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OHMIC LOSSES OF A RELATIVISTIC ELECTRON RING MOVING ALONG A CONDUCTING CYLINDER†

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A relativistic electron ring moving along a conducting wall is acted on by a retarding force due to the ohmic losses of the image currents in the wall. This force is calculated for an electron ring moving coaxially along a cylinder of arbitrary thickness. The dependence of the force on the thickness and conductivity of the cylinder and on the ring velocity is discussed.

1 INTRODUCTION

In the electron ring accelerator[1] conducting structures near the electron rings, may be favourable, e.g., to suppress the negative mass instability or to improve the focusing of the rings by means of image effects. However, due to the finite conductivity of the structure the induced image currents lead to ohmic losses, which give rise to a retarding force on the electron ring. This affects the ring dynamics. The retarding force can be of advantage for slowing down the ring, but it can have a very undesired effect if the ring is to be accelerated.

In this paper we consider the retarding force on an infinitely thin electron ring with major radius R which moves inside or outside a hollow cylinder of thickness d and conductivity σ. The electrons rotate in the ring with the velocity v_φ, and the ring moves as a whole parallel to the ring axis with the velocity v_z (Figure 1). The retarding force F_z on the electrons consists of an electric and a magnetic component:

$$F_z = eE_z - ev_\varphi B_r$$

As E_z is produced by the ring charge alone and B_r by the ring current alone; the contributions of the ring charge and ring current are additive and can be calculated separately. The retarding effect becomes particularly large if the distance a of the ring from the wall becomes small. In this approximation $(a/R \ll 1)$ the retarding force on a ring moved inside an infinitely thick cylinder has been calculated by Voskresensky *et al.*[2] Herrmann[3,4] has numerically calculated the magnetic component of the retarding force for a thin cylinder and velocities at which the penetration depth of the ring magnetic field is large relative to the cylinder thickness. But since, as we shall show later, the results strongly depend on the cylinder thickness and also on the distance of the ring from the wall, it is interesting to discuss the general case.

In Section II the retarding force component due to the ring current is calculated. Limiting cases of the ring close to the wall $(a/R \ll 1)$, and of the thin cylinder that are important for practical application are discussed. In Section III, the retarding force produced by the ring charge is given and again various limiting cases are discussed. Finally, Section IV finishes with a discussion of the application of the results to the electron ring accelerator.

2 THE RING CURRENT CASE

To take the case of a neutral ring current first, the starting point is Maxwell's equations

$$\operatorname{curl} \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \qquad \operatorname{div} \mathbf{E} = 4\pi\rho,$$
$$\operatorname{curl} \mathbf{B} = 4\pi\mathbf{j} + \frac{\partial}{\partial t}\mathbf{E}, \qquad \operatorname{div} \mathbf{B} = 0, \tag{1}$$

(velocity of light $c = 1$). Introducing cylindrical coordinates (r, φ, z), we have to solve Eqs. (1) for the following charge and current densities ρ, $\mathbf{j}$

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in the three different regions I: $0 < t < T$, II: $T < r < T + d$, III: $r > T + d$.

$$
\begin{aligned}
\text{I:}\quad & \rho = 0, \quad \mathbf{j} = J(-\sin\varphi, \cos\varphi, 0) \\
& \qquad\qquad \times\, \delta(z - v_z t)\delta(r - R), \\
\text{II:}\quad & \rho = 0, \quad \mathbf{j} = \sigma\mathbf{E}, \\
\text{III:}\quad & \rho = 0, \quad \mathbf{j} = 0,
\end{aligned}
\tag{2}
$$

where J is the ring current, σ the conductivity and d the thickness of the cylinder. The electron ring is moving inside the cylinder (Figure 1). Because of

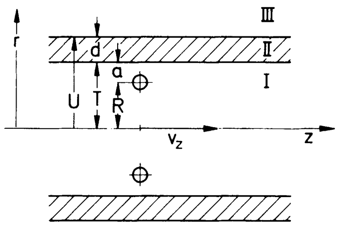

FIGURE 1 Schematic of configuration

$\partial/\partial\varphi \equiv 0$ and the special form of Eqs. (2) it can be assumed that the only nonvanishing components of the electric and magnetic fields are E_φ, B_r, B_z. Eliminating E_φ and B_z, one gets the following equations for B_r:

$$
\begin{aligned}
\text{I:}\quad & \Delta_r B_{\mathrm{I}} - \frac{\partial^2}{\partial t^2} B_{\mathrm{I}} = -\frac{4\pi}{v_z} J \frac{\partial}{\partial t} \\
& \qquad\qquad \times\, \delta(z - v_z t)\delta(r - R), \\
\text{II:}\quad & \Delta_r B_{\mathrm{II}} - \frac{\partial^2}{\partial t^2} B_{\mathrm{II}} - 4\pi\sigma \frac{\partial}{\partial t} B_{\mathrm{II}} = \sigma, \\
\text{III:}\quad & \Delta_r B_{\mathrm{III}} - \frac{\partial^2}{\partial t^2} B_{\mathrm{III}} = 0,
\end{aligned}
\tag{3}
$$

(The index r is replaced by the region label), where

$$
\Delta_r = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}.
$$

As a consequence of Maxwell's equations the tangential components of the electric and magnetic fields have to be continuous at the surface of the conducting cylinder, which leads to the following boundary conditions for B_r:

$$
\begin{aligned}
& B_{\mathrm{I}} = B_{\mathrm{II}}, \quad \frac{\partial}{\partial r} B_{\mathrm{I}} = \frac{\partial}{\partial r} B_{\mathrm{II}} \qquad \text{at } r = T, \\
& B_{\mathrm{II}} = B_{\mathrm{III}}, \quad \frac{\partial}{\partial r} B_{\mathrm{II}} = \frac{\partial}{\partial r} B_{\mathrm{III}} \qquad \text{at } r = T + d.
\end{aligned}
\tag{4}
$$

In addition, it is required that B_{I} be regular for $r = 0$, and that B_{III} vanish for $r = \infty$.

If the ring velocity v_z is kept constant, it can be assumed that the fields become stationary in the sense that all quantities only depend on $z - v_z t$ and r. Introducing the Fourier transform $B(k, r)$ defined by

$$
B(z - v_z t, r) = \int_{-\infty}^{+\infty} B(k, r) e^{ik\gamma_z(z - v_z t)}\, \mathrm{d}k,
$$

$$
\gamma_z = (1 - v_z^2)^{-1/2} \tag{5}
$$

and inserting (5) in (3), one obtains

$$
\begin{aligned}
\text{I:}\quad & \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} - k^2\right) \\
& \qquad \times\, B_{\mathrm{I}}(k, r) = 2ik\gamma_z^2 J\delta(r - R), \\
\text{II:}\quad & \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} - k^2 + 4\pi i\sigma k\gamma_z v_z\right) \\
& \qquad \times\, B_{\mathrm{II}}(k, r) = 0, \\
\text{III:}\quad & \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} - k^2\right) B_{\mathrm{III}}(k, r) = 0.
\end{aligned}
$$

Because of the reality condition $B^*(k, r) = B(-k, r)$ ($B^*(k, r)$ is the complex conjugate of $B(k, r)$) the field $B(z - v_z t, r)$ can be determined from $B(k, r)$ with $k > 0$. Therefore, in the following, it will be assumed throughout that $k > 0$. The general solution of (6), regular at $r = 0$ and vanishing for $r = \infty$, is given by

$$
\begin{aligned}
B_{\mathrm{I}}(k, r) &= B_{\mathrm{I}}(k) I_1(kr) \\
&\quad - 2ikJR\gamma_z^2 \begin{cases} I_1(kr)K_1(kR), & r < R, \\ I_1(kR)K_1(kr), & r > R, \end{cases} \\
B_{\mathrm{II}}(k, r) &= B_{\mathrm{II}}^-(k) I_1(sr) + B_{\mathrm{II}}^+(k) K_1(sr), \\
B_{\mathrm{III}}(k, r) &= B_{\mathrm{III}}(k) K_1(kr)
\end{aligned}
\tag{7}
$$

with $s = (k^2 - 4\pi i\sigma k v_z\gamma_z)^{1/2}$. The functions I_1 and K_1 are the modified Bessel functions. The coefficients $B_{\mathrm{I}}(k)$, $B_{\mathrm{II}}^-(k)$, $B_{\mathrm{II}}^+(k)$ and $B_{\mathrm{III}}(k)$ can be determined by using the four boundary conditions (4).

The force on each electron is given by $F_M = -ev_\varphi B_r(z - v_z t = 0, r = R)$. Inserting the expression for $B_{\mathrm{I}}(k, r)$ into (5) yields for the force F_M in the

z-direction.

$$F_M = -\frac{4v_\varphi \gamma_z^2 JR}{T^2} \mathscr{I}m \int_0^\infty dx I_1^2\left(x\frac{R}{T}\right) \times \frac{[sI_0(s)K_1(x) + xI_1(s)K_0(x)]A - [sK_0(s)K_1(x) - xK_0(x)K_1(s)]B}{[sI_0(s)I_1(x) - xI_1(s)I_0(x)]A - [sK_0(s)I_1(x) + xI_0(x)K_1(s)]B} \quad (8)$$

with

$$A = sK_0\left(s\frac{u}{T}\right)K_1\left(x\frac{u}{T}\right) - xK_1\left(s\frac{U}{T}\right)K_0\left(x\frac{U}{T}\right),$$
$$B = sI_0\left(s\frac{U}{T}\right)K_1\left(x\frac{U}{T}\right) + xI_1\left(s\frac{U}{T}\right)K_0\left(x\frac{U}{T}\right), \quad (9)$$

where $s = (x^2 - i\lambda Tx)^{1/2}, \lambda = 4\pi\sigma v_z\gamma_z, U = T + d$ and $x = kT$. In terms of ring data, the current J is given by $J = eN_e v_\varphi/2\pi R$, where N_e is the number of electrons in the ring, R the major ring radius and v_φ the transverse velocity of the electrons. Now, in order to discuss the dependence of the force on the longitudinal velocity v_z, one has to know the transverse velocity v_φ as a function of v_z. If electric acceleration in the z-direction is assumed, the transverse momentum is a constant of motion. From this, it follows that $v_\varphi = v_\varphi^0\gamma_z$, where v_φ^0 is the transverse velocity at the beginning of the acceleration and $\gamma_z = (1 - v_z^2)^{-1/2}$ is the longitudinal relativistic factor.

Finally, for a ring moving inside the cylinder ($R < T$), one gets for the drag force

$$F_M = -\frac{m_e r_0 N_e v_\varphi^{02}}{2\pi R(T - R)} g_i\left(\frac{R}{T}, \frac{d}{T}, \lambda T\right) \quad (10)$$

with

$$g_i\left(\frac{R}{T}, \frac{d}{T}, \lambda T\right) = \frac{4R(T - R)}{T} \mathscr{I}m \int_0^\infty dx I_1^2\left(x\frac{R}{T}\right) \times \frac{[sI_0(s)K_1(x) + xI_1(s)K_0(x)]A - [sK_0(s)K_1(x) - xK_0(x)K_1(s)]B}{[sI_0(s)I_1(x) - xI_1(s)I_0(x)]A - [sK_0(s)I_1(x) + xI_0(x)K_1(s)]B}, \quad (11)$$

where A and B are given in Eqs. (9) and $s = (x^2 - i\lambda Tx)^{1/2}, \lambda = 4\pi\sigma v_z\gamma_z$; m_e is the electron mass and r_0 the classical electron radius.

The corresponding calculation for the ring moving outside the cylinder ($R > U$) gives the following expression for the drag force:

$$F_M = -\frac{m_e r_0 N_e v_\varphi^{02}}{2\pi R(R - U)} g_0\left(\frac{R}{U}, \frac{d}{U}, \lambda U\right), \quad (12)$$

where

$$g_0\left(\frac{R}{U}, \frac{d}{U}, \lambda U\right) = \frac{4R(R - U)}{U^2} \mathscr{I}m \int_0^\infty dx K_1^2\left(x\frac{R}{U}\right) \times \frac{[\tilde{s}I_0(\tilde{s})I_1(x) - xI_0(x)I_1(\tilde{s})]C - [\tilde{s}K_0(\tilde{s})I_1(x) + xI_0(x)K_1(\tilde{s})]D}{[\tilde{s}I_0(\tilde{s})K_1(x) + xK_0(x)I_1(\tilde{s})]C - [\tilde{s}K_0(\tilde{s})K_1(x) - xK_0(x)K_1(\tilde{s})]D}, \quad (13)$$

with

$$C = \tilde{s}K_0\left(\tilde{s}\frac{T}{U}\right)I_1\left(x\frac{T}{U}\right) + xI_0\left(x\frac{T}{U}\right)K_1\left(\tilde{s}\frac{T}{U}\right),$$
$$D = \tilde{s}I_0\left(\tilde{s}\frac{T}{U}\right)I_1\left(x\frac{T}{U}\right) - xI_0\left(x\frac{T}{U}\right)I_1\left(\tilde{s}\frac{T}{U}\right), \quad (14)$$

and $\tilde{s} = (x^2 - ix\lambda U)^{1/2}, \lambda = 4\pi\sigma v_z\gamma_z$.

In Figures 2a and b and Figures 3a and b, the functions $g_i(R/T, d/T, \lambda T)$ and $g_0(R/U, d/U, \lambda U)$ are plotted as function of $p = \lambda a$ ($a = T - R$ or $a = R - U$, respectively) for different values of d/a and R/T, R/U. The plots show that for sufficiently small d/a the curves have the same shapes and are only shifted in the p-direction.

This result can be found directly by taking the limit of a thin cylinder in g_i and g_0. Assuming in these cases $(\lambda/(T - R))^{1/2}d \ll 1$ and $(\lambda/(R - U))^{1/2}d \ll 1$, respectively, one gets the following approximations for g_i and g_0:

$$g_i\left(\frac{R}{T}, \frac{d}{T}, \lambda T\right) = \frac{4R(T - R)}{T^2}\tau \times \int_0^\infty dx \frac{x^2 I_1^2(x(R/T))K_1^2(x)}{1 + \tau^2 x^2 I_1^2(x)K_1^2(x)}, \quad (15)$$

and

$$g_0\left(\frac{R}{U}, \frac{d}{U}, \lambda U\right) = \frac{4R(R - U)}{U^2}\tau \times \int_0^\infty dx \frac{x^2 I_1^2(x)K_1^2(x(R/U))}{1 + \tau^2 x^2 I_1^2(x)K_1^2(x)}, \quad (16)$$

where $\tau = 4\pi\sigma d v_z\gamma_z$.

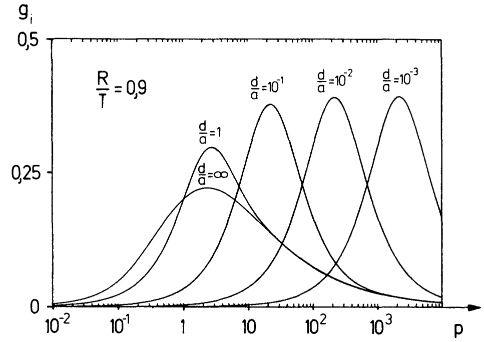

fig. 2a

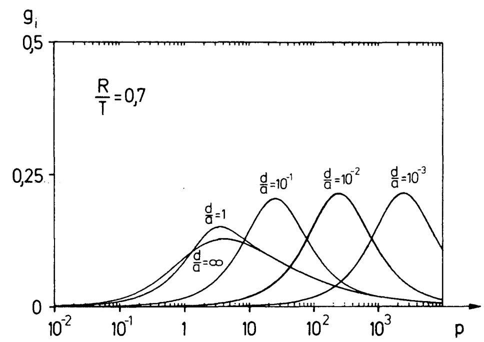

fig. 2b

FIGURES 2*a* and *b* For a ring moving inside a hollow cylinder the retarding force $F_M = -(m_e r_0 N_e v_\varphi^{02}/R(T-R)) \cdot g_i$ is plotted as function of $p = 4\pi\sigma a v_z \gamma_z$ and d/a.

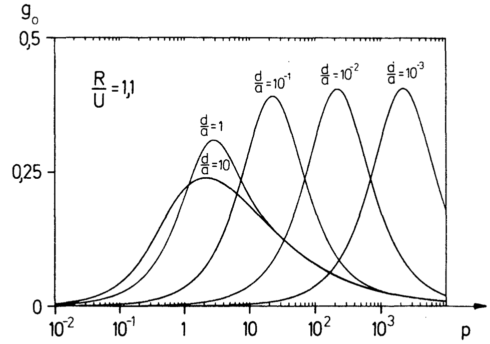

fig. 3a

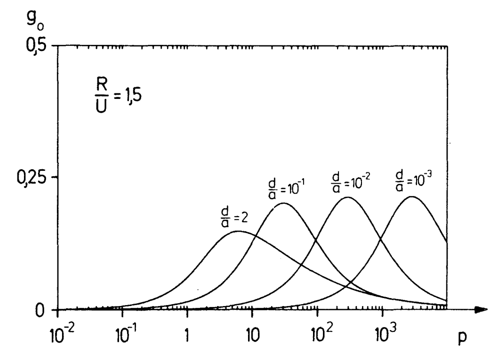

fig. 3b

FIGURES 3*a* and *b* For a ring moving outside a hollow cylinder the retarding force $F_M = -(m_e r_0 N_e v_\varphi^{02}/2\pi R(R-U))g_0$ is plotted as function of $p = 4\pi\sigma a v_z \gamma_z$ and d/a.

In this approximation, which is of practical interest, the force depends on the product of conductivity σ and thickness d or, in other words, on the surface resistance of the thin cylinder. In Figures 4 and 5, the drag force is plotted as a function of τ for different values of R/T and R/U, respectively.

If the ring moves close to the walls, $(T - R) \ll T$ or $(R - U) \ll U$ a plane approximation can be made. Then both functions g_i and g_0 have the same limit $g(p, d/a)$. The following expression is obtained for $g(p, d/a)$ (Ref. 5),

$$g\left(p, \frac{d}{a}\right) = p\mathscr{R}e \times \int_0^\infty \frac{x\,dx}{x - ip + (x^2 - 2ipx)^{1/2} \times \operatorname{ctgh}[(x^2 - 2ipx)^{1/2}(d/2a)]}, \qquad (17)$$

where $p = 4\pi\sigma a v_z \gamma_z$ and a is the distance of the ring from the cylinder. For the two limits $d/a = \infty$ and

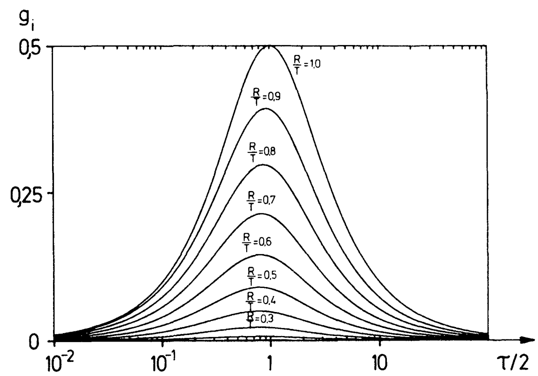

FIGURE 4 For a ring moving inside a thin hollow cylinder the retarding force $F_M = -(m_e r_0 N_e v_\varphi^{02}/2\pi R(T-R))g_i$ is plotted as function of $\tau = 4\pi\sigma d v_z \gamma_z$ and R/T.

d/a small, the integral (17) can be evaluated analytically. For $d/a = \infty$ one obtains[2]

$$g\left(p, \frac{d}{a} = \infty\right) = -\frac{1}{p} + \frac{\pi}{2}[J_1(p)\sin p - N_1(p)\cos p], \quad (18)$$

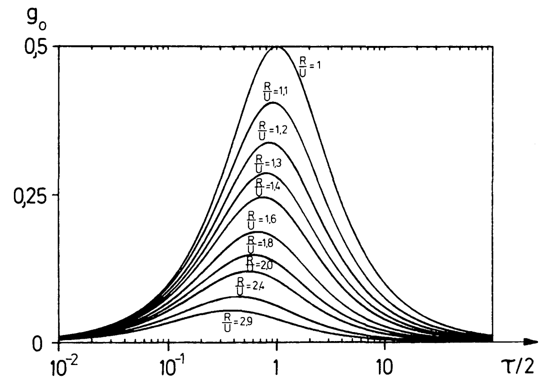

FIGURE 5 For a ring moving outside a thin hollow cylinder the retarding force $F_M = -(m_e r_0 N_e v_\varphi^{02}/2\pi R(R-U))g_0$ is plotted as function of $\tau = 4\pi\sigma d v_z \gamma_z$ and R/U.

where $J_1(p)$ and $N_1(p)$ are the Bessel and Neumann functions, respectively. For $(1+p^2)^{1/4}(d/a) \ll 1$, one finds the very simple result

$$g\left(p, \frac{d}{a}\right) = \frac{pd/2a}{1+(pd/2a)^2}. \tag{19}$$

The asymptotic behaviour $p \gg 1$ of $g(p, d/a)$ is found to be

$$g\left(p \gg 1, \frac{d}{a}\right) \approx \frac{1}{2}\left(\frac{\pi}{d}\right)^{1/2} \tag{20}$$

The function $g(p, d/a)$ is plotted in Figure 6 as a function of p for different values of d/a. For sufficiently small d/a, the maximum value of $g(p, d/a)$ is doubled relative to the $d = \infty$ case, and the maximum is shifted to higher values of p. The maxima are reached for $d = \infty$ at $2\pi\sigma a v_z \gamma_z = 1$ and for small d at $2\pi\sigma d v_z \gamma_z = 1$.

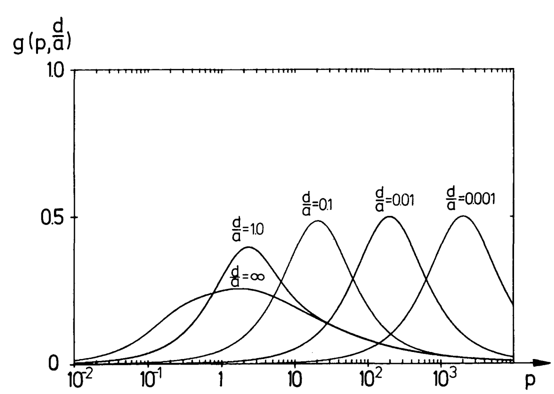

FIGURE 6 For a ring moving inside or outside close to a hollow cylinder the retarding force $F_e = -(m_e r_0 N_e v_\varphi^{02}/2\pi Ra)g$ is plotted as function of $p = 4\pi\sigma a v_z \gamma_z$ and d/a.

3 THE CASE OF A CHARGED RING

The contribution to the retarding force due to the ring charge can be similarly calculated. Assuming first that the ring is moving inside the hollow cylinder ($R < T$), we obtain for the charge and current densities in the three regions (Figure 1):

$$\begin{aligned}
\text{I:}\quad & \rho = Q\delta(z - v_z t)\delta(r - R), \\
& \mathbf{j} = (0, 0, 1)Q\delta(z - v_z t)\delta(r - R)v_z, \\
\text{II:}\quad & \rho = 0, \qquad \mathbf{j} = \sigma\mathbf{E}, \\
\text{III:}\quad & \rho = 0, \qquad \mathbf{j} = 0,
\end{aligned} \tag{21}$$

where Q is the charge line density.

Now the only nonvanishing components of the electric and magnetic fields are B_φ, E_r, E_z. To find the retarding force, we need the electric field component E_z. Inserting Eqs. (21) in Eqs. (1), eliminating B_φ and E_r and using Eq. (5) yields for the Fourier transformed E_z the equations (the index z is dropped and replaced by the region label):

$$\begin{aligned}
\text{I:}\quad & \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - k^2\right)E_{\text{I}} = 2ikQ\delta(r - R), \\
\text{II:}\quad & \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - k^2 + 4\pi i\sigma v_z \gamma_z k\right)E_{\text{II}} = 0, \\
\text{III:}\quad & \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - k^2\right)E_{\text{III}} = 0.
\end{aligned} \tag{22}$$

The conditions that B_φ and E_z be continuous at the region boundaries give the following condition for $E_z(k, r)$

$$\begin{aligned}
E_{\text{I}} = E_{\text{II}},\ \frac{\partial}{\partial r}E_{\text{I}} &= \frac{k + i4\pi\sigma/v_z\gamma_z}{k - i4\pi\sigma/v_z\gamma_z}\frac{\partial}{\partial r}E_{\text{II}} \\
E_{\text{II}} = E_{\text{III}},\ \frac{\partial}{\partial r}E_{\text{III}} &= \frac{k + i4\pi\sigma/v_z\gamma_z}{k - i4\pi\sigma/v_z\gamma_z}\frac{\partial}{\partial r}E_{\text{II}}.
\end{aligned} \tag{23}$$

Moreover it is required that E_{I} be regular at $r = 0$ and E_{III} vanish for $r \to \infty$. The solution of Eqs. (22) can be determined in the same way as in the neutral ring current case. For a ring moving inside the cylinder ($R < T$) the force on each electron $F_e = eE_z(z - v_z t = 0, r = R)$ yields

$$F_e = -\frac{m_e r_0 N_e}{2\pi R(T - R)}h_i\left(\frac{R}{T}, \frac{d}{T}, v_z\gamma_z, \sigma T\right) \tag{24}$$

with

$$h_i = \frac{4R(T-R)}{R^2}\mathscr{I}m\int_0^\infty \mathrm{d}x\,xI_0^2\left(x\frac{R}{T}\right) \times \frac{(x+4\pi i\sigma T/v_z\gamma_z)K_0(x)E_0 + sK_1(x)E_1}{(x+4\pi i\sigma T/v_z\gamma_z)I_0(x)E_0 - sI_1(x)E_1}, \tag{25}$$

and

$$\begin{aligned} E_0 &= K_1(s)F_0 + I_1(s)F_1,\\ E_1 &= -K_0(s)F_0 + I_0(s)F_1,\\ F_0 &= sI_0\left(s\frac{U}{T}\right) + \left(x + \frac{4\pi i\sigma T}{v_z\gamma_z}\right)I_1\left(s\frac{U}{T}\right),\\ F_1 &= sK_0\left(s\frac{U}{T}\right) - \left(x + \frac{4\pi i\sigma T}{v_z\gamma_z}\right)K_1\left(s\frac{U}{T}\right), \end{aligned} \tag{26}$$

where $s = (x^2 - 4\pi i\sigma T v_z\gamma_z x)^{1/2}$. The corresponding calculation for a ring moving outside a cylinder ($R > U$) gives

$$F_e = -\frac{m_e r_0 N_e}{2\pi R(R-U)}h_0\left(\frac{R}{U}, \frac{d}{U}, v_z\gamma_z, \sigma U\right) \tag{27}$$

with

$$h_0 = \frac{4R(R-U)}{U^2}\mathscr{I}m\int_0^\infty \mathrm{d}x K_0^2\left(x\frac{R}{U}\right) \tag{28}$$

$$\times \frac{(x+4\pi i\sigma U/v_z\gamma_z)I_0(x)G_0 - \tilde{s}I_1(x)G_1}{(x+4\pi i\sigma U/v_z\gamma_z)K_0(x)G_0 + \tilde{s}K_1(x)G_1},$$

and

$$\begin{aligned} G_0 &= I_1(\tilde{s})H_0 - K_1(\tilde{s})H_1,\\ G_1 &= I_0(\tilde{s})H_0 + K_0(\tilde{s})H_1,\\ H_0 &= \left(x + \frac{4\pi i\sigma U}{v_z\gamma_z}\right)K_1\left(\tilde{s}\frac{T}{U}\right) - \tilde{s}K_0\left(\tilde{s}\frac{T}{U}\right),\\ H_1 &= \left(x + \frac{4\pi i\sigma U}{v_z\gamma_z}\right)I_1\left(\tilde{s}\frac{T}{U}\right) + \tilde{s}I_0\left(\tilde{s}\frac{T}{U}\right), \end{aligned} \tag{29}$$

where $\tilde{s} = (x^2 - 4\pi i\sigma U v_z\gamma_z x)^{1/2}$.

The general evaluation of the integrals (25) and (28) will be omitted. To get an upper limit of the retarding force, it is sufficient to discuss the case when the ring moves close to the cylinder $(T-R) \ll T$ and $(R-U) \ll U$, respectively. In this plane approximation both functions h_i and h_0 have the same limit $h(\tilde{\lambda}, d/a, v_z\gamma_z)$. The drag force can then be written[5]

$$F_e = -\frac{m_e r_0 N_e}{2\pi Ra}h\left(\tilde{\lambda}, \frac{d}{a}, v_z\gamma_z\right) \tag{30}$$

with

$$h\left(\tilde{\lambda}, \frac{d}{a}, v_z\gamma_z\right) = \mathscr{I}m\int_0^\infty \mathrm{d}x\, e^{-x} \times \frac{\tilde{\lambda}[ix(v_z\gamma_z + (2/v_z\gamma_z)) - \tilde{\lambda}(2/v_z\gamma_z)]}{x^2 - ix\tilde{\lambda}(v_z\gamma_z - (2/v_z\gamma_z)) - (2\tilde{\lambda}^2/v_z^2\gamma_z^2) + (x + i(2\tilde{\lambda}/v_z\gamma_z))\hat{s}\,\mathrm{ctgh}(\hat{s}d/2a)}, \tag{31}$$

where $\hat{s} = (x^2 - 2i\tilde{\lambda}v_z\gamma_z x)^{1/2}$, $\tilde{\lambda} = 4\pi\sigma a$ and a is the distance of the ring from the cylinder. In Figure 7, the force $h(\tilde{\lambda}, d/a, v_z\gamma_z)$ is plotted as a function of $v_z\gamma_z$ for different values of d/a and $\tilde{\lambda}$.

In general, the function $h(\tilde{\lambda}, d/a, v_z\gamma_z)$ depends on the three variables $\tilde{\lambda} = 4\pi\sigma a$, d/a and $v_z\gamma_z$. But there are regions in which $h(\tilde{\lambda}, d/a, v_z\gamma_z)$ will be approximately a function of only one variable.

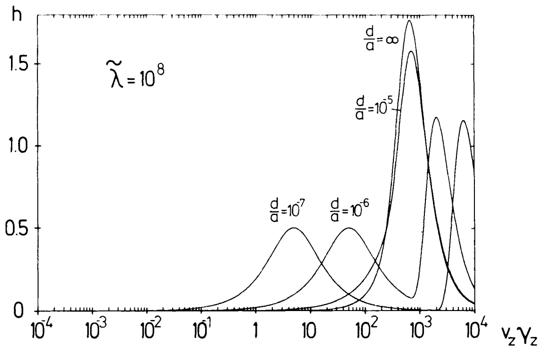

fig. 7a

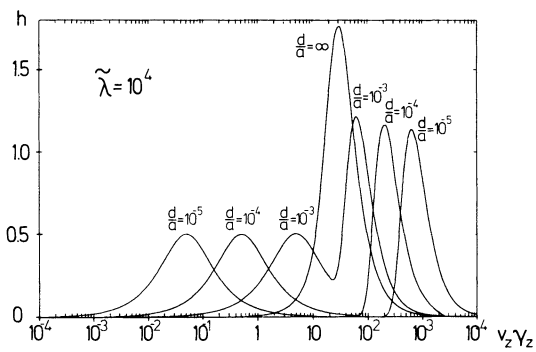

fig. 7b

FIGURE 7 For a ring moving inside or outside close to a hollow cylinder the retarding force $F_e = -(m_e r_0 N_e/2\pi Ra)h$ is plotted as function of $v_z\gamma_z$ and d/a for different values of $\tilde{\lambda} = 4\pi\sigma a$.

Beginning with an infinite thick sheet ($d = \infty$), one finds on the assumption that $\tilde{\lambda} \gg 1/v_z\gamma_z$, $\tilde{\lambda} \gg v_z\gamma_z$

$$h\left(\tilde{\lambda}, \frac{d}{a} = \infty, v_z\gamma_z\right) = \int_0^\infty \mathrm{d}x$$

$$\times\, e^{-x} \frac{2\sqrt{\kappa x}}{x - 2\sqrt{\kappa x} + 2\kappa}, \quad \kappa = \frac{\tilde{\lambda}}{(v_z\gamma_z)^3}. \tag{32}$$

The retarding force depends on $\kappa = \tilde{\lambda}/(v_z\gamma_z)^3$ and has its maximum at $\kappa \approx 0.4$

For thin sheets, there are also simple expressions for the function $h(\tilde{\lambda}, d/a, v_z\gamma_z)$. Assuming

$$(1 + (2\tilde{\lambda}v_z\gamma_z)^2)^{1/4}\frac{d}{a} \ll 1$$

one obtains for

$$\left|v_z\gamma_z + \frac{2}{v_z\gamma_z}\right| \ll \frac{2\tilde{\lambda}}{v_z^2\gamma_z^2}$$

$$h\left(\tilde{\lambda}, \frac{d}{a}, v_z\gamma_z\right) = \frac{2av_z\gamma_z/d\tilde{\lambda}}{1 + (2av_z\gamma_z/d\tilde{\lambda})^2}, \tag{33a}$$

and for

$$\left|v_z\gamma_z + \frac{2}{v_z\gamma_z}\right| \gg \frac{2\tilde{\lambda}}{v_z^2\gamma_z^2}, \frac{4a}{dv_z^2\gamma_z^2} \ll 1, \frac{2a}{d\tilde{\lambda}v_z\gamma_z} \approx 1$$

$$h\left(\tilde{\lambda}, \frac{d}{a}, v_z\gamma_z\right) = \frac{2a/d\tilde{\lambda}v_z\gamma_z}{1 + (2a/d\tilde{\lambda}v_z\gamma_z)^2}, \tag{33b}$$

and, for

$$\left|v_z\gamma_z + \frac{2}{v_z\gamma_z}\right| \gg \frac{2\tilde{\lambda}}{v_z^2\gamma_z^2}, \frac{2a}{d\tilde{\lambda}v_z\gamma_z} \ll 1,$$

$$h\left(\tilde{\lambda}, \frac{d}{a}, v_z\gamma_z\right) = \pi \exp\left(-\frac{4a}{dv_z^2\gamma_z^2}\right)\frac{4a}{dv_z^2\gamma_z^2}. \tag{33c}$$

The asymptotic behaviour $v_z\gamma_z \gg 1$ of $h(\tilde{\lambda}, d/a, v_z\gamma_z)$ is found to be

$$h\left(\tilde{\lambda}, \frac{d}{a}, v_z\gamma_z\right) = \frac{1}{2}\left(\frac{\pi}{v_z\gamma_z\tilde{\lambda}}\right)^{1/2} \tag{34}$$

For a more detailed discussion of the function $h(\tilde{\lambda}, d/a, v_z\gamma_z)$ see Ref. 6.

In Figures 7*a* and *b* the function $h(\tilde{\lambda}, d/a, v_z\gamma_z)$ is plotted as a function of $v_z\gamma_z$ for different values of d/a and $\tilde{\lambda}$. Equations (32) and (33a–c) give a good approximation of the curves near the maxima. To present an overall picture of the position of the maxima of h in Figure 8 they are plotted as functions of $v_z\gamma_z$, $\tilde{\lambda}$ and d/a.

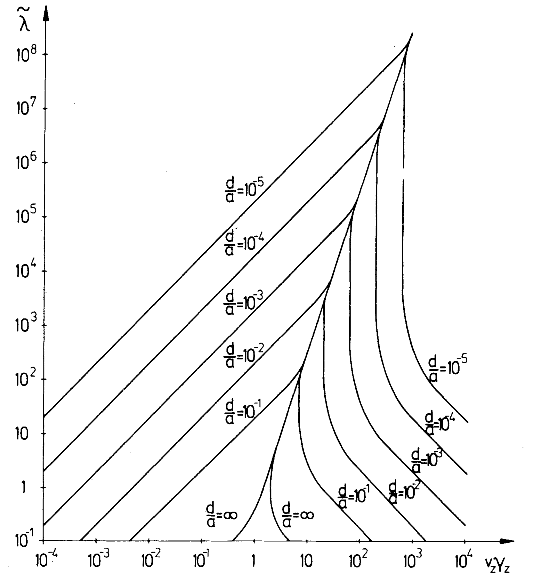

FIGURE 8 The position of the maxima of F_e is shown as function of $v_z\gamma_z$ and d/a for different values of $\tilde{\lambda} = 4\pi\sigma a$.

4 APPLICATION TO THE ELECTRON RING ACCELERATOR

Characteristic data of an electron ring device for the acceleration of heavy ions are: number of electrons $N_e = 5 \times 10^{12}$, major ring radius $R = 2.5$ cm, distance of the ring from the conducting wall $a = 1$ cm, the transverse velocity $v_\varphi^0 \approx 1$ and the longitudinal velocity $v_z: 0 < v_z\gamma_z < 1$. If the conductivity of the applied wall material is varied between copper ($\sigma = 5.2 \times 10^{17}$ sec^{-1}) and graphite ($\sigma = 7 \times 10^{12}$ sec^{-1}) the parameter λ will be in the range of $10^3 < \lambda < 2 \times 10^8$. If furthermore, the thickness d of the conducting cylinder is assumed to be $d/a \gg 10^{-3}$ the drag force F_e due to the longitudinal current of the ring electrons is very small and can be neglected. In the above described parameter region a good approximation of F_e is obtained by taking Eq. (32) in the limit

$$\kappa = \frac{\tilde{\lambda}}{(v_z\gamma_z)^3} \gg 1$$

$$F_e = -\frac{m_e r_0 N_e}{2\pi R a}\frac{(v_z\gamma_z)^{3/2}}{4(\sigma a)^{1/2}}, \tag{35}$$

which for the given ring data, the conductivity of copper and $v_z\gamma_z = 1$ yields $F_e = 0.05$ eV/cm.

The main contribution to the drag force is given by F_M due to the transverse motion of the ring electrons. For a thick cylinder ($d = \infty$) it follows from Eqs. (18) and (10) (see Figure 3) that F_M reaches its maximum for copper at $v_z \gamma_z = 10^{-8}$ and, inserting the above given ring data, the maximum force is $F_M = 11$ keV/cm.

For the more interesting case of a thin cylinder ($d/a \ll 1$) the retarding force is given by Eqs. (10) and (19). The force depends only on $v_z \gamma_z$ and the product σd. This suggests the introduction of the surface resistivity $\rho = (\sigma d)^{-1}$. Then F_M can be written as

$$F_M = -\frac{m_e r_0 N_e}{2\pi R a} \frac{2\rho[\Omega]/377 v_z \gamma_z}{1 + (2\rho[\Omega]/377 v_z \gamma_z)^2}. \quad (36)$$

With the ring data assumed the maximum force is $F_M = 22$ keV/cm at a velocity $v_z \gamma_z = \rho[\Omega]/188.5$ (v_z in units of the velocity of light). It is thus shown that the drag force F_M on an intense relativistic electron ring due to image currents in conducting walls can reach high values. This retarding force can be useful if one is interested in slowing down the longitudinal motion of an electron ring.[7] Using the retarding force [Eq. (36)] the integration of the equation of motion yields a stopping distance as a function of the velocity v_z^0

$$L = \frac{\gamma_\perp 2\pi R a}{r_0 N_e} \left(\frac{188.5}{\rho[\Omega]} \frac{v_z^{03}}{3} + \frac{\rho[\Omega]}{188.5} v_z^0 \right) \quad (38)$$

where is $\gamma_\perp = (1 - v_\varphi^{02})^{1/2}$. With $\rho = 188.5\, v_z^0/\sqrt{3}$ the stopping distance becomes a minimum. With the above ring data and $v_z^0 = 0.1$ one obtains a stopping distance $L = 3.5$ cm. This result shows that the retarding by means of a resistive cylinder can be very effective.

On the other hand, if one wants to accelerate the ring, the drag force can be very embarrassing.

The characteristic behaviour of the retarding force is, that it first increases with the velocity v_z, then reaches a maximum and finally decreases for higher v_z. It should be pointed out that above the maximum, where $\partial |F_M| / \partial v_z < 0$, the contribution F_M to the accelerating forces leads to an unstable situation for the longitudinal motion. This effect has been discussed by Herrmann[3,4] in detail.

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