2-\eta_1)}(\xi_1+\xi) & \text{for } -\xi_1<\xi<-\xi_0\,, \end{cases} \tag{26}$$

where the η subscript designates a derivative with respect to η. Equation (20) becomes

$$\mathcal{M}\,\Phi = \begin{cases} \dfrac{\Omega_3}{(\xi_1-\xi_0)(\eta_2-\eta_1)}\cdot k\eta\left(\xi_1-\xi-\frac{1}{kw}\right) & \text{for } \xi_0<\xi<\xi_1 \\[3ex] \dfrac{\Omega_2}{(\xi_1-\xi_0)(\eta_2-\eta_1)}\cdot k\eta\left(\xi_1+\xi+\frac{1}{kw}\right) & \text{for } -\xi_1<\xi<-\xi_0\,. \end{cases} \tag{27}$$

III. RELAXATION SOLUTION

In order to adapt the potential problem represented by Eqs. (21-27) to the numerical processes of the FOROCYL-GOLLYCONDER computing program[4] the variables $(\xi\,\eta)$ employed here are first replaced by (xy) where

$$x = \frac{1}{2\pi}\xi\,, \quad \text{and} \quad y = \frac{\sqrt{\frac{1}{w^2}+N^2}}{2\pi}\,\eta\,. \tag{28}$$

Equation (27) becomes

$$\frac{\partial^2 \Phi}{\partial x^2} + \left(1 + \frac{4\pi^2}{\mathcal{D}_o} y^2\right)\frac{\partial^2 \Phi}{\partial y^2} - \frac{4\pi}{w\mathcal{D}_o} y \frac{\partial^2 \Phi}{\partial x \partial y} + \frac{4\pi(k+1)}{w\mathcal{D}_o}\frac{\partial \Phi}{\partial x}$$
$$- \frac{4\pi^2}{\mathcal{D}_o}(2k+1) y \frac{\partial \Phi}{\partial y} + \frac{4\pi^2(k+1)^2}{\mathcal{D}_o}\Phi = \frac{4\pi^2}{\mathcal{D}_o}\mathcal{M}c \, , \tag{29}$$

where

$$\mathcal{D}_o \equiv \frac{1}{w^2} + N^2 \, , \tag{30}$$

and

$$\frac{4\pi^2}{\mathcal{D}_o}\mathcal{M}c = \begin{cases} \dfrac{4\pi^2 k\,\Omega_3}{(x_1-x_o)(y_2-y_1)\mathcal{D}_o} \cdot y\left(x_1 - x - \dfrac{1}{2\pi k w}\right) & \text{for } x_o < x < x_1 \\[2ex] \dfrac{4\pi^2 k\,\Omega_2}{(x_1-x_o)(y_2-y_1)\mathcal{D}_o} \cdot y\left(x_1 + x + \dfrac{1}{2\pi k w}\right) & \text{for } -x_1 < x < -x_o \, . \end{cases} \tag{31}$$

If a typical point in the (x,y) mesh is designated by (00), then a second order Taylor expansion Φ is

$$\Phi_{ij} = \Phi_{oo} + ih\,\Phi_x + j\ell\,\Phi_y + \frac{1}{2} i^2 h^2 \Phi_{xx} + ij\,h\ell\,\Phi_{xy} + \frac{1}{2} j^2 \ell^2 \Phi_{yy} \, , \tag{32}$$

where h and l are the x and y dimensions of a unit cell in the mesh. To form an algorism for solving Eq. (29), each point (ij) is given the weight N_{ij}. Multiplying Eq. (32) by N_{ij} and summing over i and j from -1 to 1 gives

$$\sum N_{ij}\Phi_{ij} = \Phi_{oo}\sum N_{ij} + h\Phi_x \sum iN_{ij} + \ell\Phi_y \sum jN_{ij}$$
$$+ \frac{1}{2}h^2\Phi_{xx}\sum i^2 N_{ij} + h\ell\,\Phi_{xy}\sum ijN_{ij} + \frac{1}{2}\ell^2\Phi_{yy}\sum j^2 N_{ij} \, . \tag{33}$$

In order to eliminate the derivative terms in Eq. (33), the coefficients are made proportional to the corresponding coefficients in Eq. (29). Thus

$$h\sum i N_{ij} = \frac{4\pi(k+1)\alpha}{w\mathcal{D}_0}, \tag{34}$$

$$\ell\sum j N_{ij} = -\frac{4\pi^2(2k+1)\alpha}{\mathcal{D}_0} y, \tag{35}$$

$$\frac{1}{2}h^2\sum i^2 N_{ij} = \alpha, \tag{36}$$

$$h\ell\sum ij N_{ij} = -\frac{4\pi\alpha}{w\mathcal{D}_0} y, \tag{37}$$

and

$$\frac{1}{2}\ell^2\sum j^2 N_{ij} = \left(1 + \frac{4\pi^2}{\mathcal{D}_0} y^2\right)\alpha. \tag{38}$$

Substituting Eqs. (34-38) into Eq. (33) and using Eq. (29) results in the 8-point algorism

$$\mathcal{D}\,\Phi_{00} = \sum N_{ij}\,\Phi_{ij} - \frac{4\pi^2\alpha}{\mathcal{D}_0} m c, \tag{39}$$

where

$$\mathcal{D} \equiv \sum N_{ij} - \frac{4\pi^2(k+1)^2}{\mathcal{D}_0}\alpha. \tag{40}$$

The solution of Eqs. (34-38) for the weights N_{ij} and α is not unique. A solution differing from Laslett's choice[3] that possesses some desirable features without appreciable change in accuracy is

$\alpha = \frac{1}{2}\ell^2$ and

$$N_{10} = \frac{1}{2}r^2 + \frac{\pi(k+1)r\ell}{w\mathcal{D}_0}, \tag{41}$$

$$N_{-10} = \frac{1}{2}r^2 - \frac{\pi(k+1)r\ell}{w\mathcal{D}_0}, \tag{42}$$

$$N_{01} = \frac{1}{2}\left(1 + \frac{4\pi^2}{\mathcal{D}_0} y^2\right) - \frac{\pi^2(2k+1)\ell}{\mathcal{D}_0} y, \tag{43}$$

$$N_{0-1} = \frac{1}{2}\left(1 + \frac{4\pi^2}{D_0} y^2\right) + \frac{\pi^2(2k+1)\ell}{D_0} y \ , \tag{44}$$

$$N_{11} = -\frac{\pi r}{2w D_0} y = N_{-1-1} = -N_{1-1} = -N_{-11} \ . \tag{45}$$

From these relations it follows that

$$D = 1 + r^2 + \frac{4\pi^2}{D_0} y^2 - \frac{2\pi^2(k+1)^2\ell^2}{D_0} \ , \tag{46}$$

and

$$D \Phi_{00} = \sum N_{ij} \Phi_{ij} + \mathcal{E} \ , \tag{47}$$

where

$$\mathcal{E} = \begin{cases} -\dfrac{2\pi^2 k \ell^2 \Omega_2}{D_0 W H} \cdot J\left(I - I_1 + \dfrac{1}{2\pi h k w}\right) & \text{for } I_1 < I < I_2 \\[2ex] -\dfrac{2\pi^2 k \ell^2 \Omega_3}{D_0 W H} \cdot J\left(I_4 - I - \dfrac{1}{2\pi h k w}\right) & \text{for } I_3 < I < I_4 \ . \end{cases} \tag{48}$$

In Eq. (48) the notation[4]

$$x = (I - I_0) h \ , \tag{49}$$

$$y = J\ell \ , \tag{50}$$

$$W = I_2 - I_1 = I_4 - I_3 \ , \tag{51}$$

and

$$H = J_2 - J_1 \tag{52}$$

is used. The algorism in Eq. (47), the boundary conditions in Eqs. (21-25), together with the usual boundary conditions[3] for Ω serve to obtain a solution to the magnetostatic problem except that, on row $y = y_1$, current values[3] are needed to account for the discontinuity in the normal derivative of the potential as required by Eq. (26).

The standard algorism of Eq. (39) requires a correction on the row $y = y_1$ since the values of Ω extrapolated into the copper region differ from the corresponding values of Φ . If the difference between the extrapolated Ω and Φ is designated by

$$\Delta \equiv \Omega - \Phi \tag{53}$$

the values of Δ at the points (-11), (01), and (11) are

$$\Delta_{01} = \frac{1}{2}(\Delta_{11} + \Delta_{-11}) = \ell\,\Delta_y + \frac{1}{2}\ell^2 \Delta_{yy}\,, \tag{54}$$

and

$$\frac{1}{2}(\Delta_{11} - \Delta_{-11}) = h\ell\,\Delta_{xy}\,. \tag{55}$$

The correction o current value required is

$$\Omega_{00} = \left[\,Std.\ Algorism\right] + \left[CV\right]\,, \tag{56}$$

where

$$\mathcal{D}\left[CV\right] = N_{01}\Delta_{01} + \frac{1}{2}(N_{11} + N_{-11})(\Delta_{11} + \Delta_{-11}) + \frac{1}{2}(N_{11} - N_{-11})(\Delta_{11} - \Delta_{-11})\,. \tag{57}$$

From Eq. (45) it is to be noted that $N_{11} + N_{-11}$ is zero. Equations (26), (20), and (49-52) may be used to obtain

$$\Delta_{01} = \begin{cases} \dfrac{\Omega_2}{WH}(I-I_1) + \dfrac{1}{2}\dfrac{\Omega_2}{WH}\cdot\dfrac{\ell^2 J_1}{\left[\dfrac{\mathcal{D}_0}{4\pi^2}+\ell^2 J_1^2\right]}\cdot\left[(k+1)(I-I_1)+\dfrac{1}{2\pi h w}\right] & \text{for } I_1 < I < I_2 \\ \dfrac{\Omega_3}{WH}(I_4-I) + \dfrac{1}{2}\dfrac{\Omega_3}{WH}\cdot\dfrac{\ell^2 J_1}{\left[\dfrac{\mathcal{D}_0}{4\pi^2}+\ell^2 J_1^2\right]}\cdot\left[(k+1)(I_4-I)-\dfrac{1}{2\pi h w}\right] & \text{for } I_3 < I < I_4, \end{cases} \tag{58}$$

and

$$\frac{1}{2}\left(\Delta_{11}-\Delta_{-11}\right) = \begin{cases} \dfrac{\Omega_2}{WH} & \text{for } I_1 < I < I_2 \\ -\dfrac{\Omega_3}{WH} & \text{for } I_3 < I < I_4 \,. \end{cases} \tag{59}$$

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