MATHEMATICAL MODELLING AND ANALYSIS
VOLUME 9 NUMBER 1, 2004, PAGES 25–38

NUMERICAL SOLUTION OF SINGLE MODE GYROTRON EQUATION

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Received October 12 2003; revised December 15 2003

Abstract. In this paper we study numerical problems arising in solving the single mode gyrotron equation. Using the method of finite differences analytical and numerical solutions are obtained. Quasistationary solutions and corresponding eigenvalues and eigenfunctions of this problem are investigated.

Key words: the spectral problems, finite-difference schemes, stability analysis, analytical solutions

1. Introduction

Gyrotrons are microwave sources whose operation is based on the stimulated cyclotron radiation of electrons oscillating in a static magnetic field. Single mode non-stationary gyrotron oscillations can be described by the following system of partial differential equations [2]:

$$\begin{cases} \dfrac{\partial p}{\partial x} + i\left(\Delta + |p|^2 - 1\right)p = i f(t,x)\,, \\ \dfrac{\partial^2 f}{\partial x^2} - i\dfrac{\partial f}{\partial t} + \delta f = \dfrac{I}{2\pi}\displaystyle\int_0^{2\pi} p\,d\theta_0. \end{cases} \tag{1.1}$$

Here $i = \sqrt{-1}$, $x \in [0, L]$ is the normalized axial coordinate, t is the normalized time, Δ is the frequency mismatch, δ describes variation of the critical frequencies, Δ and δ are real numbers, $p = p(t, x, \theta_0)$ is the dimensionless complex transverse momentum of the electron, $f = f(t, x)$ is the high-frequency field in resonator, I

is the dimensionless current, θ_0 is the parameter. The system of equations has to be supplemented by the standard initial condition

$$p(t, 0, \theta_0) = \exp(i\theta_0), \quad 0 \le \theta_0 < 2\pi, \quad f(0, x) = f_0(x)\,,$$

and by the boundary condition for the field at the entrance to the interaction space $f(t, 0) = 0$, and at the exit to the interaction space

$$\frac{\partial f(t, L)}{\partial x} = -i\gamma f(t, L),$$

where γ is a positive parameter, $f_0(x)$ is given complex function. An efficient numerical method for solving this reduced system of equations was presented in [1]. However, it was discovered that the results of the computations depend in a nontrivial manner on the chosen spatial and temporal step-lengths. So, main difficulties arises in numerical solving of Schrödingers type equation with special boundary conditions. The aim of this paper is to study in detail numerical problems for the second equation of (1.1).

2. Solution of the differential problem

We begin with the homogeneous Schrödinger type partial differential equation ($I = 0$)

$$\frac{\partial^2 f}{\partial x^2} - i\frac{\partial f}{\partial t} + \delta f = 0, \tag{2.1}$$

where $x \in (0, L)$, $t > 0$ – is time, $\delta = const$[1]. Boundary conditions can be written as

$$f(t, 0) = 0, \quad \frac{\partial f(t, L)}{\partial x} = -i\gamma f(t, L). \tag{2.2}$$

We represent the quasi-stationary solution of the problem (2.1) and (2.2) in the form

$$f(t, x) = g(x)\exp(i\alpha t), \tag{2.3}$$

where α is a complex number $\alpha = \alpha_1 + i\alpha_2$ (α_2 is a temporal damping factor: if $\alpha_2 > 0$, the solution (2.3) decreases, if $\alpha_2 < 0$, the solution increases, and for $\alpha_2 = 0$ the solution is oscillating in time). We now consider nontrivial solutions (2.3) of the differential problem by computing allowed values of the parameter α, as well as the corresponding discrete problem. Substituting the solution (2.3) into equation (2.1) and boundary conditions (2.2), we obtain the Sturm-Liouville problem for the ordinary differential equation

$$\begin{cases} g''(x) + \lambda^2 g(x) = 0, \\ g(0) = 0, \quad g'(L) = -i\gamma g(L), \end{cases} \tag{2.4}$$

[1] Using the substitution $g(t, x) = f(t, x)\exp(i\delta t)$ for function g we would obtain the boundary value problem (2.1) and (2.2) with $\delta = 0$. We don't use mentioned substitution because function f and parameter δ have the physical interpretation

where $\lambda^2 = \alpha + \delta$ is complex value. The solution of problem (2.4) is

$$g(x) = \tilde{C}_1 \sin(\lambda x),$$

where $\tilde{C}_1$ is an arbitrary constant. From boundary conditions we obtain a transcendental complex equation for calculating the eigenvalue λ:

$$\lambda \cos(\lambda L) + i\gamma \sin(\lambda L) = 0$$

or

$$z \cos z = -i\tilde{\gamma} \sin z, \tag{2.5}$$

where $z = z_1 + iz_2 = \lambda L$ and $\tilde{\gamma} = \gamma L$. It is obvious that $z = 0$ is a root of the equation. Moreover, if z is the root of (2.5), then also $-z$ is the root of this equation. Therefore we can confine ourselves to consider only $z_1 > 0$. Separating real and imaginary parts in equation (2.5), we obtain a system of two real transcendental equations

$$\begin{cases} z_1 \cos z_1 \cosh z_2 + z_2 \sin z_1 \sinh z_2 = \tilde{\gamma} \cos z_1 \sinh z_2, \\ z_2 \cos z_1 \cosh z_2 - z_1 \sin z_1 \sinh z_2 = -\tilde{\gamma} \sin z_1 \cosh z_2. \end{cases} \tag{2.6}$$

Multiplying the first equation of system (2.6) by $\sin z_1 \cosh z_2$ and the second equation by $\cos z_1 \sinh z_2$ and summing, we exclude the parameter $\tilde{\gamma}$ and obtain the relation

$$z_1 \sin(2z_1) + z_2 \sinh(2z_2) = 0\,.$$

It follows that the nontrivial roots of the last equation satisfy the inequality

$$\sin(2z_1) < 0 \ \text{ or } \ \tan(z_1) < 0\,.$$

Dividing the second equation of the system (2.6) by $\cos z_1 \cosh z_2$, we obtain

$$z_2 - z_1 \tan z_1 \tanh z_2 = -\tilde{\gamma} \tan z_1.$$

It can be seen that if $z_1 > 0$, then also $z_2 > 0$. Let us number the roots of (2.5) $z^{(k)}$, $k = 1, 2, \ldots$, whose real parts $z_1^{(k)}$ are positive, by increasing their real parts and take into account that $(k-1)\pi < z_1^{(k)} < k\pi$, $0 < z_2^{(k)} < \tilde{\gamma} + 1$. Since $\lambda = \sqrt{\alpha + \delta}$ or $z^2 = L^2(\alpha + \delta)$, we have

$$\alpha_1 = \frac{z_1^2 - z_2^2}{L^2} - \delta, \quad \alpha_2 = \frac{2z_1 z_2}{L^2}.$$

It is seen that the parameter δ affects only the values of α_1 and $\alpha_2 > 0$. The results of computations performed by means of "MAPLE" for $L = 15$, $\gamma = 2$, and $\delta = 0$ are summarized in Tab. 1 for the first eight eigenvalues and numerical values of $\alpha_1^{(k)}$ and $\alpha_2^{(k)}$, $k = \overline{1, 8}$. Fig. 1 shows the first fifty eigenvalues λ_k. It can be seen that $\alpha_2^{(k)} > 0$ and that all solutions

$$f^{(k)}(t, x) = \sin\left(\sqrt{\alpha_1^{(k)} + \delta + i\alpha_2^{(k)}}\, x\right) \exp((-\alpha_2^{(k)} + i\alpha_1^{(k)})t)$$

Table 1. The roots of equation (2.6) and values of $\alpha^{(k)}$.

k	$z_1^{(k)}$	$z_2^{(k)}$	$\alpha_1^{(k)}$	$\alpha_2^{(k)}$
1	3,1381	0,10498	0,0437	0,00293
2	6,2758	0,21232	0,1748	0,01184
3	9,4128	0,32466	0,3933	0,02716
4	12,5484	0,44547	0,6989	0,04969
5	15,6814	0,57970	1,0914	0,08080
6	18,8092	0,73539	1,5700	0,12295
7	21,9251	0,92732	2,1327	0,18073
8	25,0054	1,18594	2,7727	0,26360

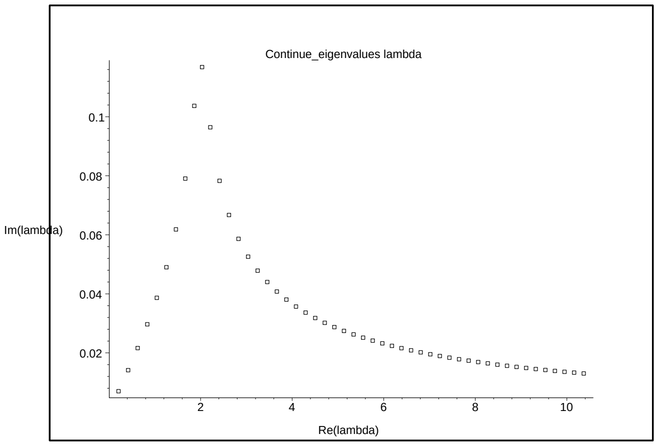

Figure 1. Eigenvalues of the continuous problem λ_k, $k = \overline{1, 59}$.

monotonically decrease in time, i.e.,

$$f^{(k)}(x,t) \to 0, \ \ t \to +\infty, \ k = 1, 2, \dots ,$$

(here $\tilde{C}_1 = 1$). Taking the square root in expression, we obtain two complex numbers in the form $\pm(a^{(k)} + ib^{(k)})$, where $a^{(k)} > 0$ and $b^{(k)} > 0$, if $\alpha_1^{(k)} + \delta > 0$ and $\alpha_2^{(k)} > 0$. Since the functions $f^{(k)}(t,x)$ contain an arbitrary constant $\tilde{C}_1$, the complex number with the minus sign does not give us any new result and can be ignored. Separating real and imaginary parts we obtain

$$|f^{(k)}(t,x)| = \exp(-\alpha_2^{(k)} t)\sqrt{\sinh^2(b^{(k)}x) + \sin^2(a^{(k)}x)}.$$

Let us note that the complex eigenfunctions $g_k(x) = \sin(\lambda_k x)$ ($\lambda_k = \sqrt{\alpha^{(k)} + \delta}$ and $\alpha^{(k)} = \alpha_1^{(k)} + i\alpha_2^{(k)}$) are orthogonal, i.e.,

$$\langle g_k, g_k \rangle = \int_0^L g_k(x) g_n(x)\, dx = 0, \quad k \neq n.$$

Correspondingly

$$\|g_k\|^2 = \langle g_k, g_k \rangle = \int_0^L g_k^2(x)\,dx = \frac{1}{2}\left(L + \frac{i\gamma}{\lambda_k^2 - \gamma^2}\right).$$

Each continuous function $\tilde{f}(x)$, $x \in (0, L)$ with boundary conditions (2.4) can be expanded in the series of the orthonormalized eigenfunctions $\tilde{g}_k(x) = g_k(x)/\|g_k\|$

$$\tilde{f}(x) = \sum_{k=1}^{\infty} c_k \tilde{g}_k(x)\,,$$

where the expansion coefficients can be found in the form $c_k = \langle \tilde{f}, \tilde{g}_k \rangle$. Calculating by means of "MAPLE", we obtain that the oscillation frequency of the functions increases and their absolute values rapidly decrease with increasing k.

3. Solution of the discrete problem

In the finite differences method we use a uniform homogeneous spatial and temporal grids:

$$\omega_h = \{\, x_j : \quad x_j = jh,\ j = \overline{1, N-1}, Nh = L \,\},\ \ \omega_\tau = \{\, t_n : \quad t_n = n\tau, n \geq 1 \,\}$$

(corresponding step-lengths are h and τ). We substitute the continuous function $f = f(t, x)$ in these grids by the discrete grid function $y = y(t, x)$, $t \in \omega_\tau$, $x \in \omega_h$ with values $y(t_n, x_j) \equiv y_j^n$. The corresponding derivatives of the function we approximate by finite–differences

$$\frac{\partial^2 f(t_n, x_j)}{\partial x^2} \approx \Lambda y_j^n \equiv \frac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{h^2}\,, \tag{3.1}$$

$$\frac{\partial f(t_n, x_j)}{\partial t} \approx \frac{y_j^{n+1} - y_j^n}{\tau}, \qquad \frac{\partial f(t_n, L)}{\partial x} \approx \frac{y_N^n - y_{N-1}^n}{h}\,. \tag{3.2}$$

Difference (3.2) approximates the first derivative only to the first order of accuracy, i.e., $O(h)$. To obtain the second order approximation, we must use the expression

$$\frac{\partial f(t_n, L)}{\partial x} \approx \frac{1.5y_N^n - 2y_{N-1}^n + 0.5y_{N-2}^n}{h}\,. \tag{3.3}$$

Substituting differences (3.1), (3.2) into the problem (2.1) – (2.2), we obtain a two-layer finite-difference scheme with weight $\sigma \in [0, 1]$

$$\begin{cases} i\dfrac{y_j^{n+1} - y_j^n}{\tau} = \sigma(\Lambda y_j^{n+1} + \delta y_j^{n+1}) + (1 - \sigma)(\Lambda y_j^n + \delta y_j^n), \quad j = \overline{1, N-1}\,, \\ y_0^{n+1} = 0, \quad \dfrac{y_N^{n+1} - y_{N-1}^{n+1}}{h} = -i\gamma y_N^{n+1}\,. \end{cases} \tag{3.4}$$

Difference equations (3.4) approximate the initial differential equation (2.1) to the second order both in space and time, if $\sigma = 1/2$, and to the first order in time, if $\sigma \neq 1/2$. Boundary conditions (2.2) are approximated only to the first order. To obtain the second order, one has to use expression (3.3). Seeking to find the discrete quasi-stationary solution by analogy to (2.3) we take

$$y_j^n = g_j \exp(i\alpha n\tau) \quad (g_j = g(x_j),\ t_n = n\tau),$$

then we obtain that the discrete function $g_j \not\equiv 0$ satisfies the three-point finite-difference scheme

$$\begin{cases} \Lambda g_j + \mu^2 g_j = 0, \quad j = \overline{1, N-1}\,, \\ g_0 = 0, \quad g_N = C g_{N-1}, \end{cases} \tag{3.5}$$

which approximates the continuous problem (2.4). Here

$$C = (1 + i\gamma h)^{-1}, \ \ \mu^2 = \tilde{\alpha} + \delta, \ \ \tilde{\alpha} = \frac{(1 - \exp(i\alpha\tau))i}{(\sigma \exp(i\alpha\tau) + 1 - \sigma)\tau} \tag{3.6}$$

are complex constants ($\tilde{\alpha} \to \alpha$, if $\tau \to 0$). Now the solution of (3.5) can be written as $g_j = \tilde{C}_1 \sin(qx_j)$, where $\tilde{C}_1$ is arbitrary constant, $1 - \mu^2 h^2/2 = \cos(qh)$ and $x_j = jh$. It follows from boundary conditions (3.5) that the complex parameter q has to be determined from the complex transcendental equation

$$\sin(qL) = C \sin(q(L - h)), \tag{3.7}$$

where the parameter q has complex values

$$q_k = a_k + ib_k, \quad k = \overline{1, N-1}. \tag{3.8}$$

If $\gamma = \infty$ (boundary conditions of the first kind), then $C = 0$ and equation $\sin(qL) = 0$ is valid, if $q_k = \frac{k\pi}{L}$ (real numbers). Then we get also real eigenvalues [4]

$$\mu_k^2 = \frac{4}{h^2} \sin^2 \frac{k\pi h}{2L}, \quad k = \overline{1, N-1}\,.$$

Therefore

$$\tilde{\alpha}_k = 2h^{-2}(1 - \cos(q_k h)) - \delta = A_k + iB_k, \tag{3.9}$$

where

$$\begin{aligned} A_k &= 2h^{-2}(1 - \cos(a_k h)\cosh(b_k h)) - \delta, \\ B_k &= 2h^{-2} \sin(a_k h) \sinh(b_k h), \quad k = \overline{1, N-1}. \end{aligned} \tag{3.10}$$

Since $C = C_1 + iC_2$, $C_1 = (1 + (\gamma h)^2)^{-1}$, and $C_2 = -\gamma h(1 + (\gamma h)^2)^{-1}$, we separate in equation (3.7) real and imaginary parts and obtain the system of two real transcendental equations

$$\begin{cases} \sin(a_k L)\cosh(b_k L) = C_1 \sin(a_k l_1)\cosh(b_k l_1) - C_2 \cos(a_k l_1)\sinh(b_k l_1), \\ \cos(a_k L)\sinh(b_k L) = C_1 \cos(a_k l_1)\sinh(b_k l_1) + C_2 \sin(a_k l_1)\cosh(b_k l_1), \end{cases} \tag{3.11}$$

where $l_1 = L - h$. If $h \to 0$, then $a_k L \to z_1$, $b_k L \to z_2$ and we obtain the system of equations (2.6). After calculation of $\tilde{\alpha}_k$, we obtain from (3.6) the approximate values

$$\alpha_k = \frac{1}{i\tau} \ln\left(1 - \frac{\tau\tilde{\alpha}_k}{i + \sigma\tau\tilde{\alpha}_k}\right), \quad k = \overline{1, N-1}. \tag{3.12}$$

It can be seen from (3.12) that the temporal step-length τ and the parameter of the scheme σ, i.e., the temporal approximation, do not affect the values of $\tilde{\alpha}_k$, their changes have to be taken into account only in the expression (3.12). Approximating boundary conditions (2.4) by the second order expression (3.3), instead of equation (3.7) we obtain the complex transcendental equation

$$\sin(qL) = C^*(2\sin(ql_1) - 0,5\sin(ql_2)),$$

where $l_2 = L-2h,\ C^* = (1,5+i\gamma h)^{-1}$. The results of computations with $L = 15$, $\gamma = 2$, $\delta = 0$, $\tau = h = 0,1$, and $\sigma = 1$ are presented in Tab. 2, where a_k and b_k are solutions of (3.11) and

$$\frac{\pi(k-1)}{L} < a_k < \frac{\pi k}{L}, \quad 0 < b_k < 1\,.$$

The values A_k and B_k were obtained from (3.9) and (3.10) $k = \overline{1,8}$. The results do not change much (five digits remain the same) by changing the temporal step-length τ in interval $(0,01, 0,1)$. More accurate results can be obtained with $\sigma = 0.5$. Comparing the solutions of the continuous and discrete problems, we see that only for the first two eigenvalues three or four digits remain the same, while for other eigenvalues the accuracy rapidly deteriorates. Using the second order approximation even for the eighth eigenvalue two digits are correct, if $\sigma = 1/2$. Considering only the spatial discretization (the variable x is discretized $x_j = jh$ and the variable t is continuous), we obtain (by means of the *method of lines*) the boundary problem for

Table 2. The discrete values q_kL, $\tilde{\alpha}_k$.

k	a_kL	b_kL	A_k	B_k
1	3,1380	0,1050	0,0437	0,0029
2	6,2745	0,2115	0,1748	0,0118
3	9,4095	0,3240	0,3929	0,0271
4	12,5400	0,4440	0,6976	0,0494
5	15,6630	0,5745	1,0879	0,0798
6	18,7725	0,7215	1,5618	0,1201
7	21,8535	0,8925	2,1154	0,1728
8	24,8805	1,0875	2,7396	0,2393

the system of complex ordinary differential equations

$$\begin{cases} i\dfrac{dy_j}{dt} = \Lambda y_j + \delta y_j, \quad j = \overline{1, N-1}\,, \\ y_0 = 0, \quad \dfrac{y_N - y_{N-1}}{h} = -i\gamma y_N\,, \end{cases}$$

where $y_j = y_j(t)$ are continuous functions of time, $j = \overline{0,N}$. Seeking the quasi-stationary solution of this system in the form

$$y_j(t) = g_j \exp(i\alpha t)$$

we obtain the system similar to (3.5) where $\mu^2 = \alpha + \delta$. This means that the quantities $A_k + iB_k$ in expression (3.10) are approximate eigenvalues α_k, $k = \overline{1, N-1}$, obtained by means of the method of lines (see Tab. 2). Using in the boundary conditions the second order approximation, we obtain an analogous problem, which, just as the grid method, gives more accurate results. In oder to increase the accuracy of discrete equation (3.5) we will use the Taylor expansion

$$\Lambda g(x_i) = g''(x_i) + \frac{h^2}{12} g^{(4)}(x_i) + \ldots + \frac{2h^{2m-2}}{(2m)!} g^{(2m)}(x_i) + O(h^{2m}),$$

where $m = 1, 2, \ldots$. From equation (2.4) it follows that

$$\Lambda g(x_i) = \frac{2g(x_i)}{h^2} \tilde{\mu}_m^2 \,,$$

where

$$\tilde{\mu}_m^2 = \left(-\frac{(\lambda h)^2}{2!} + \frac{(\lambda h)^4}{4!} + \ldots + (-1)^m \frac{(\lambda h)^{2m}}{2m!} + O(h^{2m}) \right).$$

Similarly from boundary conditions (3.5) we obtain

$$g(x_{N-1}) = g(x_N) - hg'(x_N) + \ldots + \frac{(-1)^l h^l}{l!} g^{(l)}(x_N) + O(h^{l+1}), \ l \geq 0 \,,$$

and from the boundary condition of the problem (2.4) $g(x_{N-1}) = C_k g(x_N)$, where

$$C_k = 1 - \frac{(h\lambda)^2}{2!} + \ldots + (-1)^k \frac{(h\lambda)^{2k-2}}{(2k-2)!}$$
$$+ \frac{i\gamma}{\lambda} \left(h\lambda - \frac{(h\lambda)^3}{3!} + \ldots + (-1)^k \frac{(h\lambda)^{2k-1}}{(2k-1)!} \right) + O(h^{2k}), \quad k \geq 1 \,.$$

It can be seen that the discrete problem (the errors are proportional to $O(h^{2m})$ and $O(h^{2k})$ $m, k = 1, 2, \ldots$) is given in the form

$$\begin{cases} \Lambda g_i + \tilde{\mu}_m^2 g_i = 0, \quad i = \overline{1, N-1} \,, \\ g_0 = 0, \quad g_N = C_k^{-1} g_{N-1}. \end{cases}$$

It can be seen that in the limit case ($m \to \infty$, $k \to \infty$) $\tilde{\mu}_m^2 \to \frac{2}{h^2}(1 - \cos(h\lambda))$ and $C_k \to \cos(h\lambda) + i\gamma\lambda^{-1} \sin(h\lambda)$ or we obtain the transcendental equation (2.5). Eigenfunctions of the discrete problem (3.5)

$$g_j^{(k)} \equiv g^{(k)}(x_j) = \sin(q_k x_j), \ \ x_j = jh, \ j = \overline{0, N}, \ k = \overline{1, N-1}$$

are orthogonal with respect to the scalar product $\langle g^{(k)}, g^{(n)} \rangle \equiv h \sum_{j=1}^{N} g_j^{(k)} g_j^{(n)}$, i.e., $\langle g^{(k)}, g^{(n)} \rangle = 0$, if $k \neq n$. This follows from the Green formula [4]. Evaluating $\|g^{(k)}\|^2 = \langle g^{(k)} g^{(k)} \rangle$, we obtain orthonormalized system $\tilde{g}^{(k)} = g^{(k)} / \|g^{(k)}\|$, for which $\langle \tilde{g}^{(k)}, \tilde{g}^{(n)} \rangle = \delta_{k,n}$ (the Kronecker symbol). Considering the second order approximation for the boundary condition (3.3), we cannot obtain a system of orthogonal eigenfunctions. Evaluating $\|g^{(k)}\|^2$, we obtain

$$\|g^{(k)}\|^2 = \frac{1}{2}\left(L - \frac{h\sin(q_k L)\cos(q_k(L-h))}{\sin(q_k h)}\right).$$

If $h \to 0$, then $q_k \to \lambda_k$ and $\|g^{(k)}\|^2 \to \|g_k\|^2$. If boundary conditions (3.4) are given as $y_N^{n+1} = 0$ ($\gamma = \infty$), then $q_k = \dfrac{k\pi}{L}$ and $\|g^{(k)}\|^2 = \dfrac{L}{2}$ [4]. Each grid function $\tilde{f}(x)$, $x \in \omega_h$ with boundary conditions (3.5) can be expanded as a finite sum

$$\tilde{f}(x) = \sum_{k=1}^{N} c_k \tilde{g}^{(k)}(x)$$

of orthonormalized eigenfunctions $\tilde{g}^{(k)}(x) = g^{(k)}(x)/\|g^{(k)}\|$, $x \in \omega_h$, where the expansion coefficients can be found with the help of the expressions $c_k = \langle \tilde{f}, \tilde{g}^{(k)} \rangle$. The solution of the boundary problem

$$\begin{cases} \Lambda g = -\tilde{f}(x), \quad x \in \omega_h\,, \\ g(0) = 0, \quad g(L) = Cg(L-h) \end{cases}$$

is $g(x) = \sum_{k=1}^{N} c_k \tilde{g}^{(k)}(x)/\mu_k^2$.

4. Stability of the difference scheme

To study the stability of the discrete problem (difference scheme) (3.4), we rewrite the difference equations with respect to the difference $z_j = y_j^n - f(x_j, t_n)$ in the matrix operator form

$$(E + i\tau\sigma(\Lambda + \delta))z^{n+1} = (E - i\tau(1-\sigma)(\Lambda + \delta))z^n,$$

where $z^n = (z_1^n, z_2^n, \ldots, z_{N-1}^n)^T$ is the error vector-column and E is the unit operator. Hence $z^{n+1} = Gz^n$, where

$$G = (E + i\tau\sigma(\Lambda + \delta))^{-1}(E - i\tau(1-\sigma)(\Lambda + \delta))$$

is the transition operator with the eigenvalues

$$\lambda_k = \frac{1 + i\tau(1-\sigma)(\mu_k^2 - \delta)}{1 - i\tau\sigma(\mu_k^2 - \delta)}, \quad k = \overline{1, N-1},$$

where μ_k^2 are eigenvalues of the difference operator $(-\Lambda)$ to be determined from the boundary problem (3.5) $\mu_k^2 = 2h^{-2}(1 - \cos(q_k h))$. If μ_k^2 are real numbers, e.g., in the case of the first kind boundary conditions ($\gamma = \infty$, $z_N^{n+1} = 0$) $q_k = \frac{k\pi}{L}$, then from the stability condition [4]

$$|\lambda_k|^2 = (1 + \tau^2(1-\sigma)^2(\mu_k^2 - \delta)^2)(1 + \tau^2\sigma^2(\mu_k^2 - \delta)^2)^{-1} \le 1,$$

it follows that

$$\sigma \ge \frac{1}{2} \tag{4.1}$$

independent of the size of the temporal step-length τ. Similar problem for Schrödinger type differential equation was investigated in [3]. Taking the boundary condition of the third kind in the form $z_N^{n+1} = C z_{N-1}^{n+1}$ and determining the complex parameter q_k in the form (3.8), we find from (3.11) that $\mu_k^2 = \tilde{\alpha}_k + \delta$, where $\tilde{\alpha}_k$ can be determined from (3.9) and (3.10). Then from

$$|\lambda_k|^2 = \frac{(1-\tau(1-\sigma)B_k)^2 + A_k^2\tau^2(1-\sigma)^2}{(1+\tau\sigma B_k)^2 + A_k^2\tau^2\sigma^2} \leq 1$$

it follows that

$$-2B_k + \tau(1-2\sigma)(A_k^2 + B_k^2) \leq 0.$$

If $\sigma \geq \frac{1}{2}$ and $B_k \geq 0$, then this inequality holds and the difference scheme (3.4) is stable. If $\sigma = 1$, we obtain the inequality

$$\tau \geq -2B_k(A_k^2 + B_k^2)^{-1}, \tag{4.2}$$

which is important, if $B_k < 0$. It is seen from (3.11) that, if (a_k, b_k) is a solution of this system, then also $(-a_k, -b_k)$ is a solution. The values of the coefficients A_k, B_k do not change and it is sufficient to consider only $a_k > 0$. If simultaneously $b_k > 0$, then also $B_k > 0$, and the stability condition holds in the form (4.1). If $a_k = b_k = 0$, then $B_k = 0$ and the difference scheme is stable. Calculations with the help of "MAPLE" show that positive variables a_k correspond to positive variables b_k i.e., $B_k > 0$. If the parameter $\gamma < 0$, then it can be easily seen that positive a_k correspond to negative b_k and the difference scheme (3.4) is absolutely unstable, if the temporal step-length τ is not large enough (in inequality (4.2) $B_k < 0$).

5. Method of separation of variables

Let us consider the inhomogeneous equation

$$\frac{\partial^2 f}{\partial x^2} - i\frac{\partial f}{\partial t} + \delta f = F(t,x), \tag{5.1}$$

with a given function $F(t,x)$. We seek the solution $f = f(t,x)$ with the boundary conditions (2.2) in the form of a series

$$f(t,x) = \sum_{k=1}^{\infty} a_k(t)\tilde{g}_k(x), \tag{5.2}$$

where $\tilde{g}_k(x)$ are orthonormalized eigenfunctions and $\lambda_k L = z^{(k)} = z_1^{(k)} + iz_2^{(k)}$ are solutions of (2.6). To determine functions $a_k(t)$, we use the given initial conditions $f(0,x) = f_0(x)$. Taking a scalar product of (5.2) and a fixed eigenfunction, if $t = 0$, we obtain $a_k(0) = \langle f_0, \tilde{g}_k \rangle$. By analogy expanding the right-hand side of (5.1)

$$F(x,t) = \sum_{k=1}^{\infty} F_k(t)\tilde{g}_k(x)$$

we obtain $F_k(t) = \langle F, \tilde{g}_k \rangle$. Assuming that series (5.2) and the series, differentiated twice with respect to x and once with respect to t, uniformly converge, and substituting it into (5.1), we obtain the ordinary differential equation

$$-\lambda_k^2 a_k(t) - i\dot{a}_k(t) + \delta a_k(t) = F_k(t), \quad t > 0$$

and

$$a_k(t) = a_k(0)\exp(i\alpha^{(k)}t) + i\int_0^t \exp(i\alpha^{(k)}(t-\zeta))F_k(\zeta)\,d\zeta,$$

where $\dot{a}_k(t) = \dfrac{da_k}{dt}$, $\alpha^{(k)} = \lambda_k^2 - \delta$. As example, if $f_0(x) = \sin\left(\frac{\pi x}{L}\right)$, $F = 0$, then solutions of the differential problem can be obtained in the form

$$f(t,x) = 2\pi L \sum_{k=1}^{\infty} \frac{\exp(i\alpha^{(k)}t)\sin(\lambda_k x)\sin(\lambda_k L)(\lambda_k^2 - \gamma^2)}{(\pi^2 - \lambda_k^2 L^2)(\lambda_k^2 L - \gamma^2 L + i\gamma)}\,.$$

Solving the corresponding discrete problem with the initial condition

$$y_0^j = \sin(\frac{\pi}{L}x_j) \equiv f_0(x_j), \quad j = \overline{0, N}\,,$$

we obtain $y_j^n = \sum_{k=1}^{N-1} a_k^n \tilde{g}^{(k)}(x_j)$, where $\tilde{g}^{(k)}(x_j) = \sin(q_k x_j)/\|g^{(k)}\|$ are discrete eigenfunctions. Determining a_k^0 from the initial condition

$$a_k^0 = (f_0, \tilde{g}^{(k)}) = hb_k,\ b_k = \sum_{s=1}^{N-1} \sin(\frac{\pi}{L}sh)\tilde{g}^{(k)}(sh),$$

we find from difference equations the recurrence relation $a_k^{n+1} = \rho_k a_k^n$ or $a_k^n = (\rho_k)^n a_k^0$, where $\rho_k = 1 + i\tau\tilde{\alpha}_k/(1 - i\tau\sigma\tilde{\alpha}_k)$. Hence

$$y_j^n = h\sum_{k=1}^{N-1} \frac{b_k \exp(i\alpha_k \tau n) g^{(k)}(x_j)}{\|g^{(k)}\|^2}, \tag{5.3}$$

where

$$b_k = \frac{\sin(\frac{\pi}{L}h)\sin(q_k L)}{2(\cos(q_k h) - \cos(\frac{\pi}{L}h))},$$

α_k can be determined from (3.12), and $\tilde{\alpha}_k$ from (3.9). It can be easily seen that $y_j^n \to f(t_n, x_j)$, if $h \to 0$, $\tau \to 0$, i.e., the solution of the discrete problem converges to the solution of the continuous problem. Using the method of lines (only spatial discretization), we obtain $y_j(t) = \sum_{k=1}^{N-1} a_k(t)\tilde{g}^{(k)}(x_j)$ by analogy, where functions $a_k(t)$ are solutions of the Cauchy problem

$$\begin{cases} \dfrac{da_k(t)}{dt} = i\alpha^{(k)}a_k(t)\,, \\ a_k(0) = hb_k\,, \end{cases}$$

i.e., $a_k(t) = a_k(0)\exp(i\alpha_k t)$. Hence the solution can be written as

$$y_j(t) = h\sum_{k=1}^{N-1} b_k \exp(i\alpha_k t)\sin(q_j x_j)/\|g^{(k)}\|^2,$$

i.e., analogously to (5.3) where $\alpha_k = 2h^{-2}(1 - \cos(q_k h)) - \delta$.

Table 3. The values $|f|$, $|f_h|$, $|f_l|$, $|f_{2h}|$ for $x = L$, $\tau = 0,01$.

t	$\|f\|$	$\|f_h\|$	$\|f_l\|$	$\|f_{2h}\|$
0,1	0,05087	0,05085	0,05162	0,04413
0,2	0,06217	0,06224	0,06269	0,05785
0,3	0,06871	0,06862	0,06899	0,06557
0,4	0,07289	0,07286	0,07311	0,07056
0,5	0,07574	0,07593	0,07590	0,07403
0,6	0,07806	0,07827	0,07808	0,07672
0,7	0,08041	0,08013	0,07977	0,07884
0,8	0,08103	0,08164	0,08225	0,08066
0,9	0,08214	0,08291	0,08338	0,08208
1,0	0,08310	0,08398	0,08403	0,08327
10,0	0,09468	0,09526	0.09457	0,09615
20,0	0,08996	0,08994	0,08996	0,09128
30,0	0,10289	0,10299	0,10289	0,10294
40,0	0,09560	0,09561	0,09559	0,09616
50,0	0,09127	0,09127	0,09125	0,09107

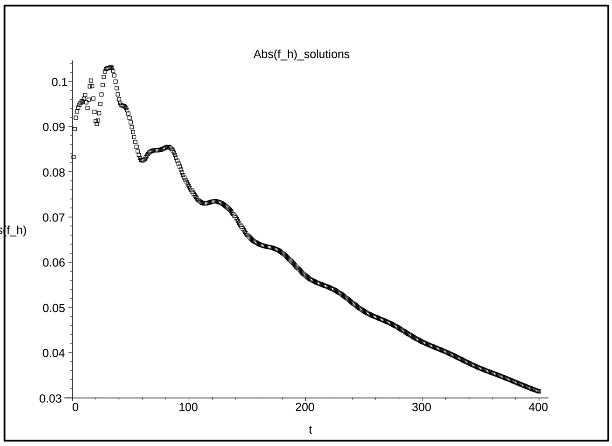

Figure 2. Solution of the grid problem $|f_h|$, $N = 750$, $x = L$.

6. Numerical results and conclusions

Computations were carried out with the following values of the parameters $\delta = 0$, $L = 15$, $\gamma = 2$, $h = 1/10;\ 1/50$, $N = 150;\ 750$, $\tau = 0,1;\ 0,01$, $\sigma = 1$, and $f_0(x) = \sin\left(\frac{\pi x}{L}\right)$. The finite difference scheme was realized by means of the FORTRAN code and analytically by using the expansion (finite series) in the form of a sum. The results coincide up to seven digits. The discrete solutions $|f_h|$ and $|f_{2h}|$ for $x = L$ were compared with the solution $|f|$ of the continuous problem which was obtained from the series at fixed time moments $t \le 50$ ($|f_{2h}|$ is the discrete solution obtained by means of the FORTRAN code and the second order approximation of the boundary condition). In computing the series the terms were summed up to the term whose modulus was smaller than $\varepsilon = 10^{-8}$ (the number of included terms was in all cases smaller than 1000). In Tab. 3 we also present the solution $|f_l|$, which was obtained by means of the analytic expansion of the line method. It does not depend on the temporal step-length τ. It is obvious that the results coincide up to two or three digits. In Fig. 2 we show the numerical solution in the interval $\tau \in (0, 400)$. The solution oscillates up to $t \approx 50$, after which it rapidly approaches zero. Calculations show that reducing the spatial step-length h in the grid method improves the accuracy. For example, if $t = \tau = 0,1$, then

$$|f_h| = \; 0,0480\,(h = 0,1); \;\; 0,0520\,(h = 0,05); \\ 0,0510\,(h = 0,025); \;\; 0,0509(h = 0,02)\,.$$

It follows from the results presented in Tab. 3 that the scheme with the first order approximation of the boundary condition is even more accurate. This is due to the orthogonality of the corresponding eigenfunctions and this fact is important in improving accuracy of numerical methods.

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Girotrono lygties vienos modos skaitiniai sprendiniai

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Straipsnyje nagrinėjami skaitiniai sprendiniai gauti tiriant girotrono lygties vieną modą. Analitiniai ir skaitiniai sprendiniai gauti taikant baigtinių skirtumų metodą. Ištirti kvazistacionarieji sprendiniai ir atitinkamos tokio uždavinio tikrinės reikšmės ir tikrinės funkcijos.