XVII. CIRCUIT THEORY*

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A. PHASE INVARIANTS

The circuit significance of the conjunctive, or inert, transformation was discussed in Quarterly Progress Report No. 56 (page 198). It was shown that imbedding an arbitrary network, with admittance [Y], in an inert network (that is, a network that conserves complex power) transforms the admittance by the conjunctive transformation.

$$[Y_1] = [T]^x[Y][T]$$
$$[T]^x = \text{conjugate transpose of } [T] \tag{1}$$

The purpose of this report is to show some interesting invariant properties of this transformation.

If Y is the admittance of a one-port network, then Eq. 1 may be written $Y_1 = |T|^2 Y$. It is thus apparent that the magnitude of Y_1 can be varied at will by varying the magnitude of T, but the phase angle of Y_1 must equal the phase angle of Y. Since the phase angle is the only invariant for a one-port network, we are encouraged to look for a generalized phase invariant for an n-port network.

Since the eigenvalues of [Y] have many interesting properties, we are led to wonder whether the phase angles of these eigenvalues are ever invariant under a conjunctive transformation. The answer is that they are invariant if $[Y][Y]^x = [Y]^x[Y]$. A matrix that commutes with its conjugate transpose is called "normal"; this class includes as important special cases matrices that are Hermitian, skew-Hermitian, symmetric, skew-symmetric, and/or unitary. The proof that the phase angles of the eigenvalues of a normal matrix are conjunctively invariant is straightforward.

PROOF 1. If [U] is unitary, then $[U]^x = [U]^{-1}$. By a classical theorem, a [U] can always be found to satisfy the relation $[U]^x[Y][U] = [D]$, where [D] contains the eigenvalues of [Y] on the diagonal and zeros elsewhere (1). By dividing the i^{th} row and column by the square root of the magnitude of the i^{th} eigenvalue, we can conjunctively transform Y into a diagonal matrix with the i^{th} diagonal element having unit magnitude and the same phase angle as the i^{th} eigenvalue.

As a practical example of this transformation, we can consider a 2 × 2 symmetric matrix.

*This work was supported in part by Purchase Order DDL B-00306 with Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology with the joint support of the U.S. Army, Navy, and Air Force under Air Force Contract AF 19(604)-5200.

Let $[Y] = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$; $[Y_1] = [T]^x[Y][T]$

If $[T] = \begin{bmatrix} |w|^2x^x & wz \\ w^xz^x & |z|^2y \end{bmatrix}$; with x, y chosen to satisfy relations

$$ax + b(1+xy) + cy = 0$$

$$a^x x + b^x(1+xy) + c^x y = 0$$

and

$$|w|^{-2} = |a|x|^2+b(x+x^x)+c|$$

$$|z|^{-2} = |a+b(y+y^x)+c|y|^2|$$

then $[Y_1] = \begin{bmatrix} \frac{A}{|A|} & 0 \\ 0 & \frac{B}{|B|} \end{bmatrix}$; A, B = eigenvalues of $[Y] = \frac{a+c}{2} \pm \left[\left(\frac{a-c}{2}\right)^2 + b^2\right]^{1/2}$

Since most nonsymmetric admittance matrices are not normal, we need to find a more general phase invariant. One approach is by analogy with the procedure for finding the phase angle of a scalar. If y is a scalar, we can write

$$y = \exp(\alpha+j\beta); \quad y^x = \exp(\alpha-j\beta); \quad y^{x,-1} = \exp(-\alpha+j\beta)$$

$$yy^{x,-1} = \exp(2j\beta)$$

Thus $\frac{1}{2j}\ln(yy^{x,-1}) = \beta \pm n\pi$, with n an integer. Except for the ambiguity of $\pm n\pi$, we can determine β from $yy^{x,-1}$. We might expect that the eigenvalues of $[Y][Y]^{x,-1}$ are invariant and could be interpreted as $\exp(2j\beta_i)$, where the β_i are the generalized phase invariants. The proof that the eigenvalues of $[Y][Y]^{x,-1}$ are invariant is relatively simple.

PROOF 2. If $[Y_1] = [T]^x[Y_o][T]$

$$\text{then } [Y_1][Y_1]^{x,-1} = [T]^x[Y_o][Y_o]^{x,-1}[T]^{x,-1}$$

$$\text{Hence } \mathrm{ev}\left([Y_1][Y_1]^{x,-1}\right) = \mathrm{ev}\left([Y_o][Y_o]^{x,-1}\right)$$

For the scalar phase relation, $yy^{x,-1}$ always has unit magnitude so that β is a real number. In the matrix relation, however, β may also occur in conjugate pairs. A proof that β must be real or occur in conjugate pairs follows.

PROOF 3. $(\mathrm{ev}[Y][Y]^{x,-1})^{x,-1} = \mathrm{ev}([Y][Y]^{x,-1})^{x,-1} = \mathrm{ev}[Y]^{x,-1}[Y] = \mathrm{ev}[Y][Y]^{x,-1}$

Hence, if $e^{j\beta_i} = \mathrm{ev}_i[Y][Y]^{x,-1}$, then there must be a β_k that satisfies the relation

$$\left(e^{j\beta_i}\right)^{x,-1} = \left(e^{j\beta_i}\right)^{x} = e^{j\beta_k} = \mathrm{ev}_k[Y][Y]^{x,-1}$$

and thus β must be real or occur in conjugate pairs.

One possible interpretation for a complex phase invariant can be deduced from an example. Let

$$[Y] = \begin{bmatrix} 0 & e^{a} \\ -e^{-a} & 0 \end{bmatrix} je^{j\beta}; \quad [Y]^{x,-1} = \begin{bmatrix} 0 & e^{-a} \\ -e^{a} & 0 \end{bmatrix} je^{j\beta} \tag{2}$$

$$[Y][Y]^{x,-1} = \begin{bmatrix} e^{2a} & 0 \\ 0 & e^{-2a} \end{bmatrix} e^{j2\beta}; \quad \frac{1}{2j} \ln\, (\mathrm{ev}[Y][Y])^{x,-1} = \beta \pm ja \pm n\pi$$

The network described by [Y] in Eq. 2 has the property that if it is terminated by equal source and load impedances with phase angles β, then the voltage gain will be e^{a} in one direction and e^{-a} in the other. Thus the imaginary part of the phase invariants is related to the gain of nonreciprocal networks.

The conjugate-pair phase invariants, $\beta \pm ja$, for a two-port network can be calculated as follows:

$$\text{If } [Y] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \quad \mathrm{ev}[Y][Y]^{x,-1} = (\cosh 2a \pm \sinh 2a)\, e^{j2\beta}$$

where

$$\cosh 2a = \frac{|b|^2 + |c|^2 - (ad^x + a^x d)}{2\,|ad-bc|}; \quad \cos 2\beta = \frac{\mathrm{Re}\,(ad-bc)}{|ad-bc|} \tag{3}$$

If $\cosh 2a$ in Eq. 3 is less than unity, then the phase invariants are both real. Defining these real invariants as β_1 and β_2, we find that

$$\cos(\beta_1 - \beta_2) = \frac{(ad^x + a^x d) - |b|^2 - |c|^2}{2\,|ad-bc|}; \quad \cos(\beta_1 + \beta_2) = \frac{\mathrm{Re}\,(ad-bc)}{|ad-bc|}$$

The eigenvalues of $[Y][Y]^{x,-1}$ alone do not represent a complete set of invariants, and other methods must be used to determine completely a canonic form for Y. The problem of developing a complete and simple canonic form has not been fully solved.

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