

## X. MODULATION THEORY AND SYSTEMS

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### A. SIGNAL-TO-NOISE RATIOS IN LIMITERS WITH REGENERATIVE FEEDBACK

Consider the block diagram of Fig. X-1 in which the  $S$  and  $N$  symbols represent the mean-square values of signal and noise at the principal points, when a sinusoid of constant amplitude  $E_s$  and instantaneous frequency  $\omega_1$  plus an independent random-fluctuation

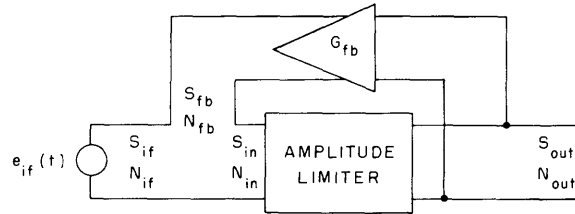


Fig. X-1. Limiter with regenerative feedback.

noise of total mean-square value  $N_{if}$  is applied across the i-f terminals. To start with, if  $n_{if}(t)$  denotes the i-f sample noise time function and  $n_{fb}(t)$  the noise fed back, then we have

$$N_{in} = \overline{[n_{if}(t) + n_{fb}(t)]^2} = N_{if} + N_{fb} + 2\rho(N_{if}N_{fb})^{1/2} \quad (1)$$

where  $\rho$  is the correlation coefficient of i-f and feedback noises, and the bar denotes the time average over a suitably long interval. Similarly, if we assume that the feedback phase shift  $\phi_{fb}(\omega_i)$  at the instantaneous frequency of the signal is negligible, we have

$$S_{in} = S_{if} + G_{fb}^2 S_{out} + 2G_{fb}(S_{if}S_{out})^{1/2} \quad (2)$$

The  $S/N$  mean power ratio at the limiter input is given by  $S_{in}/N_{in}$ .

If we now write

$$\frac{S_{out}}{N_{out}} = \gamma \frac{S_{in}}{N_{in}} \quad (3)$$

then substitution from Eqs. 1 and 2 in Eq. 3, followed by a rearrangement of the terms, leads to a quadratic equation in  $(S_{out}/N_{out})^{1/2}$ . Since the square-root quantity in the third term of Eq. 2 is strictly intended to be positive, so is  $(S_{out}/N_{out})^{1/2}$ , and of the two roots of the quadratic equation only the positive one is acceptable. This argument leads to

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$$\left(\frac{S_{\text{out}}}{N_{\text{out}}}\right)^{1/2} / \left(\frac{S_{\text{if}}}{N_{\text{if}}}\right)^{1/2} = -\rho(S_{\text{fb}}/S_{\text{if}})^{1/2} + \left[ \gamma \left\{ 1 + (S_{\text{fb}}/S_{\text{if}})^{1/2} \right\}^2 - (1-\rho^2)(S_{\text{fb}}/S_{\text{if}}) \right]^{1/2} \quad (4)$$

The right-hand side of Eq. 4 gives the factor by which the ratio of rms signal to rms noise has been altered in going from the input to the output of the limiter with regenerative feedback under the assumption (to be examined later) that a feedback quasi-stationary state exists in which signal and noise have the mean-square values that appear at the output in Fig. X-1. In discussing the significance of this result, it is helpful to have an expression for the mean-square value of the noise fed back. Such an expression follows from the relations  $N_{\text{fb}} = G_{\text{fb}}^2 N_{\text{out}}$ ,  $N_{\text{out}} = S_{\text{out}}/(S_{\text{out}}/N_{\text{out}})$ , and Eq. 4. The result is

$$N_{\text{fb}} = \frac{N_{\text{if}}}{\left[ -\rho + \left\{ \gamma \left[ 1 + (S_{\text{if}}/S_{\text{fb}})^{1/2} \right]^2 - (1-\rho^2) \right\}^{1/2} \right]^2} \quad (5)$$

$$= \frac{N_{\text{if}}}{\left[ -\rho + \left\{ \gamma + (\rho^2 - 1) \right\}^{1/2} \right]^2} \quad \text{for large values of } (S_{\text{fb}}/S_{\text{if}})^{1/2} \quad (6)$$

$$= (S_{\text{fb}}/\gamma S_{\text{if}}) N_{\text{if}} \quad \text{for small values of } (S_{\text{fb}}/S_{\text{if}})^{1/2} \quad (7)$$

The expression on the right in Eq. 4 involves three quantities that we must examine closely. First, there is the quantity  $\gamma$  that relates the signal-to-noise mean power ratios at the input and output of the limiter in accordance with Eq. 3. This quantity,  $\gamma$ , is a complicated function of  $S_{\text{in}}/N_{\text{in}}$ . For the case of gaussian noise at the input of the limiter,  $\gamma$  was computed by Davenport (1). But, in the present problem, the limiter action alters the statistics of the noise so that it is no longer gaussian. Consequently, the noise at the input of the limiter is the sum of gaussian noise from the i-f amplifier and non-gaussian noise from the feedback terminals, and the two add up to non-gaussian noise. Therefore  $\gamma$  for the limiter in the feedback steady state cannot be expected to be exactly the same as the function computed by Davenport. But it can be expected to behave in a similar manner as a function of  $(S/N)_{\text{in}}$ ; in particular, it exceeds unity for the larger values of  $(S/N)_{\text{in}}$  and becomes less than unity for very low  $(S/N)_{\text{in}}$ . Note from Eq. 7 that when the signal component fed back is weak relative to the signal coming from the i-f amplifier, the noise fed back is small compared with the i-f noise from the i-f terminals. Under this condition the resultant noise at the limiter input is largely made up of the i-f noise, and its statistics are therefore approximately gaussian if the i-f noise is gaussian. Consequently, under conditions of strong input signal relative to the amount of feedback present,  $\gamma$  is given very closely by Davenport's curve. The problem of determining  $\gamma$  analytically in the feedback steady state when  $S_{\text{fb}}$  is not small compared

with  $S_{if}$  is formidable and it remains unsolved.

It is important to observe that the right-hand side of Eq. 4 will exceed unity, provided that

$$\gamma > \frac{1 + 2\rho(S_{fb}/S_{if})^{1/2} + \frac{S_{fb}}{S_{if}}}{\left[1 + (S_{fb}/S_{if})^{1/2}\right]^2} \quad (8)$$

Fortunately, the right-hand member of this inequality is  $\leq 1$  for all  $\rho \leq 1$ .

Second, there is the correlation coefficient,  $\rho$ , of the noise coming from the i-f amplifier and the noise coming through the feedback amplifier to the input of the limiter. Decorrelation between the i-f and feedback noises is caused by two mechanisms that are present in the loop. First, there is decorrelation by the memoryless amplitude limiting of the sum of signal plus noise. The amount of noise decorrelation that results from memoryless amplitude limiting of signal plus noise is a function of the S/N ratio at the limiter input. For gaussian noise at the limiter input, the correlation coefficient of input and output noise can be computed directly from expressions derived by Davenport (1). The results show that the correlation coefficient is slightly below 0.95 for  $(S/N)_{in} < 1/10$  and approximately 0.71 for  $(S/N)_{in} > 10$ , with a smooth transition in between. Second, there is decorrelation as a result of delay in transmission around the loop. The autocorrelation function of filtered fluctuation noise falls off at a rate that depends upon the noise bandwidth, the exact shape of the curve being a function of the shape of the noise-power spectral density at the filter output. The mathematical reason for this lies, of course, in the fact that the autocorrelation function of the noise is the inverse Fourier transform of its power spectral density. But the physical reason can perhaps be seen by tracing the noise back to its source, where it is generated as a random superposition of short pulses, with a resultant power spectral density that is essentially uniform over the passband of the intervening concatenation of filters that precede the point of observation. At the point of observation, the noise is therefore a superposition of the impulse responses of the over-all filter to the noise "impulses" from the source. The ability to extrapolate the noise waveform into the future is therefore essentially lost if the "future" comes one or more overall-filter time constants later.

Now, the resultant noise at the limiter input is the sum of the i-f noise and the noise fed back. Since the correlation coefficient of the i-f noise and the limiter input noise is given by

$$\rho_{if, in} = (N_{if}/N_{in})^{1/2} + \rho(N_{fb}/N_{in})^{1/2}$$

decorrelation by the limiter and by the delay in transmission around the loop ensures that

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the limiter input noise starts the cycle around the loop with only a partial correlation with the i-f noise. When the level of the voltage fed back is large in comparison with the level at the i-f terminals, approximation 6 shows that the resultant noise at the limiter input is, to a large extent, made up of the noise fed back. Therefore in the feedback steady state, the instantaneous behavior of the noise present across the feedback terminals tends to be little influenced by the instantaneous behavior of the i-f noise over a past that may stretch over intervals that are in excess of the decorrelation time of the i-f noise.

Thus, even though no analytical solution now exists for the exact behavior of the correlation coefficient ( $\rho$  in Eq. 4) of the i-f and feedback noises in the feedback steady state as a function of the relative i-f signal and noise conditions,  $\rho^2$  can certainly be expected to be negligible as compared with unity and can be dropped in the second term in Eq. 4. It may even be assumed to be so low as not to materially influence the result in Eq. 4, even though it multiplies a quantity for which large values are desirable (as we shall indicate presently). It is interesting to observe that negative values of  $\rho$  (which are not unlikely) are advantageous because the first term in Eq. 4 is thereby transformed into an asset. Note that nonzero  $\rho$  of either sign is an asset in the second term in Eq. 4.

Finally, there is the quantity  $S_{fb}/S_{if}$  on the right-hand side of Eq. 4. This quantity represents the ratio of mean-square values of the signal component fed back and the signal component introduced at the i-f terminals. The larger this ratio is, the greater the improvement in S/N ratio indicated in Eq. 4. Large values of this ratio also ensure smaller values of  $\rho$ , and should increase the value of  $\gamma$ . But there are bounds (imposed by the feedback phase shift at the instantaneous frequency of the signal) on how large we can allow  $S_{fb}/S_{if}$  to be before we begin to violate the assumption of feedback quasi-stationary state which underlies the validity of the argument leading to Eq. 4.

When the S/N mean power ratio at the input of the limiter is large, the mean-square value,  $S_{out}$ , of the signal component can be expected to be essentially  $k^2/2$ , where  $k$  is the amplitude that the signal component at the limiter output would have in the absence of the noise. Using Eq. 3, we can then write

$$N_{out} = (k^2/2\gamma)/(S_{in}/N_{in}) \quad (9)$$

Under these conditions the limiter operates on the sum of signal plus noise present at its input in a manner that improves the signal relative to the noise by keeping the amplitude of the signal component substantially constant and depressing the relative mean-square value of the noise. The condition that leads to this (with  $\rho = 0$ , for simplicity) can be expressed in the form

$$\frac{S_{in}}{N_{if}} > b + \frac{S_{fb}}{\gamma N_{if}} \quad (10)$$

where  $b$  is a threshold value that can be determined from the curve that describes  $S_{\text{out}}$  as a function of  $(S/N)_{\text{in}}$  by seeking the smallest value of  $(S/N)_{\text{in}}$  for which  $S_{\text{out}} \approx k^2/2$ . For gaussian noise (plus a sinusoid) at the input of the limiter, the value of  $b$  is around 3. If we set  $S_{\text{out}} = k^2/2$  in expression 10 and write  $E_{\text{fb}} = kG_{\text{fb}}$ , we have

$$\frac{(E_s + E_{\text{fb}})^2 / 2}{N_{\text{if}}} > b + \frac{E_{\text{fb}}^2 / 2}{\gamma N_{\text{if}}} \quad (11)$$

By straightforward manipulations, inequality 11 can be reduced to

$$\left( \frac{E_s^2 / 2}{N_{\text{if}}} \right)^{1/2} > \left( b + \frac{E_{\text{fb}}^2 / 2}{\gamma N_{\text{if}}} \right)^{1/2} - \left( \frac{E_{\text{fb}}^2 / 2}{N_{\text{if}}} \right)^{1/2} \quad (12)$$

Inequality 12 defines the threshold that must be exceeded by the ratio of rms signal and rms noise at the i-f terminals shown in Fig. X-1 in order that  $(S/N)_{\text{in}} > b$ , which, in turn, ensures that  $S_{\text{out}} \approx k^2/2$ . In deriving this condition, we have assumed that a feedback quasi-stationary state exists, and that  $\rho = 0$ .

It is clear from the preceding discussion that substantial signal improvement in the presence of relatively strong random noise (and full limiter saturation) can be guaranteed only by choosing  $E_{\text{fb}}$  to be large compared with  $E_s$  (and greater than the threshold of limiter saturation). The regenerative signal-booster arrangement of Fig. X-1 should therefore oscillate (2) in the absence of a signal, and the oscillation should be sufficiently strong relative to the input noise to bring about an acceptable degree of FM noise quieting. By "FM noise quieting" we mean a compression of the instantaneous-frequency excursions of the output from the scheme in the manner experienced when a sinusoid of adequate strength is added to the noise – which amounts to a narrowing of the probability density function of the instantaneous frequency of the output in the absence of an input signal. If, in the absence of signal, the amplitude of the oscillation is reduced below the adequate-quieting level relative to the total applied noise, the level of the noise disturbance on the oscillation frequency will rise. The output from the oscillator eventually loses all semblance of coherence when the noise takes effective control.

The improvement in  $S/N$  ratio that results from the regenerative feedback may or may not be sufficient to raise the signal from below to above a preassigned threshold of acceptable reception, subject to the value of  $(S_{\text{fb}}/S_{\text{if}})^{1/2}$ . The theoretical upper limit on the permissible value of  $(S_{\text{fb}}/S_{\text{if}})^{1/2}$  in a given circuit is set by the maximum value of feedback phase shift encountered by the signal around the loop. This limitation is imposed by the fact that – oscillation or no oscillation – a feedback quasi-stationary state must first be established around the loop. For example, suppose that the oscillation amplitude (as seen at the feedback terminals) must exceed a value  $E_{\text{osc, th}}$  in order that

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the output S/N ratio in Eq. 4 exceed the threshold of acceptable noise performance of the FM demodulator used after the oscillating limiter. An input sinusoidal carrier whose frequency coincides with the frequency of in-phase feedback is indistinguishable from the oscillation at the output of the limiter. Therefore, the threshold that the input amplitude  $E_s$  of this sinusoid must exceed is zero if  $E_{osc} > E_{osc, th}$ . However, if the frequency of the applied sinusoid is changed to a value  $\omega_1$  at which a phase-shift deviation  $\phi_{fb}(\omega_1)$  from in-phase feedback is experienced, then as long as

$$(E_{osc}/E_s) \sin |\phi_{fb}(\omega_1)| < 1 \quad (13)$$

the signal should override the noise in the output when  $E_{osc} > E_{osc, th}$ . Satisfaction of this condition by the input sinusoid enables this sinusoid to shift the mean frequency about which the compressed fluctuations in the instantaneous frequency of the output occur to the desired value,  $\omega_1$ . The regenerative action boosts the signal amplitude at the limiter input to a value that fluctuates with time in synchronism with a slow modulation in  $\omega_1(t)$ , but this magnitude will, at worst, very nearly equal  $E_{osc}$  at instants of time when  $\phi_{fb}(\omega_1)$  attains its maximum permissible value. Therefore, the relatively weak signal, which (by itself) would have been suppressed by the total noise, when applied to a conventional FM demodulator, is boosted by the regenerative action of the "locked" oscillating limiter to the higher level that it must have in order to override the noise. If  $E_{osc}$  is adjusted so that it equals the threshold value  $E_{osc, th}$ , then  $E_s$  must satisfy the condition

$$E_s > E_{osc, th} \sin |\phi_{fb}(\omega_1)|_{max} \quad (14)$$

In order to determine the amount of reduction in the threshold of satisfactory reception indicated by condition 14, we must first relate  $E_{osc, th}$  to the noise threshold,  $E_{n, th}$ , of the FM demodulator. This can be done with the aid of Eq. 4. Thus, in terms of  $E_{n, th}$ , condition 14 can be written as

$$E_s > \left\{ \frac{\rho + \left( \rho^2 + \gamma(1 + \sin |\phi_{fb}(\omega_1)|_{max})^2 - 1 \right)^{1/2}}{\gamma(1 + \sin |\phi_{fb}(\omega_1)|_{max})^2 - 1} \sin |\phi_{fb}(\omega_1)|_{max} \right\} E_{n, th} \quad (15)$$

If we assume that  $\rho = 0$ , and design the feedback loop so that  $|\phi_{fb}(\omega_1)|_{max} \ll 1$ , condition 15 can be simplified in the form

$$E_s > \left\{ \frac{\sin |\phi_{fb}(\omega_1)|_{max}}{(\gamma-1)^{1/2}} \right\} E_{n, th} \quad (16)$$

The noise threshold that  $E_s$  must exceed is thus reduced by the factor in braces in conditions 15 and 16. If  $|\phi_{fb}(\omega_1)|_{max} = 5^\circ$  (or 0.087 rad),  $E_s$  can be nearly -20 db below  $E_{n, th}$ .

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From the definition of  $E_{osc, th}$ , it is clear that the signal need only override the noise components that fall in the immediate neighborhood of its instantaneous frequency in order to satisfy the "locking" condition. In a sense, this indicates that in the oscillating limiter the signal, in effect, combats noise density rather than total noise.

E. J. Baghdady

### References

1. W. B. Davenport, Jr., Signal-to-noise ratios in bandpass limiters, J. Appl. Phys. 24, 720-727 (1953); see especially Fig. 5.
2. E. J. Baghdady, FM interference and noise suppression properties of the oscillating limiter, IRE National Convention Record, Part 8, 1959, pp. 13-39; Trans. IRE, PGVC-13, pp. 37-63 (1959).

