XXV. NETWORK SYNTHESIS

Prof. E. A. Guillemin
R. O. Duda
H. B. Lee
W. C. Schwab

RESEARCH OBJECTIVES

The primary objective as stated last year is still our major one, although some of the results obtained during the past year have so broadened our point of view that a whole new approach to passive bilateral, as well as to active nonbilateral, synthesis has presented itself. Essentially this new attitude approaches the realization problem from a parameter matrix point of view, rather than from the conventional one in which network construction is coupled directly with a mathematical expansion of the rational impedance function. Complete topological generality is one of the features of this new approach, as well as the fact that all linear systems, whether passive or active, bilateral or nonbilateral, are included (1).

Solutions to several subproblems essential to this main topic, solved during the past year, are reported below (Secs. XXV-A and XXV-B), as is also a preliminary study of the workability of the new approach in terms of some familiar problems in passive synthesis (Sec. XXV-C).

Research on a closely related problem dealing with the construction of a network graph from a stated cut-set or tie-set matrix (mentioned in last year's Research Objectives) resulted in a paper (2) that was presented at the International Symposium on Circuit and Information Theory, in Los Angeles, California, June 16-18, 1959.

E. A. Guillemin

References

1. E. A. Guillemin, An approach to the synthesis of linear networks through use of normal coordinate transformations leading to more general topological configurations, a paper to be presented at the IRE National Convention, New York, March 1960, describes this method.

2. E. A. Guillemin, How to grow your own trees from cut-set or tie-set matrices, Trans. IRE, vol. CT-6, pp. 110-126 (May 1959).

A. ON THE ANALYSIS AND SYNTHESIS OF SINGLE-ELEMENT-KIND NETWORKS

This report pertains to networks containing only one kind of element R, L or C. To be specific, we shall consider the element kind to be resistive and regard the branch parameters as given in terms of their conductance values, since the node basis will be chosen to characterize network equilibrium. No loss in generality is thereby implied, it being understood that familiar methods of source transformation and the duality principle are available for generalizing end results in ways that are appropriate to the effective handling of any given physical situation.

1. Determination of the Node Conductance Matrix from Branch Conductances or Vice Versa

The first topic that we shall consider deals with establishing simple ways of relating topology and branch conductance values to the node conductance matrix and vice versa.

The node conductance matrix is the parameter matrix characterizing the equilibrium equations in terms of a chosen set of node-pair voltages. The number of these may be equal to or less than the number of branches in a tree appropriate to the given network graph, since we need not consider all of the independent node pairs to be points of access. If we do consider all of them to be accessible node pairs, then a node conductance matrix of order $\underline{n}$ pertains to a network having a total of n + 1 nodes (for which the number of branches in any tree equals n). We shall begin by considering this special situation because familiarity with it provides an essential background for the more general problem.

If the total number of branches is $\underline{b}$ and the cut-set matrix having n rows and b columns is denoted by a, then the node conductance matrix is given by the familiar expression

$$G = a\, g_b\, a_t \tag{1}$$

in which g_b is the diagonal branch conductance matrix of order $\underline{b}$, and a_t is the transpose of a. Since a can be chosen in a large number of ways for the same given network, characterization of the precise form and properties of G is not simple. However, for a fixed geometrical tree configuration, a is essentially fixed in form, except for a rearrangement of rows (the arrangement of columns being immaterial because it affects only the identities of the diagonal elements in g_b). Hence for a fixed tree geometry, G can vary only in a rearrangement of its rows and columns; and a change in the reference direction for a tree-branch (node-pair) voltage merely causes all element values in a corresponding row and column to change. We shall regard such changes in G as not altering its fundamental form, and hence consider the number of distinct fundamental forms of G as being equal to the number of distinct geometrical tree configurations constructible for a given n (the number of distinct patterns constructible with n match sticks).

Out of the variety of possible tree configurations, two particular ones are of primary interest: the "starlike" tree, in which all branches have the same node in common; and the "linear" tree, in which successive branches have one node in common. The first of these implies a node-to-datum set of node-pair voltages and gives rise to the so-called dominant matrix G in which all nondiagonal terms are negative, the diagonal ones positive, and sums of elements in any row or column are non-negative. As we have mentioned, sign changes resulting from the multiplication of corresponding rows and columns by -1 are disregarded because they can easily be recognized and corrected.

Figure XXV-1 shows a graph for n = 5, in which the branches of the starlike tree are numbered 11, 22, 33, 44, 55. This is a "full" graph, in that branches connect each node with all other nodes, the total number of branches being n(n+1)/2, which is the same as the total number of distinct elements in the symmetrical matrix G of order n.

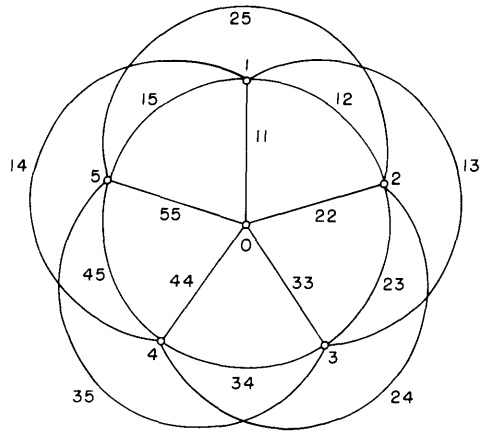

Fig. XXV-1. Branch numbering in a full graph for n = 5 with a starlike tree.

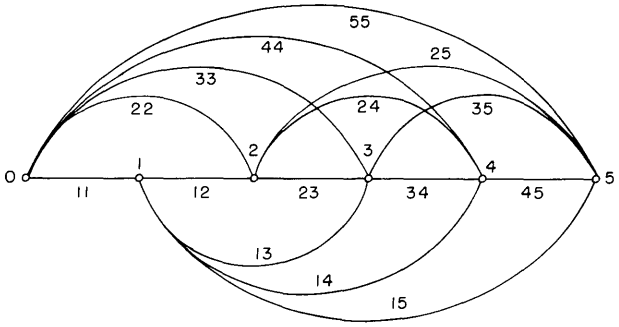

Fig. XXV-2. The graph of Fig. XXV-1 redrawn with the nodes arranged upon a straight line.

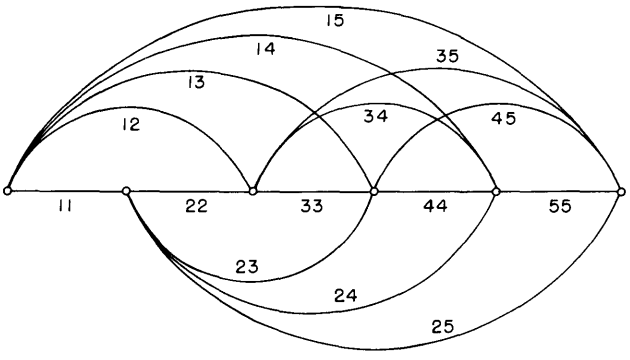

Fig. XXV-3. Branch numbering in a full graph for n = 5 with a linear tree.

Branches connecting nodes $\underline{i}$ and $\underline{k}$ are labeled ik; and we shall denote conductance values of the branches by g_{ik} to correspond to the branch numbering given in this graph.

In Fig. XXV-2 this same graph is redrawn with the nodes arranged upon a straight line so that comparison with the branch numbering for the choice of a linear tree becomes easier. The branch numbering for the linear tree is shown in Fig. XXV-3.

In the node-to-datum arrangement with the starlike tree, the algebraic relations between the branch conductances g'_{ik} and the elements G'_{ik} of the dominant matrix G' are obvious, and are given by

$$g'_{ik} = -G'_{ik} \qquad \text{for } i \neq k \tag{2}$$

and

$$g'_{ii} = \sum_{k=1}^{n} G'_{ik} \quad \text{or} \quad G'_{ii} = \sum_{k=1}^{n} g'_{ik} \tag{3}$$

Here primes are used on these quantities because we want to reserve the same notation without a prime for the corresponding quantities appropriate to a linear tree.

If we denote the node-to-datum voltages by a matrix e', the pertinent equilibrium equations read

$$G' e' = i'_s \tag{4}$$

in which the elements of i'_s are current sources across tree branches. For the linear tree of Fig. XXV-3, the corresponding equilibrium equations are written

$$G e = i_s \tag{5}$$

and if we denote the transformation from one set of node-pair voltages to the other by the matrix equation

$$e = T e' \tag{6}$$

then inspection of Figs. XXV-2 and XXV-3 shows that the transformation matrix T has the form

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & & -1 & 1 \end{bmatrix} \tag{7}$$

with the inverse

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \tag{8}$$

From Eqs. 5 and 6 we obtain

$$T_t\, G\, T\, e' = T_t\, i_s = i'_s \tag{9}$$

(the identification of $T_t\, i_s$ with i'_s being required by the condition of power invariance), and thus by comparison with Eq. 4 we have

$$G' = T_t\, G\, T \tag{10}$$

in which T_t is the transpose of T.

Through the use of Eqs. 7 and 8 we can now find

$$T\, G\, T = T\, T_t^{-1}\, G' = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & -1 & 0 \end{bmatrix} \times G' \tag{11}$$

whereupon relations 2 and 3 yield

$$T\, G\, T = \begin{bmatrix} g'_{11} & g'_{22} & g'_{33} & \cdots & g'_{nn} \\ g'_{12} & g'_{13} & \cdots & & g'_{1n} \\ g'_{23} & \cdots & & & g'_{2n} \\ \cdots & & & & \\ & & & & g'_{n-1,n} \end{bmatrix} \tag{12}$$

in which the cross-hatched region below the principal diagonal contains elements in

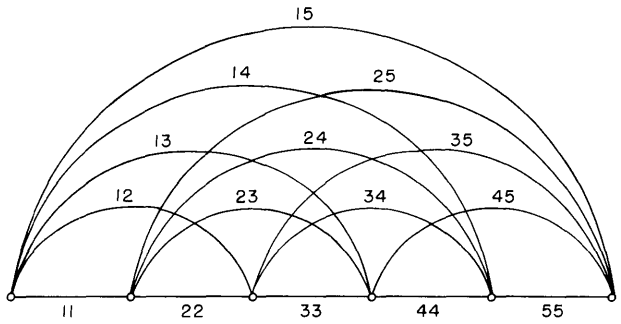

Fig. XXV-4. Redrawn version of the graph of Fig. XXV-3.

which we have no interest. From a comparison of Figs. XXV-2 and XXV-3, noting particularly the differences in notation for the branch conductances, we see that the result just obtained may be written

$$T\,G\,T = \begin{bmatrix} g_{11} & g_{12} & g_{13} & \cdots & g_{1n} \\ & g_{22} & g_{23} & \cdots & g_{2n} \\ & & g_{33} & \cdots & g_{3n} \\ & & & \cdots & \\ & & & & g_{nn} \end{bmatrix} \tag{13}$$

which represents relationships between branch conductances in the graph of Fig. XXV-3 and elements in the node conductance matrix appropriate to a linear tree that are comparable in simplicity with the familiar relations pertinent to a conductance matrix based upon a node-to-datum set of voltage variables. In fact, if we tilt matrix 13 so that its principal diagonal becomes horizontal, and redraw Fig. XXV-3 in the pyramidal form shown in Fig. XXV-4, then the identification of elements in this matrix with pertinent branches in the associated network becomes strikingly evident, as does also the fact that the number of distinct elements in a conductance matrix G of order n exactly equals the number of branches in a full graph with n tree branches.

Recognition of the last relationship, incidentally, makes it clear that we can always obtain branch conductance values appropriate to a given cut-set matrix α and node conductance matrix G, by writing the matrix Eq. 1 in the equivalent algebraic form

$$G_{ik} = \sum_{\nu=1}^{b} \alpha_{i\nu}\, \alpha_{k\nu}\, g_{\nu} \tag{14}$$

(in which g_ν, the branch conductances, are elements in the diagonal matrix g_b) and solving this set of n(n+1)/2 simultaneous equations for the same number of unknown g_ν's. When a is based upon a linear tree, we arrive in this way (after appropriate manipulations) at the same result that is evident in Eq. 13 and in Fig. XXV-4.

This result affords an equally simple numerical procedure for computing G_{ik}'s from g_{ik}'s, or vice versa, since the operation on rows and columns demanded by the transformation matrix T or T^{-1} is so easy to carry out. Suppose we denote matrix 13 by g, and assume, as an example, that

$$g = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ & 1 & 2 & 2 & 1 \\ & & 1 & 2 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \tag{15}$$

Then by inspection

$$T^{-1}\, g = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ & 3 & 5 & 4 & 2 \\ & & 6 & 6 & 3 \\ & & & 7 & 4 \\ & & & & 5 \end{bmatrix} \tag{16}$$

and so we have the conductance matrix

$$G = \begin{bmatrix} 9 & 8 & 6 & 3 & 1 \\ & 14 & 11 & 6 & 2 \\ & & 15 & 9 & 3 \\ & & & 11 & 4 \\ & & & & 5 \end{bmatrix} \tag{17}$$

In Eqs. 15 and 16 the elements below the principal diagonal are of no interest and are not involved in the manipulations. In Eq. 17 they are of interest, of course, and can readily be inserted, since we know that G is symmetrical.

Proceeding in the opposite direction, we obtain from Eq. 17

$$T\, G = \begin{bmatrix} 9 & 8 & 6 & 3 & 1 \\ & 6 & 5 & 3 & 1 \\ & & 4 & 3 & 1 \\ & & & 2 & 1 \\ & & & & 1 \end{bmatrix} \tag{18}$$

and so

$$T\,G\,T = g = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ & 1 & 2 & 2 & 1 \\ & & 1 & 2 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \tag{19}$$

which is what we started out with.

In going from Eq. 15 to Eq. 16 we write down the first row in Eq. 15, then add to this one the second row in Eq. 15 to form the second row in Eq. 16, then add to this one the third row in Eq. 15 to form the third in Eq. 16, and so forth. Having formed Eq. 16, we construct Eq. 17 by columns, starting by writing down the fifth column in Eq. 16, then add to this one the fourth column in Eq. 16 to form the fourth in Eq. 17, and so forth, as was done with the rows.

In the opposite direction, starting with Eq. 17, the first row is the same as in Eq. 18. The second row in Eq. 18 is the second minus the first in Eq. 17 (ignoring the absent term); the third row in Eq. 18 is the third minus the second in Eq. 17; the fourth is the fourth minus the third in Eq. 17, and so forth. Analogous operations on columns, starting with the fifth and working towards the first, yield the transformation from Eq. 18 to Eq. 19. With a little practice, these operations are very simple to carry out. Only addition and subtraction are involved.

The process is about as easy to perform, and the conditions leading to positive elements in g are almost as easily recognizable, as are those pertaining to a dominant G-matrix appropriate to a graph with starlike tree. These conditions on a G-matrix appropriate to a linear tree will be implied by designating G as a "uniformly tapered" matrix, the reason for the choice of this term being evident in the numerical example just given.

2. Response Functions Directly Related to Branch Conductances

In an analysis problem, having formed the matrix G for a given graph and its branch conductance values, we are next interested in evaluating elements in the inverse of G — the open-circuit driving-point and transfer impedances for the chosen terminal pairs. The following manipulations are aimed at devising a computational scheme whereby these impedances can be obtained from the branch conductance values with a minimum number of additions, multiplications, and divisions.

We begin by writing for the symmetrical matrix G the representation

$$G = A \times A_t \tag{20}$$

and assume for A the triangular form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \tag{21}$$

Since a triangular matrix is easy to invert, the inverse of G, which is given by

$$G^{-1} = A_t^{-1} \times A^{-1} \tag{22}$$

is likewise more easily obtainable from representation 20, inasmuch as the elements in A are readily computed. (Note that G is the Grammian matrix formed from the rows of A.) Thus the elements in A are found by a simple recursion process, and so are those in the inverse of A, which again is triangular and has the same form as A.

Instead of pursuing the details of this process, however, we wish to show how we can tie the branch conductance values into this representation so that, in an analysis problem, we can omit the formation of the matrix G altogether and proceed directly with the formation of G^{-1}. Thus, with Eqs. 13 and 20 in mind, and the specific form of T given by Eq. 7 before us, we construct the products

$$T \times A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ (a_{21}-a_{11}) & a_{22} & 0 & \cdots & 0 \\ (a_{31}-a_{21}) & (a_{32}-a_{22}) & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (a_{n1}-a_{n-1,1}) & (a_{n2}-a_{n-1,2}) & (a_{n3}-a_{n-1,3}) & \cdots & a_{nn} \end{bmatrix} \tag{23}$$

and

$$A_t \times T = \begin{bmatrix} (a_{11}-a_{21}) & (a_{21}-a_{31}) & (a_{31}-a_{41}) & \cdots & (a_{n-1,1}-a_{n1}) & a_{n1} \\ -a_{22} & (a_{22}-a_{32}) & (a_{32}-a_{42}) & \cdots & (a_{n-1,2}-a_{n2}) & a_{n2} \\ 0 & -a_{33} & (a_{33}-a_{34}) & \cdots & (a_{n-1,3}-a_{n3}) & a_{n3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n}-a_{nn}) & a_{nn} \end{bmatrix} \tag{24}$$

The second of these matrices is almost the negative transpose of the first. In fact, if we add to the first matrix a last row with the elements $-a_{n1}, -a_{n2}, \ldots -a_{nn}$, and then ignore the first row, its negative transpose is the second matrix. This fact suggests that we consider matrix 23 with the stated additional row, namely, the matrix with n + 1 rows and n columns given by

$$H = \begin{bmatrix} a_{11} & 0 & 0 & \cdot & 0 \\ (a_{21}-a_{11}) & a_{22} & 0 & \cdot & 0 \\ (a_{31}-a_{21}) & (a_{32}-a_{22}) & a_{33} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (a_{n1}-a_{n-1,1}) & (a_{n2}-a_{n-1,2}) & (a_{n3}-a_{n-1,3}) & \cdot & a_{nn} \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdot & -a_{nn} \end{bmatrix} \tag{25}$$

In this matrix all columns add to zero; and if the vector set defined by rows is denoted by $h_o, h_1, h_2, \ldots h_n$, then these vectors form the sides of a closed polygon in n-dimensional space. The branch conductance values in matrix 13 are seen to be given by the scalar products

$$g_{ik} = -h_{i-1} \cdot h_k \qquad \text{for } i \leqslant k \leqslant n \tag{26}$$

and i = 1, 2, ... n. More specifically, if we designate the elements in H as indicated in the matrix

$$H = \begin{bmatrix} h_{01} & 0 & 0 & \cdot & 0 \\ h_{11} & h_{12} & 0 & \cdot & 0 \\ h_{21} & h_{22} & h_{23} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & h_{n-1,n} \\ h_{n1} & h_{n2} & h_{n3} & \cdot & h_{nn} \end{bmatrix} \tag{27}$$

then, for the g_{ik} expressions, we obtain

$$g_{ik} = -\sum_{\nu=1}^{i} h_{i-1,\nu} h_{k\nu} \qquad \text{for } k \geqslant i \tag{28}$$

Since we are interested in computing elements in the matrix H from g_{ik} values in a given network (because the elements of A and of its inverse are then readily obtainable) we manipulate Eq. 28 as follows. First, we split off the last term in the sum and have

$$g_{ik} = -\sum_{\nu=1}^{i-1} h_{i-1,\nu}h_{k\nu} - h_{i-1,i}h_{ki} \tag{29}$$

or

$$-h_{i-1,i}h_{ki} = g_{ik} + \sum_{\nu=1}^{i-1} h_{i-1,\nu}h_{k\nu} \tag{30}$$

For a particular column of H (fixed value of i) it will be noticed that the right-hand side of this equation involves only coefficients h in preceding columns (up to and including column i-1). Formula 30 would thus be suited for the sequential computation of the h-coefficients if it were not for the factor $h_{i-1,i}$ on the left. This awkwardness can be removed by introducing the quantities

$$p_{ki} = -h_{i-1,i}h_{ki} = g_{ik} + \sum_{\nu=1}^{i-1} h_{i-1,\nu}h_{k\nu} \tag{31}$$

Since the columns of H add to zero, we have

$$\sum_{k=i-1}^{n} h_{ki} = 0, \quad \text{or} \quad h_{i-1,i} = -\sum_{k=i}^{n} h_{ki} \tag{32}$$

and thus

$$\sum_{k=i}^{n} p_{ki} = -h_{i-1,i}\sum_{k=i}^{n} h_{ki} = h_{i-1,i}^2 \tag{33}$$

From Eq. 31,

$$h_{ki} = \frac{p_{ki}}{-h_{i-1,i}} \tag{34}$$

If we use relation 34 and Eq. 33 to form

$$h_{i-1,\nu}h_{k\nu} = \frac{p_{i-1,\nu}p_{k\nu}}{h_{\nu-1,\nu}^2} = \frac{p_{i-1,\nu}p_{k\nu}}{\sum_{k=\nu}^{n} p_{k\nu}} \tag{35}$$

substitution in Eq. 31 yields the result

$$p_{ki} = g_{ik} + \sum_{\nu=1}^{i-1} \left(\frac{p_{i-1,\nu} p_{k\nu}}{\sum_{k=\nu}^{n} p_{k\nu}} \right); \quad (k \geq i = 1, 2, \ldots n) \tag{36}$$

Note that for i = 1 the sum drops out, and we have simply $p_{k1} = g_{1k}$. The matrix with coefficients p_{ki} of the form

$$P = \begin{bmatrix} p_{11} & 0 & 0 & \cdot & 0 \\ p_{21} & p_{22} & 0 & \cdot & 0 \\ p_{31} & p_{32} & p_{33} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \vdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & 0 \\ p_{n1} & p_{n2} & p_{n3} & \cdot & p_{nn} \end{bmatrix} \tag{37}$$

can be calculated sequentially, starting with the elements in the first column, then those in the second, and so forth, since formula 36 for any fixed value of the index i involves only elements in the columns 1 to i - 1. The sum in the denominator of the summand in Eq. 36 is simply the sum of all elements in the ν^{th} column. Hence, in the computational procedure, each time the elements in an additional column are calculated, their sum may also be recorded below it, so that the value of this sum is readily available for computation of the next column.

The elements of matrix A, Eq. 21, can now be expressed directly in terms of the p_{ki}. From the form of matrix H, Eq. 25, and the notation in Eq. 27, we have

$$a_{ik} = \sum_{\nu=k-1}^{i-1} h_{\nu k} = h_{k-1,k} + \sum_{\nu=k}^{i-1} h_{\nu k} = \sum_{\nu=k}^{i-1} h_{\nu k} - \sum_{\nu=k}^{n} h_{\nu k} = -\sum_{\nu=i}^{n} h_{\nu k} \tag{38}$$

in which the relation in Eq. 32 is used. Substituting for $h_{\nu k}$ from Eq. 34, with assistance from Eq. 33, we have

$$a_{ik} = \frac{\sum_{\nu=i}^{n} p_{\nu k}}{h_{k-1,k}} = \frac{\sum_{\nu=i}^{n} p_{\nu k}}{\left(\sum_{\nu=k}^{n} p_{\nu k} \right)^{1/2}} \qquad \text{for } i \geq k = 1, 2, \ldots n \tag{39}$$

Appearance of the square root in this expression does not contradict the well-known requirement that response in a lumped network be a rational function of the branch conductances, since the impedances that we shall presently compute are quadratic functions of the a_{ik}, as is also evident from Eq. 22. For this reason, it is advisable not to compute

the a_{ik}'s until an evaluation of the desired response function in terms of these coefficients is made and the radicals are eliminated.

If the inverse of matrix A is denoted by

$$A^{-1} = B = \begin{bmatrix} b_{11} & 0 & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & 0 & \cdots & 0 \\ b_{31} & b_{32} & b_{33} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & b_{n3} & & \cdots & b_{nn} \end{bmatrix} \tag{40}$$

then we have the following relations between the a_{ik} and b_{ik}:

$$a_{11}b_{11} = 1 \tag{41}$$

$$\left.\begin{aligned} a_{11}b_{21} + a_{21}b_{22} &= 0 \\ a_{22}b_{22} &= 1 \end{aligned}\right\} \tag{42}$$

$$\left.\begin{aligned} a_{11}b_{31} + a_{21}b_{32} + a_{31}b_{33} &= 0 \\ a_{22}b_{32} + a_{32}b_{33} &= 0 \\ a_{33}b_{33} &= 1 \end{aligned}\right\} \tag{43}$$

and so forth. Or, starting at the opposite end, we have

$$a_{nn}b_{nn} = 1 \tag{44}$$

$$\left.\begin{aligned} a_{n-1,n-1}b_{n,n-1} + a_{n,n-1}b_{nn} &= 0 \\ a_{n-1,n-1}b_{n-1,n-1} &= 1 \end{aligned}\right\} \tag{45}$$

$$\left.\begin{aligned} a_{n-2,n-2}b_{n,n-2} + a_{n-1,n-2}b_{n,n-1} + a_{n,n-2}b_{nn} &= 0 \\ a_{n-2,n-2}b_{n-1,n-2} + a_{n-1,n-2}b_{n-1,n-1} &= 0 \\ a_{n-2,n-2}b_{n-2,n-2} &= 1 \end{aligned}\right\} \tag{46}$$

and so forth.

The open-circuit resistance matrix, which is the inverse of G, is given by

$$R = G^{-1} = B_t \times B \tag{47}$$

The elements of R in which we are particularly interested are

$$r_{nn} = b_{nn}^2 \tag{48}$$

$$r_{n-1,n} = b_{n,n-1} b_{nn} \tag{49}$$

$$r_{n-2,n} = b_{n,n-2} b_{nn} \tag{50}$$

The first of these is a driving-point function; the other two are transfer functions. Because of the implied linear tree upon which the terminal pairs are based, we see more particularly that Eq. 49 is the open-circuit transfer impedance of a grounded two terminal-pair (three-terminal network), while Eq. 50 is the open-circuit transfer impedance of an arbitrary two terminal-pair network because the two terminal-pairs involved are not adjacent, and hence do not have a terminal in common.

From Eqs. 44, 48, and 39 we have for the driving-point impedance (which, incidentally, can through appropriate branch numbering be the impedance across any chosen branch in the given network), the surprisingly simple result

$$r_{nn} = \frac{1}{a_{nn}^2} = \frac{1}{p_{nn}} \tag{51}$$

For the transfer function, Eq. 49, we find straightforwardly

$$\frac{r_{n-1,n}}{r_{nn}} = -\frac{a_{n,n-1}}{a_{n-1,n-1}} = \frac{-p_{n,n-1}}{(p_{n-1,n-1}+p_{n,n-1})} \tag{52}$$

and for the transfer function given by Eq. 50, we obtain

$$\frac{r_{n-2,n}}{r_{nn}} = \frac{a_{n,n-1}a_{n-1,n-2} - a_{n,n-2}a_{n-1,n-1}}{a_{n-1,n-1}a_{n-2,n-2}}$$

$$= \frac{p_{n,n-1}p_{n-1,n-2} - p_{n,n-2}p_{n-1,n-1}}{(p_{n-1,n-1}+p_{n,n-1})(p_{n-2,n-2}+p_{n-1,n-2}+p_{n,n-2})} \tag{53}$$

Regarding numerical computation of these quantities, the determination of all elements in matrix P, Eq. 37, by formula 36 is found to involve

$$(n-1)\cdot 1 + (n-2)\cdot 2 + (n-3)\cdot 3 + \ldots + 1\cdot(n-1) = \sum_{x=1,2\ldots}^{n-1} \frac{x(x+1)}{2} = \frac{n(n-1)(n+1)}{6} \tag{54}$$

multiplications, the same number of divisions, and

$$\frac{n(n-1)(n+1)}{6} + \frac{n(n-1)}{2} = \frac{n(n-1)(n+4)}{6} \tag{55}$$

additions, or a total number of operations

$$\frac{n(n-1)(n+2)}{2} \tag{56}$$

The number of additional operations involved in the computation of driving-point or transfer functions 48, 49 or 50 is small, and is evident from relations 51, 52, and 53 in which it should be noted that the sums appearing in the denominators of the last two of these are already available and do not require further addition. Thus computation of r_{nn} requires one additional division; $r_{n-1,n}$ requires one additional multiplication and one division; and $r_{n-2,n}$ requires four additional multiplications, a subtraction and one division.

This method for the calculation of network response thus appears to be computationally more economical than any other known method, especially the recently revived and much talked of Kirchhoff combinatorial method and trivial variations thereof. The common denominator in the expressions obtained by the latter method has as many terms as the graph has enumerable trees (each term being the product of n branches). Thus in a full graph with n = 3 there are 16 trees and this denominator alone involves 15 additions plus 32 multiplications, while our formula 56 yields 15 for all operations (additions, multiplications, and divisions).

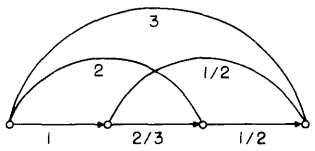

Fig. XXV-5. A full graph for n = 3 according to the pattern set in Fig. XXV-3. Branch numbers are conductance values in mhos.

As a simple example, let us take the graph of Fig. XXV-5 for which n = 3. Numbers on the branches are conductance values in mhos, and the branch numbering is understood to follow the pattern set in Fig. XXV-4. Let the problem be to find the input impedance across the 1/2-mho tree-branch number 3 (which by rearrangement could be any other branch) and the open-circuit transfer impedances between this branch and the other two tree branches (which can also be any other two branches). Construction of the P-matrix according to formula 36 takes the form indicated in the schedule

$$\begin{array}{lll} 1 & & \\ 2 & \frac{2}{3} + \frac{1}{3} = 1 & \\ 3 & \frac{1}{2} + \frac{1}{2} = 1 \quad \frac{1}{2} + 1 + \frac{1}{2} = 2 & (57) \\ \hline 6 & & 2 \end{array}$$

from which Eqs. 51, 52, and 53 yield

$$r_{33} = 1/2 \text{ ohm}$$

$$2r_{23} = -1/2 \text{ ohm}$$

$$2r_{13} = \frac{2 - 3}{2 \cdot 6} = -1/12 \text{ ohm}$$

The minus signs in the last two results arise from the implied reference directions for the tree branches as shown in Fig. XXV-5.

3. Realization of an n^{th}-order Matrix G by an n + 1-node Network

If a given node conductance matrix G is known to be appropriate to a starlike tree, then the necessary and sufficient realizability conditions (in the form of an n + 1-node network) are simply that it be a dominant matrix; and if it is known to be based upon a linear tree, then the uniformly tapered property described above is necessary and sufficient for its realization. For any given order n there is a finite number of distinct geometrical tree configurations; and the given G matrix must be based upon one of these and satisfy pertinent realizability conditions if a corresponding network with all positive elements is to exist.

If we know the geometrical tree configuration upon which the matrix G is based, then we can readily write the appropriate transformation matrix T connecting the node-pair voltages in that tree to those in a starlike or a linear tree and, by means of a congruent transformation like that expressed in Eq. 10, transform G to a form appropriate to either of these basic tree configurations, whereupon the question regarding its realizability is readily answered, and, if it is answered in the affirmative, the corresponding network is constructible straightforwardly.

The problem of testing and realizing a given G matrix by an n + 1-node network, therefore, will be solved if we can devise a method for discovering the geometrical tree configuration upon which that G matrix is based. This we shall now proceed to do; and we will see, incidentally, that the pertinent tree configuration is unique if G is nondegenerate, in the sense that all of its elements are nonzero. Thus our method, as we shall see, is based upon the recognition that there exists a one-to-one correspondence

(properly interpreted, of course) between the pertinent tree configuration and the algebraic sign distribution among elements in the G matrix. This algebraic sign pattern enables us to recognize the geometrical configuration of the tree upon which G is based, and G must be realizable with that tree configuration if it is realizable at all.

Obviously, if G contains zeros, there is an ambiguity in the algebraic sign pattern, and the possibility exists that more than one tree may be appropriate. Although we can still apply the method to obtain systematically all appropriate trees in a situation of this sort, the process loses its compactness, and therefore we shall assume, for the time being, that G has no zero elements.

The easiest way to see that a definite algebraic sign pattern among the elements in G is linked with a given geometrical tree configuration, is through physical, rather than analytical, reasoning. Since G is a short-circuit driving-point and transfer matrix, we visualize all n terminal pairs across tree branches provided with short-circuiting links and remind ourselves that the values of currents in these links, per volt of ideal voltage source in one of them, equal numerically the elements in G inclusive of their algebraic signs relative to chosen reference arrows on the tree branches.

Thus, with the unit voltage source in the short-circuiting link across tree branch No. 1, the currents in the various links numerically equal the elements in row 1 of G. With the unit voltage source in the link across tree branch No. 2, the currents in the n short-circuiting links, in value, equal the elements in row 2 of G, and so on. The voltage rise of the source is made to coincide in direction with the reference arrow on the tree branch in parallel with it, and any resulting current is positive if its direction in the pertinent short-circuiting link agrees with the reference arrow on the tree branch alongside it, and it is a negative current if its direction is opposite to this reference arrow.

As an example, Fig. XXV-6 shows a tree for n = 5 with short-circuiting links across its branches and a voltage source applied to terminal pair No. 1. If we visualize the presence of the rest of the branches associated with this tree in a full graph, it is clear, by inspection, that the various short-circuit currents have directions as indicated by

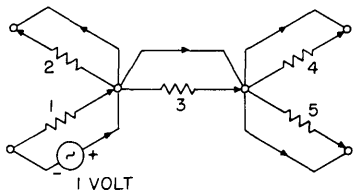

Fig. XXV-6. A geometrical tree configuration for a 6-node network showing short-circuiting links across node pairs and reference arrows for establishing the algebraic sign pattern of the pertinent conductance matrix.

arrows in the short-circuiting links; and thus we see that all elements in the first row of a G matrix appropriate to this tree (with the assumed reference arrows) are positive.

If we shift the voltage sources into the short-circuiting link across branch 2 with its polarity in the same relation to the reference arrow on that branch as is shown for branch No. 1 in Fig. XXV-6, then it is equally simple to see, by inspection, that the resulting currents for branches 1 and 2 are in positive directions, while those in all the other short-circuiting links are negative. In the second row of G, therefore, the first two elements are positive and the rest are negative. Thus it is a simple matter to establish all algebraic signs for the elements of G; and we become convinced, incidentally, that this sign pattern has nothing to do with element values but is uniquely fixed by the geometrical tree configuration, except for interchanging rows and columns (renumbering of tree branches), and the multiplication of rows and columns by minus signs (changing reference arrows on the tree branches). These, as we shall see presently, are trivial operations, as far as recognition of the pertinent tree configuration is concerned.

In order to facilitate the correlation of algebraic sign patterns with geometrical tree configurations, we observe that we can dispense with drawing the tree branches in sketches like the one in Fig. XXV-6. It suffices to draw lines for branches, as we are accustomed to do in network graphs, and to regard these as the short-circuiting links, their reference arrows being included in the usual manner. A voltage source, as well as all other branches in the full graph, can easily be imagined to be in their proper places and the directions of pertinent currents can, with a little practice,

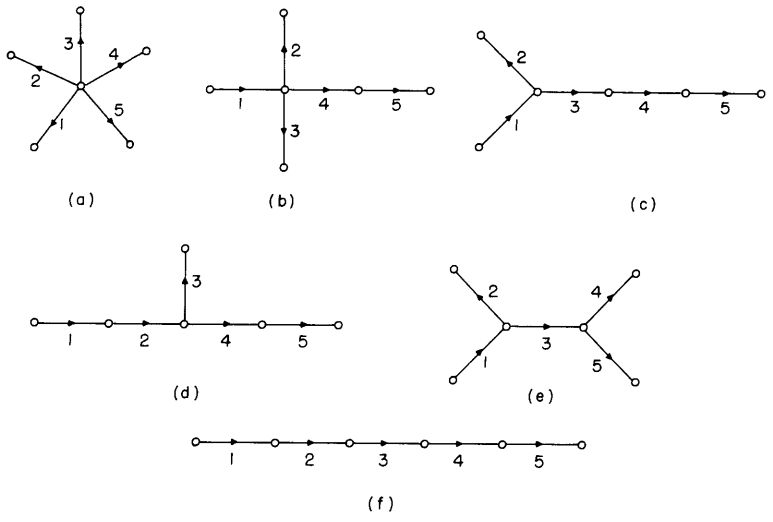

Fig. XXV-7. Complete set of geometrical tree patterns that are possible with a 6-node network.

be deduced by inspection.

In Fig. XXV-7 all of the six tree configurations for n = 5 are drawn. We can imagine the branches as being water pipes. If in the starlike tree, Fig. XXV-7a, we squirt water into pipe No. 1 in its reference direction, it will obviously flow in the positive reference directions in all of the four other pipes. Hence in the first row of G all elements are plus. If we squirt water in the reference direction through pipe No. 2, it flows positively through pipe No. 1, and negatively through the other three. Thus in the second row of G the first two elements are plus and the rest are minus. This homely analogy makes the process of establishing sign patterns for all trees fast and effortless. The resulting sign patterns thus obtained are indicated as follows by what we might call "sign matrices"

$$S_a = \begin{bmatrix} + & + & + & + & + \\ & + & - & - & - \\ & & + & - & - \\ & & & + & - \\ & & & & + \end{bmatrix}; \quad S_b = \begin{bmatrix} + & + & + & + & + \\ & + & - & - & - \\ & & + & - & - \\ & & & + & + \\ & & & & + \end{bmatrix}; \quad S_c = \begin{bmatrix} + & + & + & + & + \\ & + & - & - & - \\ & & + & + & + \\ & & & + & + \\ & & & & + \end{bmatrix} \tag{59}$$

$$S_d = \begin{bmatrix} + & + & + & + & + \\ & + & + & + & + \\ & & + & - & - \\ & & & + & + \\ & & & & + \end{bmatrix}; \quad S_e = \begin{bmatrix} + & + & + & + & + \\ & + & - & - & - \\ & & + & + & + \\ & & & + & - \\ & & & & + \end{bmatrix}; \quad S_f = \begin{bmatrix} + & + & + & + & + \\ & + & + & + & + \\ & & + & + & + \\ & & & + & + \\ & & & & + \end{bmatrix} \tag{60}$$

Since these matrices are symmetrical, we need only record signs above the principal diagonal.

Signs on the principal diagonal are always plus for obvious reasons. Observe, also, that we have chosen reference arrows for the tree branches in such a way that the first row is always a row of plus signs. In an arbitrarily given G matrix this state of affairs need not be fulfilled, but we can always multiply rows (and corresponding columns) by minus signs to fulfill this condition, and thus convert the given matrix to a sort of normal or basic form with regard to its algebraic sign pattern. This step eliminates once and for all those trivial variants of the given matrix G which might stem from sign changes of its elements in rows (and in corresponding columns). We may regard the

process of making all signs in the first row positive as one of reducing the reference arrows on the branches of the implied tree to a common basic pattern.

Except for row and column interchanges (renumbering of tree branches), each sign matrix uniquely specifies a geometrical tree pattern and vice versa; and we shall presently describe a simple way of constructing the tree from a given sign matrix. In the meantime, it is interesting to observe that the linear tree is the only one for which all signs in the G matrix are positive. If a given G matrix has all positive elements (when its sign pattern is normalized as just described), then it must be realizable with a linear tree if it is realizable at all; that is to say, it must be a uniformly tapered matrix or else it has no realization in an n + 1-node network.

4. How to Grow a Tree from a Given Sign Matrix

The procedure for constructing a tree from a given sign matrix is similar in some respects to a method already developed for determining the graph pertinent to a given cut-set matrix (1). There, however, the growth pattern for the tree must first be established, while in the present situation we can proceed at once with the growth process for the pertinent tree, since any given sign matrix, in contrast with a cut-set matrix, may always be regarded as having an appropriately "ordered form" to begin with.

Since the sign matrices 59 and 60 for the tree configurations of Fig. XXV-7 are too simple to represent worth-while examples, and, moreover, are based upon a particular branch-numbering sequence that need not be fulfilled in an arbitrarily given situation, we choose to illustrate the method of tree construction proposed here by the following more elaborate sign matrix

$$S = \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} + & + & + & + & + & + & + & + & + \\ & + & + & + & + & + & + & - & + \\ & & + & - & - & + & - & + & + \\ & & & + & + & - & + & + & + \\ & & & & + & - & - & + & + \\ & & & & & + & - & + & + \\ & & & & & & + & + & + \\ & & & & & & & + & - \\ & & & & & & & & + \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \qquad (61)$$

Numbering of the rows and columns is done to facilitate identification with

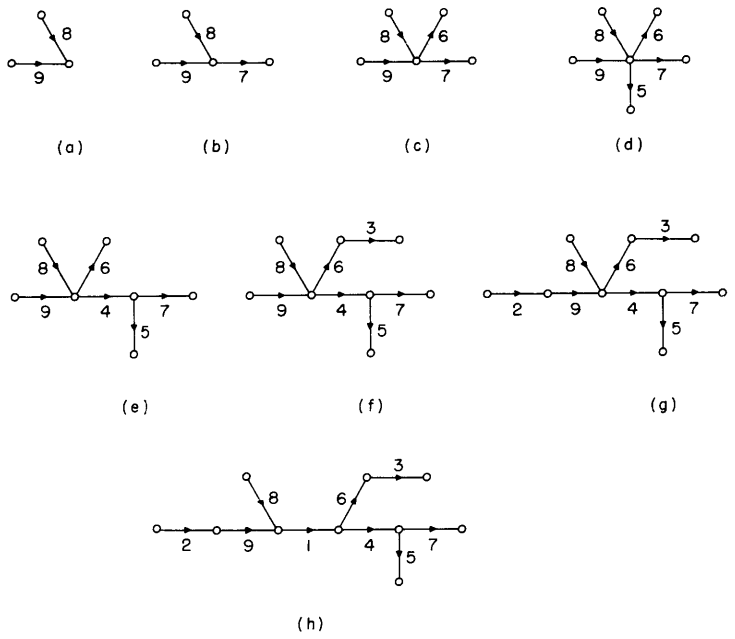

Fig. XXV-8. Growth of a tree according to the sign matrix given in Eq. 61.

correspondingly numbered tree branches.

Construction of the tree is accomplished by starting with the last branch, No. 9, and successively adding branches 8, 7, 6, ...; we are guided in assigning their relative positions by the confluence or counterfluence of reference arrows as demanded by the signs in the respective rows and columns of matrix S. Thus branches 8 and 9 must be counterfluent, while branch 7 is confluent with both 8 and 9. This state of affairs can be satisfied only by having branches 7, 8, and 9 meet in a common point as shown in Fig. XXV-8, in which the various sketches (Fig. XXV-8a-8h) show the growth of the tree, branch by branch, the position of each added branch being uniquely determined (except for trivial variants) by the signs in the pertinent row of S.

Construction of the tree is thus straightforward and always possible, unless a contradiction is encountered, in which case no tree exists and the given G matrix has no realization.

5. Final Realization Procedure for the Matrix G

Once the tree for a given G matrix has been found, it is a simple matter to write the transformation matrix T which, in a congruent transformation, carries G over into a form appropriate either to a starlike tree (the dominant form) or to a linear tree (the uniformly tapered form). In either case, the conditions for its realization present no further problem. Finally, the terminal pairs appropriate to the given matrix can

readily be determined from the geometrical picture upon which construction of the matrix T is based.

Observe that the existence of a tree appropriate to the given G matrix is not sufficient to ensure its realization. Observe, also, that although we can construct many different T matrices connecting the tree of the given G matrix with a starlike or a linear tree, it is sufficient to try only one, for if this one fails to yield a realizable G matrix (either dominant or uniformly tapered), then no other transformation T can do so, since a contrary assumption leads to a contradiction. Hence the realization procedure discussed here does not require repeated trials; one straightforward attack tells the whole story.

6. Concluding Remarks

When the given G matrix contains zero elements (2), then the sign matrix correspondingly contains blank spaces that may be interpreted either as plus or minus signs. If, in the construction of the tree in the manner just described, either sign is admissible for the insertion of a particular tree branch, then a variant in the tree's tentative configuration is possible. If such a variant is nontrivial, and if nontrivial variants occur again in subsequent steps of the tree-growth process, then more than one geometrical tree pattern can be associated with the given G matrix. However, it need not follow that the realizability conditions appropriate to these various trees are fulfilled for the given element values in G.

Although several possibilities need now to be investigated, their number is greatly reduced from the totality of algebraic sign arrangements that are possible on a purely combinatorial basis by reason of the step-by-step nature of the tree-growing process, which enables one by inspection to rule out the majority of trials and hence keep the procedure well within reasonable bounds even in the consideration of degenerate cases.

Collaterally, it may be interesting to point out that the present discussion offers an alternative method for the construction of trees (and hence graphs) from given cut-set matrices. Forming the Grammian from the rows of α yields a G matrix appropriate to a network with all 1-ohm branches. From its sign matrix the pertinent tree can be constructed (if one exists); and situations for which other methods fail [see, in particular, the one pertinent to a graph published elsewhere (3)] do not necessarily lead to a degenerate G matrix, and hence are strictly routine when handled by the method presented here.

At this point we need to remind ourselves that if a given G matrix is not realizable by an n + 1-node network, it may still be realizable by a 2n-node network in which the terminal pairs to which G pertains can be nonadjacent, as they manifestly must be in the n + 1-node graph in which each tree branch presents a terminal pair. In a 2n-node full graph one can always choose a linear tree, and if we identify alternate branches as

terminal pairs, we have a topological situation that is completely general, as far as the realizability of G is concerned. In other words, a linear tree in a 2n-node full graph with every other tree branch yielding an accessible terminal pair is the most general topological structure upon which the realization of G can be based. Additional nodes beyond the number 2n can do no good because any G realizable with more than 2n nodes is also realizable with 2n nodes, since the familiar star-mesh transformation process eliminates the extra nodes and leaves all elements positive.

Before the realization techniques discussed here can be applied to this more general situation, however, one must augment the given $n \times n$ G matrix to one having $2n - 1$ rows and columns. This augmentation process must, of course, fulfill the condition that subsequent abridgement to the accessible terminal pairs regains the originally given $n \times n$ G matrix.

Such an augmentation process can be formulated in simple terms and, as might be expected, it is not unique. Herein lies the greater realization capability of the 2n-node network over the n + 1-node network, for one can use the freedom inherent in the nonuniqueness of the augmentation process to help fulfill realization conditions for the implied 2n-node graph with linear tree.

The details of this process, which presents some difficulties not yet fully resolved, will be presented here in the near future.

E. A. Guillemin

References

1. E. A. Guillemin, How to grow your own trees from cut-set or tie-set matrices, Trans. IRE, vol. CT-6, pp. 110-126 (May 1959).

2. We omit the consideration of situations in which the network consists of more than one separate part, the test for which is described in reference 1.

3. E. A. Guillemin, op. cit., p. 126, see Fig. 8.

B. THE NORMAL-COORDINATE TRANSFORMATION OF A LINEAR SYSTEM WITH AN ARBITRARY LOSS FUNCTION

The normal-coordinate transformation in classical dynamics is well known to be restricted to conservative systems; that is to say, systems involving potential and kinetic energy but no loss. This situation is easily understood if we remind ourselves that a nonsingular congruent transformation of the pertinent variables can simultaneously reduce no more than two quadratic forms to sums of squares, and this only if at least one of them is nonsingular and positive (or negative) definite.

Trivial exceptions occur if a third quadratic form (which in a dynamic system may be the loss function) is linearly dependent upon the other two, or if the principal axes

of its associated quadric surface are coincident with those of one of the two other forms. It is likewise obvious that the normal-coordinate transformation is possible for systems in which the two quadratic forms involved are potential energy and loss, or kinetic energy and loss (as in the analogous electric RC or RL networks). However, it has not been shown, thus far, how a normal-coordinate transformation can generally be accomplished for a dynamic system with arbitrary loss. This is our objective in the present report.

1. Equilibrium Equations in "Mixed" Form and Their Transformation to Normal Coordinates

Since it is mathematically impossible to reduce simultaneously more than two quadratic forms to sums of squares, it is clear that, to accomplish our objective, we must find a way to represent the equilibrium of an arbitrary dynamic system with loss in terms of only two quadratic forms. Through a proper choice of variables this objective can be achieved; and it turns out that one of the quadratic forms represents all of the stored energy (kinetic and potential) while the other one is the total instantaneous rate of energy dissipation. Although in the following detailed discussion we shall use the linear passive electrical circuit as a vehicle for expression of pertinent relationships, it is clear that the results apply just as well to any analogous linear system.

In the usual approach, equilibrium conditions for an electric network are expressed either in terms of loop current or node-pair voltage variables. To achieve our objective, we must use both loop currents and node-pair voltages as variables. Although there is nothing new about establishing equilibrium equations on such a "mixed" basis, our particular approach is chosen in a special manner that is completely general and also particularly suited to accommodating a commonly encountered practical situation.

We begin by defining the branches of our network as consisting of series combinations of R and L, or parallel combinations of G and C. Letting the differential operator d/dt be denoted p, the self and mutual inductances of branches by ℓ_{ks}, branch resistances by r_k, capacitances by c_k, and conductances by g_k, we define the branch operators

$$z_{ks} = \ell_{ks}\, p + r_k \tag{1}$$

and

$$y_k = c_k\, p + g_k \tag{2}$$

Pure resistive branches involve degenerate forms of either of these; therefore no loss in generality is involved by this special branch designation, and thus an important practical situation is more readily accommodated.

Following a common practice (1), we number the RL branches consecutively from

1 to λ, and the GC branches from $\lambda + 1$ to $\lambda + \sigma$. Voltage drops in the RL branches are elements in a column matrix v_λ, and currents in the GC branches are those in the column matrix j_σ. Then, denoting a matrix of order λ with the elements z_{ks} as z_λ, and a diagonal matrix of order σ with the elements y_k as y_σ, we have

$$v_\lambda = z_\lambda\, j_\lambda \text{ and } j_\sigma = y_\sigma\, v_\sigma \tag{3}$$

in which j_λ is a column matrix representing currents in the RL branches, and v_σ is one representing voltage drops across the GC branches.

The columns of the tie-set matrix β and of the cut-set (2) matrix a are partitioned into groups of λ and σ, and the resulting submatrices identified by subscripts that indicate their row and column structure in the usual manner. Voltage sources acting around loops and current sources acting across node pairs are elements of column matrices e_s and i_s, respectively. The Kirchhoff voltage and current laws are then expressed by the matrix equations

$$\beta_{\ell\lambda}\, v_\lambda + \beta_{\ell\sigma}\, v_\sigma = e_s \tag{4}$$

and

$$a_{n\lambda}\, j_\lambda + a_{n\sigma}\, j_\sigma = i_s \tag{5}$$

As usual, we have

$$j_\lambda = (\beta_{\ell\lambda})_t\, i \text{ and } v_\sigma = (a_{n\sigma})_t\, e \tag{6}$$

in which the subscript t indicates the transposed matrix, and the column matrices i and e contain the loop-current and node-pair voltage variables defined by the chosen tie-set and cut-set matrices, in the familiar manner.

Substituting from Eqs. 1, 2, and 3, in Eqs. 4 and 5, and also making use of Eq. 6, we get the following equilibrium equations on a mixed basis:

$$\beta_{\ell\lambda}\, z_\lambda\, (\beta_{\ell\lambda})_t\, i + \beta_{\ell\sigma}\, (a_{n\sigma})_t\, e = e_s \tag{7}$$

$$a_{n\lambda}\, (\beta_{\ell\lambda})_t\, i + a_{n\sigma}\, y_\sigma\, (a_{n\sigma})_t\, e = i_s \tag{8}$$

It is well known that, for a given network graph, the rows of a tie-set matrix are orthogonal to the rows of a cut-set matrix. Hence we have either

$$\beta_{\ell\lambda}\, (a_{n\lambda})_t + \beta_{\ell\sigma}\, (a_{n\sigma})_t = 0 \tag{9}$$

or

$$a_{n\lambda}\, (\beta_{\ell\lambda})_t + a_{n\sigma}\, (\beta_{\ell\sigma})_t = 0 \tag{10}$$

If we let

$$\gamma_{\ell n} = \beta_{\ell\sigma}\,(a_{n\sigma})_t = -\beta_{\ell\lambda}\,(a_{n\lambda})_t \tag{11}$$

$$-(\gamma_{\ell n})_t = a_{n\lambda}\,(\beta_{\ell\lambda})_t = -a_{n\sigma}\,(\beta_{\ell\sigma})_t \tag{12}$$

then the equilibrium equations (Eqs. 7 and 8) may be written

$$\beta_{\ell\lambda}\, z_\lambda\, (\beta_{\ell\lambda})_t\, i + \gamma_{\ell n}\, e = e_s \tag{13}$$

$$-(\gamma_{\ell n})_t\, i + a_{n\sigma}\, y_\sigma\, (a_{n\sigma})_t\, e = i_s \tag{14}$$

Recalling Eqs. 1 and 2, and introducing the parameter matrices (3)

$$L = \beta_{\ell\lambda}\,[\ell_{sk}]\,(\beta_{\ell\lambda})_t; \quad R = \beta_{\ell\lambda}\,[r_k]\,(\beta_{\ell\lambda})_t \tag{15}$$

$$C = a_{n\sigma}\,[c_k]\,(a_{n\sigma})_t; \quad G = a_{n\sigma}\,[g_k]\,(a_{n\sigma})_t \tag{16}$$

we can write Eqs. 13 and 14 in the form

$$\left\{\begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} p + \begin{bmatrix} R & \gamma_{\ell n} \\ -\gamma_{n\ell} & G \end{bmatrix}\right\} \times \begin{bmatrix} i \\ e \end{bmatrix} = \begin{bmatrix} e_s \\ i_s \end{bmatrix} \tag{17}$$

in which, for convenience, we have denoted the transpose of $\gamma_{\ell n}$ as $\gamma_{n\ell}$. Although this submatrix depends only upon network topology, it seems more reasonable to associate it with the loss parameter matrices R and G because the differential operator p is associated with L and C.

The reason for choosing the mixed basis for expressing equilibrium is now clear because the form of matrix Eq. 17 is essentially that of a system characterized by two energy functions.

In contemplating a normal coordinate transformation we are reminded, at this point, that a linear transformation applied to the variables in a quadratic form results in a congruent transformation of its matrix. A normal-coordinate transformation must, therefore, be accomplished by a congruent transformation of the matrices in Eq. 17. However, the simultaneous diagonalization of these two matrices by such a transformation requires that both be symmetrical (besides stipulating that one of them be definite and nonsingular). This condition is not met by the matrix containing loss elements, which in partitioned form has skew-symmetric character.

We should therefore become aware of the fact that it is illogical to expect that the desired coordinate transformation will be accomplished by a real transformation. Since the latent roots involved are natural frequencies of our network, and these are surely complex in any general RLC situation, we expect that the pertinent congruent

transformation will involve a matrix with complex elements.

In anticipation of these things it is therefore not inconsistent to do a little multiplying by the operator $j = \sqrt{-1}$, and put Eq. 17 into the modified, but equivalent, form

$$\left\{\left[\begin{array}{c|c} Lp & 0 \\ \hline 0 & Cp \end{array}\right] + \left[\begin{array}{c|c} R & j\gamma_{\ell n} \\ \hline j\gamma_{n\ell} & G \end{array}\right]\right\} \times \left[\begin{array}{c} j\ i \\ \hline e \end{array}\right] = \left[\begin{array}{c} j\ e_s \\ \hline i_s \end{array}\right] \tag{18}$$

With regard to the required nonsingular character of the first matrix, we shall assume that if any linear constraint relations exist among the loop currents or node-pair voltages, they have been used to eliminate superfluous variables so that all are now dynamically independent. (Incidentally, the matrix involving R and G is then, in general, also nonsingular.) If we are considering a passive system, then the positive definiteness of the quadratic form for the stored energy is assured. Hence the problem of carrying out a normal-coordinate transformation upon the equilibrium equation (Eq. 18) is now routine.

We can carry it out in two steps: first, by transforming the matrix with L and C congruently to its canonic form, and then subjecting the resulting loss matrix to an orthogonal transformation with its modal matrix, thereby converting it to the diagonal form and leaving the canonic matrix unaltered. These two steps may be combined into a single congruent transformation, with a matrix A characterizing the coordinate transformation indicated by

$$\left[\begin{array}{c} j\ i \\ \hline e \end{array}\right] = A \times \left[\begin{array}{c} j\ i' \\ \hline e' \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} j\ e_s \\ \hline i_s \end{array}\right] = A_t^{-1} \times \left[\begin{array}{c} j\ e'_s \\ \hline i'_s \end{array}\right] \tag{19}$$

which convert the equilibrium equation (Eq. 18) into

$$A_t \times \left\{\left[\begin{array}{c|c} Lp & 0 \\ \hline 0 & Cp \end{array}\right] + \left[\begin{array}{c|c} R & j\gamma_{\ell n} \\ \hline j\gamma_{n\ell} & G \end{array}\right]\right\} \times A \times \left[\begin{array}{c} j\ i' \\ \hline e' \end{array}\right] = \left[\begin{array}{c} j\ e'_s \\ \hline i'_s \end{array}\right] \tag{20}$$

with

$$A_t \times \left[\begin{array}{c|c} Lp & 0 \\ \hline 0 & Cp \end{array}\right] \times A = \left[\begin{array}{c|c} u_\ell & 0 \\ \hline 0 & u_n \end{array}\right] p \tag{21}$$

in which u_ℓ and u_n are unit matrices of order ℓ and n, respectively, and

$$A_t \times \left[\begin{array}{c|c} R & j\gamma_{\ell n} \\ \hline j\gamma_{n\ell} & G \end{array}\right] \times A = D = \begin{bmatrix} d_1 & 0 & 0 & . & 0 \\ 0 & d_2 & 0 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & . & . & d_b \end{bmatrix} \tag{22}$$

Diagonal elements $d_1 \ldots d_b$ are the complex natural frequencies of the system. Thus Eq. 20 expresses equilibrium in the normal coordinates to which the primed variables refer, and Eqs. 19 are the pertinent transformation relations for the variables and sources.

With regard to energy relations, we observe that premultiplication on both sides of Eq. 18 by $\left[-j\, i_t \;\vdots\; e_t\right]$, the transposed conjugate of the column matrix for the variables in Eq. 18, gives

$$i_t\, Lp\, i + e_t\, Cp\, e + i_t\, R\, i + e_t\, G\, e = i_t\, e_s + e_t\, i_s \tag{23}$$

because

$$-e_t\, \gamma_{n\ell}\, i + i_t\, \gamma_{\ell n}\, e = 0 \tag{24}$$

as is evident from the fact that these terms are each other's transpose, but are matrices of order one. The right-hand side of Eq. 23 represents power supplied by the sources; the first two terms on the left-hand side represent rate of energy storage, and the remaining two terms are rate-of-energy dissipation in the lossy elements.

Analogous energy relations are obtained for the transformed Eq. 20 through premultiplication on both sides by $\left[-j\, i'_t \;\vdots\; e'_t\right]$. Use of Eqs. 21 and 22 then yields a conservation-of-energy expression, analogous to Eq. 23, for the system in terms of its normal-coordinate representation.

2. Related Impedance and Admittance Transformations

In Eq. 18 we make the familiar substitutions

$$i = I\, e^{st}, \quad e = E\, e^{st}, \quad i_s = I_s\, e^{st}, \quad e_s = E_s\, e^{st} \tag{25}$$

and introduce the notation

$$U = \left[\begin{array}{c} j\, E_s \\ \hline I_s \end{array}\right]; \quad V = \left[\begin{array}{c} j\, I \\ \hline E \end{array}\right] \tag{26}$$

$$W = \left[\begin{array}{c|c} L\, s + R & j\gamma_{\ell n} \\ \hline j\gamma_{n\ell} & C\, s + G \end{array}\right] \tag{27}$$

whereupon these equations are given by

$$W \times V = U \tag{28}$$

and Eqs. 21 and 22 yield

$$A_t \times W \times A = W_d = \begin{bmatrix} (d_1+s) & 0 & \cdots & 0 \\ 0 & (d_2+s) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (d_b+s) \end{bmatrix} \tag{29}$$

Solution of Eq. 28 involves the inverse matrix

$$W^{-1} = A \times W_d^{-1} \times A_t = X \tag{30}$$

If we partition this inverse matrix as indicated in

$$X = \left[\begin{array}{c|c} X_{\ell\ell} & j\,X_{\ell n} \\ \cdots & \cdots \\ -j\,X_{n\ell} & X_{nn} \end{array}\right] \tag{31}$$

Then the inverse of Eq. 28 with Eq. 26 substituted gives

$$\begin{aligned} X_{\ell\ell}\, E_s + X_{\ell n}\, I_s &= I \\ X_{n\ell}\, E_s + X_{nn}\, I_s &= E \end{aligned} \tag{32}$$

Observe that the elements in $X_{\ell\ell}$ are short-circuit driving-point and transfer admittances; those in X_{nn} are open-circuit driving-point and transfer impedances; and those in $X_{\ell n}$ and $X_{n\ell}$ are dimensionless transfer functions. Therefore, X may be appropriately referred to as an immittance matrix.

If we write for matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1b} \\ a_{21} & a_{22} & \cdots & a_{2b} \\ \cdots & \cdots & \cdots & \cdots \\ a_{b1} & a_{b2} & \cdots & a_{bb} \end{bmatrix} \tag{33}$$

then Eqs. 29 and 30 show that a typical element x_{ik} of matrix X is given by

$$x_{ik} = \sum_{\nu=1}^{b} \frac{k_\nu}{s + d_\nu} \tag{34}$$

with

$$k_\nu = a_{i\nu}\, a_{k\nu} \tag{35}$$

Equation 34 we recognize as a partial-fraction expansion of the immittance x_{ik}, which is a rational function of the complex frequency s. The residues k_ν in this expansion are related to elements in the i^{th} and k^{th} rows of transformation matrix A by the simple expression Eq. 35. This situation is ideally suited to the synthesis problem, for it enables one to construct the matrix A to suit the requirements of a stated x_{ik} function. For a driving-point function (i=k) the residues fix only a single row of A; for a transfer function, only the products of respective elements in two rows are fixed. The remainder of matrix A may be freely chosen so as to yield parameter matrices fulfilling realizability conditions.

3. A Transformation Having More General Applicability to Synthesis

Use of the present results in this novel approach to the synthesis problem is discussed in another paper, and so we shall not pursue their implications here. Rather, we wish to point out that to achieve results suitable for this kind of an attack upon the synthesis problem, we do not necessarily have to accomplish a normal coordinate transformation, but only the simultaneous diagonalization of the parameter matrices in the equilibrium equations (Eqs. 17); this is less restrictive, and hence allows greater latitude in the conditions under which it can be carried out.

If diagonalization of the matrices in Eq. 17 is our only objective, then we are not limited to a congruent transformation but can premultiply and postmultiply by any nonsingular matrices that accomplish the desired end. In this situation, the positive definiteness of neither of the two matrices – nor their symmetry – is required, and so we can, if we wish, include equilibrium equations for linear systems that are active and/or nonbilateral.

For the moment, however, suppose that any active or nonbilateral elements are resistive in character, so that the matrix with L and C in Eq. 17 is still symmetrical and pertinent to a positive definite quadratic form. Assume, also, that any linear constraint relations among the variables have been used to eliminate dynamically superfluous ones, so that this quadratic form is nonsingular. Then we can straightforwardly construct a matrix P which congruently transforms the first matrix in Eq. 17 to its canonic form, and the second to some other dissymmetrical matrix, B. Thus

$$P_t \times \left[\begin{array}{c|c} L & 0 \\ \hline 0 & C \end{array}\right] \times P = U_b = \begin{array}{l}\text{unit matrix} \\ \text{of order } b\end{array} \tag{36}$$

and

$$P_t \times \left[\begin{array}{c|c} R & \gamma_{\ell n} \\ \hline -\gamma_{n\ell} & G \end{array}\right] \times P = B = \begin{array}{l}\text{dissymmetrical} \\ \text{matrix of order} \\ \text{and rank } b\end{array} \tag{37}$$

A colinear transformation of B with an appropriate matrix Q now yields a diagonal matrix D (whose elements are the latent roots of B) and leaves the canonic matrix 36 unchanged. This colinear transformation of B reads

$$Q^{-1} \times B \times Q = D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & \cdots & d_b \end{bmatrix} \tag{38}$$

in which the diagonal elements, as in Eq. 22, are the complex natural frequencies of our linear system, and matrix Q is, in general, complex.

The variables in Eq. 17 are transformed as indicated by the relations

$$\left[\begin{array}{c} i \\ \hline e \end{array}\right] = P\,Q \left[\begin{array}{c} i' \\ \hline e' \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} e_s \\ \hline i_s \end{array}\right] = P_t^{-1}\,Q \left[\begin{array}{c} e'_s \\ \hline i'_s \end{array}\right] \tag{39}$$

although these are not now of any particular interest.

Considering the substitutions (Eqs. 25) and the notation

$$U = \left[\begin{array}{c} E_s \\ \hline I_s \end{array}\right]; \quad V = \left[\begin{array}{c} I \\ \hline E \end{array}\right] \tag{40}$$

and

$$W = \left[\begin{array}{c|c} L\,s + R & \gamma_{\ell n} \\ \hline -\gamma_{n\ell} & C\,s + G \end{array}\right] \tag{41}$$

which are inappreciably different from Eqs. 26 and 27, we again have our equilibrium

equations in the form of Eq. 28. Instead of Eqs. 29 and 30, however, we now find that

$$Q^{-1} \times P_t \times W \times P \times Q = W_d = \begin{bmatrix} (d_1+s) & 0 & 0 & \cdots & 0 \\ 0 & (d_2+s) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & (d_b+s) \end{bmatrix} \tag{42}$$

and

$$W^{-1} = X = P\, Q\, W_d^{-1}\, Q^{-1}\, P_t \tag{43}$$

With the matrix X partitioned as indicated in

$$X = \left[\begin{array}{c|c} X_{\ell\ell} & X_{\ell n} \\ \hline X_{n\ell} & X_{nn} \end{array}\right] \tag{44}$$

we again have solutions in the form of Eqs. 32; and elements of X have the representation shown in Eq. 34. However, in place of Eq. 35, the residues are now given by the expression

$$k_\nu = h_{i\nu}\, t_{k\nu} \tag{45}$$

in which $h_{i\nu}$ and $t_{k\nu}$ are elements in the matrices

$$H = P \times Q \text{ and } T = P \times Q_t^{-1} \tag{46}$$

An approach to the synthesis of active and/or nonbilateral networks through construction of parameter matrices from given rational driving-point and transfer functions may thus be formed to follow essentially the same pattern as for passive bilateral networks.

It is important, however, to observe that the transformation of variables indicated in Eqs. 39, even though it results in the simultaneous diagonalization of the pertinent parameter matrices, is not a normal-coordinate transformation, and does not preserve the invariance of the associated energy forms. To further our objective, as far as synthesis is concerned, we do not need the energy invariance, and are only interested in achieving the diagonal forms in Eqs. 29 and 42. In a somewhat less rigid sense, we might still regard this process as a normal-coordinate transformation because it bears a close resemblance to it, and may be given a similar physical or geometrical interpretation.

E. A. Guillemin

References

1. E. A. Guillemin, Introductory Circuit Theory (John Wiley and Sons, Inc., New York, 1953), Chapter X, articles 2, 3, 4; pp. 491-502.

2. Ibid., pp. 496-499.

3. The quantities ℓ_{sk}, r_k, etc., in square brackets denote the branch parameter matrices having these elements. See E. A. Guillemin, op. cit., pp. 492-495, and allow for the difference in the way resistances are dealt with here.

C. MATRIX SYNTHESIS OF TWO-ELEMENT-KIND DRIVING-POINT IMPEDANCES

An important transformation in network analysis is the normal coordinate transformation, which places the network in a canonic form and displays its normal modes or natural frequencies. Since these quantities are known in a synthesis problem, the possibility exists of starting with the canonic form and working back to the network by means of matrix operations. It is our purpose to show that this can always be done, and that among the resulting networks are the familiar Foster and Cauer forms. The demonstration of this fact is given for RC networks, although the same basic approach holds for all two-element-kind impedances or admittances.

Suppose that we have an RC network and we write its equilibrium equations in matrix form as

$$I = YE \tag{1}$$

where

$$Y = G + sC \tag{2}$$

If we assume that C is positive definite, it is possible to find a real nonsingular transformation matrix A with the property that (1)

$$AYA_t = Y_d = G_d + sU \tag{3}$$

where

$$G_d = \begin{bmatrix} s_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & s_n \end{bmatrix} \tag{4}$$

Then

$$Y = A^{-1}Y_dA_t^{-1} \tag{5}$$

$$Y^{-1} = A_tY_d^{-1}A \tag{6}$$

Let

$$Z = Y^{-1} = [z_{ij}] \tag{7}$$

$$A = [a_{ij}] \tag{8}$$

Then

$$z_{ij} = \sum_{\nu=1}^{n} \frac{a_{\nu i} a_{\nu j}}{s + s_\nu} \tag{9}$$

Let the RC driving-point impedance that is to be synthesized be expanded in partial fractions as

$$z = \sum_{\nu=1}^{n} \frac{k_\nu}{s + s_\nu} + k_{n+1} \tag{10}$$

where k_ν, k_{n+1}, and s_ν are all positive real numbers (2). Since k_{n+1} merely represents an added series resistor, we subtract it from z and define

$$z_{11} = \sum_{\nu=1}^{n} \frac{k_\nu}{s + s_\nu} \tag{11}$$

so that

$$a_{\nu 1}^2 = k_\nu \tag{12}$$

Thus we can quickly find Y_d and the first column of matrix A from the given impedance function. Our problem is to complete matrix A in such a way that Y, as given by Eq. 5, is physically realizable. This is most easily done by taking

$$A = \begin{bmatrix} \sqrt{k_1} & 0 & 0 & \cdots & 0 \\ \sqrt{k_2} & \sqrt{k_2} & 0 & \cdots & 0 \\ \sqrt{k_3} & \sqrt{k_3} & \sqrt{k_3} & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \sqrt{k_n} & \sqrt{k_n} & \sqrt{k_n} & \cdots & \sqrt{k_n} \end{bmatrix} \tag{13}$$

Then it can readily be verified that

$$A^{-1} = \begin{bmatrix} \frac{1}{\sqrt{k_1}} & 0 & 0 & \cdots & 0 \\ \frac{-1}{\sqrt{k_1}} & \frac{1}{\sqrt{k_2}} & 0 & \cdots & 0 \\ 0 & \frac{-1}{\sqrt{k_2}} & \frac{1}{\sqrt{k_3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{k_n}} \end{bmatrix} \tag{14}$$

and

$$Y = \begin{bmatrix} y_1 & -y_1 & 0 & \cdots & 0 \\ -y_1 & y_1 + y_2 & -y_2 & \cdots & 0 \\ 0 & -y_2 & y_2 + y_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & y_{n-1} + y_n \end{bmatrix} \tag{15}$$

where

$$y_i = \frac{s + s_i}{k_i} \tag{16}$$

which leads to the network shown in Fig. XXV-9, which is the familiar Foster form associated with the partial-fraction expansion of z. The simplicity of the matrices A and A^{-1} is due to the fact that the partial-fraction expansion leads immediately to the Foster form, and that the Foster form is intimately connected with the normal coordinates (3).

Next, we shall give a procedure that yields the Cauer form and some related forms; the proof that this procedure always works is somewhat lengthy and will not be included.

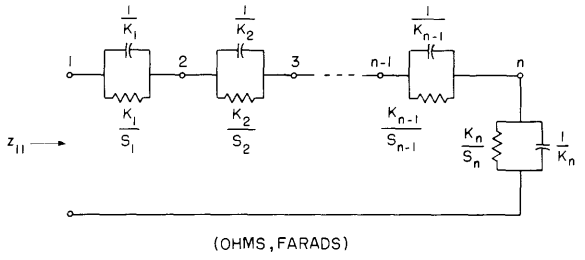

Fig. XXV-9. Foster form associated with $z_{11}(s)$.

The method splits the matrix A^{-1} into the product of a diagonal matrix B, with all positive elements, and an orthogonal matrix Q, whose rows form a set of orthonormal vectors. The first of these vectors can be found from the first column of A, and the remainder can be generated by the Schmidt orthogonalization process. While this determines Q uniquely, it is possible to select B with some degree of freedom and thus to obtain realizations other than the Cauer form.

Assume that

$$A = (BQ)^{-1} \tag{17}$$

where

$$Q^{-1} = Q_t \tag{18}$$

and

$$B = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_n \end{bmatrix} \tag{19}$$

Then

$$Q = (AB)_t \tag{20}$$

Let

$$Q = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix} = \begin{bmatrix} \overline{q}_1 \\ \vdots \\ \overline{q}_n \end{bmatrix} \tag{21}$$

where any one of the vectors in the orthonormal set is

$$\overline{q}_i = [q_{i1}, \ldots, q_{in}] \tag{22}$$

Then

$$\overline{q}_i = [a_{1i}, \ldots, a_{ni}]\, b_i \tag{23}$$

We know $a_{11}, \ldots, a_{n1}$ from Eq. 12, and, since the vectors $\overline{q}_1, \ldots, \overline{q}_n$ form an orthonormal set, we can find both $\overline{q}_1$ and b_1 from

$$\overline{q}_1 \cdot \overline{q}_1 = \left(a_{11}^2 + \ldots + a_{n1}^2\right) b_1^2 = 1 \tag{24}$$

in which we resolve algebraic sign ambiguities by taking the positive square root. To find the rest of the vectors comprising Q, we define

$$\overline{d}_i = [q_{i1}s_1, \ldots, q_{in}s_n] \tag{25}$$

Then, by the Schmidt process,

$$\left.\begin{aligned}
&\bar{q}_1 = \text{known vector}\\
&\bar{q}_2^* = \bar{d}_1 - (\bar{q}_1 \cdot \bar{d}_1)\,\bar{q}_1\\
&\bar{q}_2 = -\bar{q}_2^* \big/ |\bar{q}_2^*|\\
&\bar{q}_3^* = \bar{d}_2 - (\bar{q}_1 \cdot \bar{d}_2)\,\bar{q}_1 - (\bar{q}_2 \cdot \bar{d}_2)\,\bar{q}_2\\
&\bar{q}_3 = -\bar{q}_3^* \big/ |\bar{q}_3^*|\\
&\quad\vdots\\
&\bar{q}_n^* = \bar{d}_{n-1} - (\bar{q}_1 \cdot \bar{d}_{n-1})\,\bar{q}_1 - \cdots - (\bar{q}_{n-1} \cdot \bar{d}_{n-1})\,\bar{q}_{n-1}\\
&\bar{q}_n = -\bar{q}_n^* \big/ |\bar{q}_n^*|
\end{aligned}\right\} \tag{26}$$

It follows from these equations that the vectors $\bar{q}_i$ form an orthonormal set, and that

$$\left.\begin{aligned}
&\bar{q}_3 \cdot \bar{d}_1 = \bar{q}_4 \cdot \bar{d}_1 = \cdots = \bar{q}_n \cdot \bar{d}_1 = 0\\
&\bar{q}_4 \cdot \bar{d}_2 = \cdots = \bar{q}_n \cdot \bar{d}_2 = 0\\
&\quad\vdots\\
&\bar{q}_n \cdot \bar{d}_{n-2} = 0\\
&\bar{q}_2 \cdot \bar{d}_1 = -|\bar{q}_2^*|\\
&\bar{q}_3 \cdot \bar{d}_2 = -|\bar{q}_3^*|\\
&\quad\vdots\\
&\bar{q}_n \cdot \bar{d}_{n-1} = -|\bar{q}_n^*|
\end{aligned}\right\} \tag{27}$$

From Eqs. 2, 3, 5, 17, and 18 it follows that

$$Y = G + sC = BQG_dQ_tB_t + sBB_t \tag{28}$$

so that if we define

$$G^* = QG_dQ_t \tag{29}$$

then

$$G = BG^*B_t \tag{30}$$

and

$$C = BB_t \tag{31}$$

From Eq. 29 we see that G^* is a symmetrical matrix with eigenvalues $s_1, \ldots, s_n$. We shall assume, for simplicity, that $s_i > 0$ so that G^* is positive definite. We shall mention later the modifications that are required when one of the characteristic frequencies is zero. Expansion of Eq. 29 yields

$$G^* = \begin{bmatrix} \bar{q}_1 \cdot \bar{d}_1 & \cdots & \bar{q}_1 \cdot \bar{d}_n \\ \vdots & & \vdots \\ \bar{q}_n \cdot \bar{d}_1 & \cdots & \bar{q}_n \cdot \bar{d}_n \end{bmatrix} \tag{32}$$

From the symmetry of G^* and the results of Eqs. 27 we conclude that

$$G^* = \begin{bmatrix} \bar{q}_1 \cdot \bar{d}_1 & -|\bar{q}_2^*| & 0 & \cdots & 0 \\ -|\bar{q}_2^*| & \bar{q}_2 \cdot \bar{d}_2 & -|\bar{q}_3^*| & \cdots & 0 \\ 0 & -|\bar{q}_3^*| & \bar{q}_3 \cdot \bar{d}_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \bar{q}_n \cdot \bar{d}_n \end{bmatrix} \tag{33}$$

where $\bar{q}_i \cdot \bar{d}_i > 0$ because G^* is positive definite. It can be shown by induction that these properties of G^* are sufficient for every element of $(G^*)^{-1}$ to be positive, and that this in turn guarantees the existence of a diagonal matrix B, all of whose elements are positive, such that

$$G = BG^*B_t \tag{30}$$

is dominant and hence physically realizable. The elements of B can be obtained from an arbitrary set of positive constants k_i^* which comprise the elements of a column matrix K^*,

$$K^* = \begin{bmatrix} k_1^* \\ \vdots \\ k_n^* \end{bmatrix} \tag{34}$$

and from a column matrix B^*,

$$B^* = \begin{bmatrix} b_1^* \\ \vdots \\ b_n^* \end{bmatrix} = (G^*)^{-1} K^* = Q G_d^{-1} Q_t K^* \tag{35}$$

the relation between the numbers b_i^* and b_i being

$$b_i = \frac{b_1}{b_1^*} b_i^* \tag{36}$$

Different choices of the constants k_i^* lead to different networks, the choice $k_1^* = k_2^* = \ldots = k_{n-1}^* = 0$ and $k_n^* = 1$ yielding the Cauer form.

EXAMPLE:

$$z_{11}(s) = \frac{s^2 + 6s + 8}{s^3 + 9s^2 + 23s + 15} = \frac{\frac{3}{8}}{s+1} + \frac{\frac{1}{4}}{s+3} + \frac{\frac{3}{8}}{s+5}$$

$$a_{11} = \frac{\sqrt{6}}{4} \qquad s_1 = 1$$

$$a_{21} = \frac{1}{2} \qquad s_2 = 3$$

$$a_{31} = \frac{\sqrt{6}}{4} \qquad s_3 = 5$$

$$\bar{q}_1 = \left[\frac{\sqrt{6}}{4}, \frac{1}{2}, \frac{\sqrt{6}}{4}\right] b_1$$

$$\bar{q}_1 \cdot \bar{q}_1 = \left(\frac{3}{8} + \frac{1}{4} + \frac{3}{8}\right) b_1^2 = 1$$

$$b_1 = 1$$

$$\bar{q}_1 = \left[\frac{\sqrt{6}}{4}, \frac{1}{2}, \frac{\sqrt{6}}{4}\right]$$

$$\bar{d}_1 = \left[\frac{\sqrt{6}}{4}, \frac{3}{2}, \frac{5\sqrt{6}}{4}\right]$$

$$\bar{q}_1 \cdot \bar{d}_1 = 3$$

$$\bar{q}_2^* = \bar{d}_1 - (\bar{q}_1 \cdot \bar{d}_1)\,\bar{q}_1 = \left[-\frac{\sqrt{6}}{2},\quad 0,\quad \frac{\sqrt{6}}{2}\right]$$

$$|\bar{q}_2^*| = \sqrt{3}$$

$$\bar{q}_2 = \left[\frac{\sqrt{2}}{2},\quad 0,\quad -\frac{\sqrt{2}}{2}\right]$$

$$\bar{d}_2 = \left[\frac{\sqrt{2}}{2},\quad 0,\quad -\frac{5\sqrt{2}}{2}\right]$$

$$\bar{q}_1 \cdot \bar{d}_2 = -\sqrt{3}$$

$$\bar{q}_2 \cdot \bar{d}_2 = 3$$

$$\bar{q}_3^* = \bar{d}_2 - (\bar{q}_1 \cdot \bar{d}_2)\,\bar{q}_1 - (\bar{q}_2 \cdot \bar{d}_2)\,\bar{q}_2 = \left[-\frac{\sqrt{2}}{4},\quad \frac{\sqrt{3}}{2},\quad -\frac{\sqrt{2}}{4}\right]$$

$$|\bar{q}_3^*| = 1$$

$$\bar{q}_3 = \left[\frac{\sqrt{2}}{4},\quad -\frac{\sqrt{3}}{2},\quad \frac{\sqrt{2}}{4}\right]$$

$$Q = \begin{bmatrix} \frac{\sqrt{6}}{4} & \frac{1}{2} & \frac{\sqrt{6}}{4} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} \end{bmatrix}$$

$$G^* = \begin{bmatrix} 3 & -\sqrt{3} & 0 \\ -\sqrt{3} & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$(G^*)^{-1} = \begin{bmatrix} \frac{8}{15} & \frac{\sqrt{3}}{5} & \frac{\sqrt{3}}{15} \\ \frac{\sqrt{3}}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{\sqrt{3}}{15} & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

If we take $k_1^* = k_2^* = 0$ and $k_3^* = 1$, we obtain

$$b_1^* = \frac{\sqrt{3}}{15} \qquad b_1 = 1$$

$$b_2^* = \frac{1}{5} \qquad b_2 = \sqrt{3}$$

$$b_3^* = \frac{2}{5} \qquad b_3 = 2\sqrt{3}$$

$$Y = BG^*B_t + sBB_t$$

$$= \begin{bmatrix} 3 & -3 & 0 \\ -3 & 9 & -6 \\ 0 & -6 & 36 \end{bmatrix} + s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

which is the admittance matrix for the Cauer realization of $z_{11}(s)$. If instead we take $k_1^* = \sqrt{3}$, $k_2^* = 1$, and $k_3^* = 4$, we obtain an alternate realization,

$$b_1^* = \sqrt{3} \qquad b_1 = 1$$

$$b_2^* = 2 \qquad b_2 = \frac{2\sqrt{3}}{3}$$

$$b_3^* = 2 \qquad b_3 = \frac{2\sqrt{3}}{3}$$

$$Y = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 4 & -\frac{4}{3} \\ 0 & -\frac{4}{3} & 4 \end{bmatrix} + s \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{3} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

The networks corresponding to these matrices are shown in Figs. XXV-10 and XXV-11.

The procedure just described fails if one of the characteristic frequencies is zero, since G^* is then singular, and Eq. 35 is no longer valid. In such a case we can find G^* as before, and find B by taking b_i proportional to G_{ni}, the cofactors of the elements in the last column of G^*, the proportionality factor being so chosen that b_1 assumes its known value.

Two methods have been described for the matrix synthesis of RC driving-point impedances. Some of the extensions of these methods are obvious, such as the use of similar operations on impedance matrices to obtain the second Foster form, the interchanging of G and C to obtain the second Cauer form, and similar applications to

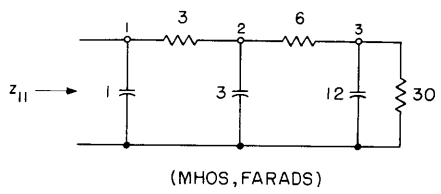

Fig. XXV-10. Cauer form associated with $z_{11}(s)$.

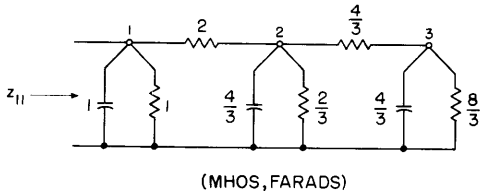

Fig. XXV-11. An alternative realization of $z_{11}(s)$.

RL and LC networks. These applications are only of academic interest, however, since the well-known procedures for two-element-kind synthesis are computationally much simpler. Work is being done to extend this synthesis approach to less well-explored areas, including the problems of equivalent networks, the grounded two-terminal-pair RC network, and active and/or nonbilateral network synthesis.

R. O. Duda

References

1. E. A. Guillemin, The Mathematics of Circuit Analysis (The Technology Press, Cambridge, Mass., and John Wiley and Sons, Inc., New York, 1949), p. 156.

2. E. A. Guillemin, Synthesis of Passive Networks (John Wiley and Sons, Inc., New York, 1957), p. 64.

3. Ibid., p. 96.