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From bubbles to foam: dilute to dense evolution of hadronic wave function at high energy

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Abstract: We derive the evolution of a hadronic light cone wave function with energy at weak coupling. Our derivation is valid both in the high and the low partonic density limit, and thus encompasses both the JIMWLK and the KLWMIJ evolution. The hadronic wave function is shown to evolve by the action of the Bogoliubov-type operator, which diagonalizes on the soft gluon sector the light-cone hamiltonian in the presence of an arbitrary valence charge density. We find explicitly the action of this operator on the soft as well as the valence degrees of freedom of the theory.

Keywords: QCD, Deep Inelastic Scattering, Parton Model, NLO Computations.

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Contents

1. Introduction

The problem of calculating hadronic scattering amplitudes at high energy is an old one. It goes back to classical works of Gribov on Reggeon Field Theory [1] in the pre-QCD days. Within the framework of QCD this question has been addressed from different points of view [2 – 5].

In the last ten years or so the subject has seen new developments. Some of these have been triggered by Mueller's reformulation of the BFKL equation [2] in terms of the dipole model [6, 7] with additional input provided by the functional approach of [8]. The result was the derivation of the functional evolution equation for the hadronic amplitude - the so-called Balitsky hierarchy [9] or JIMWLK equation [10, 11]; and its simplified mean field version due to Kovchegov [12]. This evolution takes into account coherent emission effects in the dense hadronic wave function, or in other words partonic saturation effects. These effects lead to unitarization of the scattering amplitude. Although the language of this approach is different from the original Reggeon Field Theory, a direct relation between the JIMWLK evolution and the QCD Reggeon Field Theory has been investigated recently [13, 14].

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In the last couple of years spurred on by observations of [15], the realization has emerged that the existing evolution equations, which are tailored to describe the situation when a small perturbative projectile scatters off a large dense target, do not include the so called Pomeron loop effects. The effort to account for the Pomeron loops using the probabilistic view of the evolution [16] has lead to interesting analogies between QCD and statistical systems [22]. Alternative approaches based on effective Lagrangian can be found in [17, 18, 20, 21, 19]

Another avenue that has been explored in this context is the direct approach to the evolution of the hadronic wave function [23 – 26]. This approach yielded the evolution equation valid in the limit opposite to that of JIMWLK, namely when the hadronic wave function is dilute. This so called KLWMIJ equation [24] is related to the JIMWLK equation by the dense-dilute duality transformation [25]. The basic strategy of this approach is to calculate the light cone hadronic wave function of soft gluonic modes, given the color charge density $j^a(x)$ due to the 'valence' modes - the modes with large longitudinal momentum. When the hadron is boosted, the longitudinal momentum of the soft modes is increased and they contribute to the scattering matrix and other physical observables [27]. The evolution of any physical observable is therefore in principle completely determined once we know the soft part of the wave function.

So far the hadronic wave function has been calculated only in the KLWMIJ limit, namely when the valence charge density is small; $j^a(x) \sim g$. The JIMWLK evolution on the other hand is valid when $j^a(x) \sim 1/g$, but no wave function evolution is available in the JIMWLK regime. The derivation of [9] is given directly for the scattering matrix. The original derivation of [10] is not far in spirit from the wave function form of the evolution, however it involves additional approximations which do not allow to read off the evolution of the wave function directly from the JIMWLK equation.

The main motivation to know the explicit form of the wave function evolution comes from the possibility to use it to derive the generalization of the JIMWLK/KLWMIJ evolution that includes the Pomeron loops. The knowledge of the wave function is also crucial to be able to address a wide range of semi-inclusive observables [28].

In the present paper we derive the soft gluon wave function valid at any physically interesting value of the valence color charge density. The expression we derive is valid both in the JIWMLK and the KLWMIJ limits as well as at any value of the valence charge density which interpolates between the two: $g \leq j \leq 1/g$. We do this by diagonalizing the leading part of the light cone Hamiltonian on the soft gluon sector. The transformation that diagonalizes the Hamiltonian turns out to be of the Bogoliubov type with parameters depending on the valence color charge density operator. We find explicitly the action of this transformation on the basic quantum degrees of freedom: the soft components of the vector potential $A_i^a(x, x^-)$ and the valence color charge density $j^a(x)$.

We show that the expression for the wave function indeed reproduces the JIMWLK and the KLWMIJ evolution equations. To reproduce the KLWMIJ equation one simply neglects the coherent emission effects in the wave function. Thus the evolution of the wave function in this limit is strictly perturbative [24]. The nontrivial physics in this limit is entirely due to the multiple scattering corrections in the scattering amplitude. On

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the other hand to derive the JIMWLK limit we keep all the coherent emission effects in the wave function. However as we show explicitly below, in this limit we neglect certain multiple scattering corrections in the scattering amplitude. Physically this is justified in the situation where our hadron scatters on a perturbatively small target, which is when the JIMWLK evolution is valid.

To derive the evolution equation which includes Pomeron loops exactly we have to keep both types of effects in the evolution of the scattering amplitude. Within the present framework this looks like a tractable problem. It is however beyond the scope of the present paper and is left for future work.

The paper is structured as follows. In section II we recall the general framework of the high energy evolution. Sections III and IV are the main part of this paper. Section III is devoted to the derivation of the "vacuum" wave function of the soft gluon Hilbert space in the presence of the valence color charge density. In Section IV we show that this diagonalization is achieved by the action on the free vacuum of a Bogoliubov type operator and derive explicitly the action of this operator on the soft and valence degrees of freedom. In sectionV we show how both the JIMWLK and the KLWMIJ evolution equations follow from the wave function we have found in section III in different limits. Finally a discussion is presented in section VI.

2. High energy evolution

The logic of our approach is the same as described in [24, 27]. Suppose that at some initial rapidity Y_0 we know the wave function of a hadron. In the gluon Fock space it has a generic form (we work in the $A^- = 0$ gauge)

$$|P\rangle_{Y_0} \,=\, \Psi[a^{\dagger a}(x,k^+)]\,|0\rangle \quad . \tag{2.1}$$

There is some minimal longitudinal momentum $k^+ = \Lambda$ below which there are no gluons in this wave function. More precisely, the number of soft gluons with $k^+ \leq \Lambda$ is not zero but is perturbatively small so that their contribution to the scattering amplitude at Y_0 is a small perturbative correction and can be neglected.

We are interested in describing the scattering of this hadron on some target. The target is described by some distribution of color fields $\alpha_T \equiv A^+$ with a probability density distribution $W_T[\alpha_T]$. The second-quantized S-matrix operator in the eikonal approximation (in which we are working throughout this paper) is given by

$$\hat{S} \,=\, \exp\left[i\int d^2x\, j^a(x)\,\alpha_T^a(x)\right] , \tag{2.2}$$

where

$$j^a(x) \,=\, g\int_{k^+>\Lambda}\frac{dk^+}{2\pi}\,a^{\dagger b}(x,k^+)\,T^a_{bc}\,a^c(x,k^+) \tag{2.3}$$

is the color charge density operator at the transverse position x (with $T^a_{bc} = if^{abc}$ - the generator of the color group in the adjoint representation). After scattering on a particular

configuration of the target field the hadronic wave function becomes

$$\hat{S}|P\rangle_{Y_0} \;=\; \Psi[S^{ab}(x)\, a^{\dagger b}(x,k^+)]\,|0\rangle\,, \tag{2.4}$$

where $S^{ab}(x)$ is a unitary matrix - the single gluon scattering matrix. Since the scattering amplitude is sensitive only to the color charge density in the hadronic wave function and not to any other characteristic of the hadron, we can think of this wave function as being specified by some distribution of $j^a(x)$. The color charge density correlators are determined in terms of the 'probability density functional' $W[j]$ via

$$\langle j^{a_1}(x_1)\dots j^{a_n}(x_n)\rangle_{Y_0} \;=\; \int Dj\, W_{Y_0}[j]\, j^{a_1}(x_1)\dots j^{a_n}(x_n)\,. \tag{2.5}$$

The forward scattering amplitude is then given by

$$\mathcal{S} \;=\; \int D\alpha_T^a\; W^T_{Y-Y_0}[\alpha_T(x)]\;\Sigma^P_{Y_0}[\alpha_T(x)]\,, \tag{2.6}$$

where

$$\Sigma^P_{Y_0}[\alpha_T] = \langle 0|\Psi^*[a(x,k^+)]\hat{S}\Psi[a^\dagger(x,k^+)]|0\rangle = \int Dj\;\; W_{Y_0}[j]\;\exp\left[i\int d^2x\, j^a(x)\,\alpha_T^a(x)\right]. \tag{2.7}$$

The total rapidity of the process is Y while the target is assumed to be evolved to rapidity $Y-Y_0$. Here, W^T characterizes the distribution of color fields α_T in the target, while $W\,[j]$ characterizes the distribution of color charges in the projectile. Due to Lorentz invariance $\mathcal{S}$ is Y_0 independent.

The evolution of the S-matrix (2.6) with energy in the high energy limit has the generic form

$$-\frac{d}{d\,Y}\,\mathcal{S} \;=\; \int D\alpha_T^a\; W^T_{Y-Y_0}[\alpha_T(x)]\;\; H^{\rm RFT}\left[\alpha_T,\frac{\delta}{\delta\,\alpha_T}\right]\;\;\Sigma^P_{Y_0}[\alpha_T(x)]\,, \tag{2.8}$$

where $H^{\rm RFT}$ is the Hermitian kernel of high energy evolution, which can be viewed as acting either to the right or to the left:

$$-\frac{\partial}{\partial Y}\,\Sigma^P = H^{\rm RFT}\left[\alpha_T,\,\frac{\delta}{\delta\alpha_T}\right]\;\Sigma^P[\alpha_T]\,;\quad -\frac{\partial}{\partial Y}\,W^T = H^{\rm RFT}\left[\alpha_T,\,\frac{\delta}{\delta\alpha_T}\right]\;W^T[\alpha_T]\,. \tag{2.9}$$

The color charge density operators are the generators of the $SU(N_c)$ algebra and as such do not commute

$$[j^a(x),\;j^b(y)] \;=\; i\,f^{abc}\,j^c(x)\,\delta^2(x-y)\,.$$

As explained in detail in [24], to properly take into account the non commuting nature of the charge density operators $j(x)$ and to still be able to represent wave function averages in terms of the functional integral over 'classical' fields j^a, one has to assign to j an additional 'longitudinal' coordinate. Thus in effect $j^a(x)\to j^a(x,x^-)$, where the value of x^- simply keeps track of the order of the operators j in the correlation function eq. (2.5). An analogous 'longitudinal coordinate' should be assigned to the target field α_T. Since in this paper we

work in the Hamiltonian formalism and explicitly keep track of the commutation relations of the quantum operators $j^a(x)$, we will not need to dwell on this additional longitudinal coordinate.

The preceding discussion is given in the situation when the increase of rapidity is assigned to the target. One can equally well boost the projectile. The evolution of the projectile probability density functional $W[j]$ is related to that of $\Sigma[\alpha_T]$ since the two are related by the functional Fourier transform eq. (2.7)

$$-\frac{\partial}{\partial Y} W[j], = \; H^{\rm RFT}\left[\frac{\delta}{\delta j}, -j\right] \; W[j]\,. \tag{2.10}$$

As the hadron is boosted by rapidity ΔY, the longitudinal momenta of the gluons in its wave function are scaled by the boost parameter $k^+ \to e^{\Delta Y} k^+$. Thus some gluons in the wave function emerge after boost with the longitudinal momenta above the cutoff Λ and have to be taken into account in the calculation of the scattering amplitude. The number of thus 'produced' additional gluons in the wave function is proportional to the total longitudinal phase space $\int \frac{dk^+}{k^+} = \Delta Y$.

To find the evolution of the scattering amplitude we need two ingredients. First we have to solve for the initial hadronic wave function with greater accuracy on the soft gluon Hilbert space than is necessary to calculate the scattering amplitude at the initial rapidity Y_0. Second we need to take into account the contribution of these soft gluons into the scattering amplitude at the rapidity $Y = Y_0 + \Delta Y$, which amounts to the transformation

$$j^a(x) \;\to\; j^a(x) \,+\, j^a_{\rm soft}(x)\,, \quad j^a_{\rm soft}(x) \,=\, g\int_{\Lambda\, e^{-\Delta Y}}^{\Lambda} \frac{dk^+}{2\pi} a^{\dagger b}(x,k^+)\, T^a_{bc}\, a^c(x,k^+) \tag{2.11}$$

in eq. (2.2). This transformation is conveniently represented in terms of the charge density shift operator (which also has the meaning of the 'dual' to the Wilson line operator [25])

$$\hat{R}_a \;=\; \exp\left[\int d^2z\, j^c_{\rm soft}(z)\, \frac{\delta}{\delta j^c(z)}\right]\,, \quad j^c(x) \to \hat{R}_a\, j^c(x)\,. \tag{2.12}$$

The crucial part of this program is the knowledge of the wave function on the soft gluon part of the Hilbert space, $k^+ \le \Lambda$ with some minimal accuracy. The calculation of this wave function is the subject of the next section.

The QCD light cone Hamiltonian H responsible for the dynamics of the soft modes is diagonalized by the action of a unitary operator $\Omega_{\Delta Y}$, where ΔY corresponds to the phase space volume occupied by the soft modes. Equivalently, the vacuum wave function of the soft modes in the presence of the valence color charges is $\Omega\,|P\rangle$. The kernel of the high energy evolution, $H^{\rm RFT}$ is related to Ω as [14]:

$$H^{\rm RFT} \,=\, -\lim_{\Delta Y\to 0} \frac{\langle 0_a|\Omega^\dagger_{\Delta Y}(j,a)\,\left(\hat{R}_a \,-\, 1\right)\,\Omega_{\Delta Y}(j,a)|0_a\rangle}{\Delta Y}\,. \tag{2.13}$$

We will find below that Ω is an operator of the Bogoliubov type for any physically interesting j:

$$\Omega \,=\, \mathcal{C}\,\mathcal{B}$$

with $\mathcal{C}$ denoting a coherent operator, which is the exponential of an operator linear in the soft fields A, whereas $\mathcal{B}$ is an exponential of an operator quadratic in A. In the dilute limit $j \sim g$ we have $\mathcal{B} = 1$ and the coherent operator $\mathcal{C}$ leads to the KLWMIJ evolution [24]. For dense systems $j \sim 1/g$, the Bogoliubov operator $\mathcal{B}$ also contributes to the leading order evolution kernel $H^{\rm RFT}$. We derive the action of Ω on both the valence and soft degrees of freedom, which enter equation (2.13). The JIMWLK Hamiltonian [10] is obtained from the general expression (2.13) in the limit of weak target fields α_T expanding $\hat{R}_a$ to second order in $\delta/\delta j$.

3. Diagonalizing the soft gluon hamiltonian

We will proceed in the following steps. In section 3.1 we formulate the light-cone Hamiltonian for soft gluon modes $k^+ < \Lambda$, coupled to the color charge density of the hard modes $k^+ > \Lambda$. We observe that the zero modes of the vector potential are not independent degrees of freedom, but are constraint by the residual gauge fixing and the requirement of finiteness of energy. In section 3.2, we solve the resulting constraints. In section 3.3, we diagonalize the resulting Hamiltonian, by first finding the complete set of solutions to the classical equations of motion, and then expanding the field operators in this basis. To ensure the canonical commutation relations for the creation and annihilation operators associated with these basis functions, a proper normalization of the classical solutions is needed. This normalization is found in section 3.4.

3.1 The hamiltonian and the canonical structure

The starting point of our approach is the light cone hamiltonian of QCD [29]

$$H = \int_{k^+>0} \frac{dk^+}{2\pi}\, d^2x \left(\frac{1}{2}\Pi_a^-(k^+,x)\,\Pi_a^-(-k^+,x) + \frac{1}{4}G_a^{ij}(k^+,x)\,G_a^{ij}(-k^+,x) \right) , \quad (3.1)$$

where the electric and magnetic pieces have the form

$$\begin{aligned} \Pi_a^-(x^-,x) &= -\frac{1}{\partial^+}\left(D^i\partial^+ A_i\right)^a(x^-,x)\,, \\ G_a^{\mu\nu}(x^-,x) &= \partial^\mu A_\nu^a(x^-,x) - \partial^\nu A_\mu^a(x^-,x) - g f^{abc}\, A_\mu^b(x^-,x)\, A_\nu^c(x^-,x)\,, \end{aligned} \quad (3.2)$$

and the covariant derivative is defined as

$$D_i^{ab}\ \Phi^b \ = \ \left(\partial_i\,\delta^{ab} \ - \ g\, f^{acb}\, A_i^c\right)\ \Phi^b\,. \quad (3.3)$$

Our aim is to diagonalize this Hamiltonian on the Hilbert space of soft gluon modes - those with longitudinal momenta smaller than some scale Λ. We assume that the valence part of the wave function (the component of the full wave function which does not contain soft modes) is known and is completely specified by the correlation function of the color charge density

$$j^a(x) \equiv igf^{abc}\int_{k^+>\Lambda} \frac{dk^+}{2\pi}\, a_i^{b\,\dagger}(k^+,x)\, a_i^c(k^+,x)\,. \quad (3.4)$$

The soft modes are the interesting dynamical degrees of freedom of our problem, and they interact with the valence ones via eikonal coupling in the Hamiltonian. The Hamiltonian for the soft modes is then given by eq. (3.1) with the substitution

$$\Pi_a^-(k^+,x) = \frac{1}{i(k^+ + i\epsilon)}\partial^i\partial^+ A_i^a(k^+,x) + \frac{1}{-i(k^+ + i\epsilon)} j^a(x) + g\frac{1}{-i(k^+ + i\epsilon)}\, f^{abc}\int_{|p^+|<\Lambda}\frac{dp^+}{2\pi} A_i^b(k^+ - p^+, x)\,(-ip^+)\,A_i^c(p^+,x)\,. \quad (3.5)$$

The soft fields A are defined only below the longitudinal momentum cutoff Λ, but we will not explicitly indicate it in the following.

The canonical structure of the theory is determined by the commutation relations of the fields. As we will see, the zero momentum mode of the field A is non dynamical and is determined by the residual gauge fixing (still not specified so far on top of the usual light cone gauge condition $A^+ = 0$) and the constraint of finiteness of energy. We denote by $\tilde{A}$ the part of the field that does not contain the mode with vanishing longitudinal momentum - the zero mode.

The canonical commutators of the field $\tilde{A}$ are [30]

$$[\tilde{A}_i^a(x^-,x), \tilde{A}_j^b(y^-,y)] = -\frac{i}{2}\epsilon(x^- - y^-)\delta_{ij}^{ab}(x-y)\,, \quad (3.6)$$

with

$$\epsilon(x) = \frac{1}{2}[\Theta(x) - \Theta(-x)]\,. \quad (3.7)$$

One defines the light cone canonical creation and annihilation operators as usual through

$$\tilde{A}_i^a(x^-,x) = \int_0^\infty \frac{dk^+}{2\pi}\frac{1}{\sqrt{2k^+}}\left\{a_i^a(k^+,x)\,e^{-ik^+x^-} + a_{i\,\dagger}^a(k^+,x)\,e^{ik^+x^-}\right\},$$
$$\left[a_i^a(k^+,x), a_j^{b\,\dagger}(p^+,y)\right] = (2\pi)\,\delta^{ab}\,\delta_{ij}\,\delta(k^+ - p^+)\,\delta^{(2)}(x-y)\,. \quad (3.8)$$

This translates into ($k^+ \neq 0$):

$$\left[\tilde{A}_i^a(k^+,x), \tilde{A}_j^b(p^+,y)\right] = \frac{\pi}{2}\left(\frac{1}{k^+ + i\epsilon} + \frac{1}{k^+ - i\epsilon}\right)\,\delta(k^+ + p^+)\,\delta^{ab}\,\delta_{ij}\,\delta^{(2)}(x-y)\,. \quad (3.9)$$

The Hamiltonian eq. (3.1) commutes with the generator of the x^- - independent gauge transformation, which on physical states should vanish:

$$\int dx^-(D_i\partial^+ A_i)^a - j^a(x) = 0\,. \quad (3.10)$$

Following the standard procedure we should fix this residual gauge freedom by imposing a gauge fixing condition. We will be working in the gauge (same as in [10])

$$\partial_i\, A_i^a(x^- \to -\infty) = 0\,. \quad (3.11)$$

¿From previous analysis of the behavior of the field in this gauge [10], we know that the vector potential vanishes at $x^- \to -\infty$ but approaches a non vanishing asymptotic

value at $x^- \to \infty$, which we denote by $A_i^a(x^- \to \infty, x) = \gamma_i^a(x)$. Separating the nonzero momentum modes, we thus write

$$A_i^a(x^-, x) = \frac{1}{2}\gamma_i^a(x) + \tilde{A}_i^a(x^-, x)\,. \tag{3.12}$$

Even though $\tilde{A}$ has no zero momentum mode, its asymptotics is not vanishing but is rather given by $\pm\frac{1}{2}\gamma_i^a$. It is thus convenient to define a field c which has regular behavior at infinity by

$$\begin{aligned} A_i^a(x^-, x) &= \theta(x^-)\gamma_i^a(x) + c_i^a(x^-, x)\,, \\ \tilde{A}_i^a(x^-, x) &= \epsilon(x^-)\gamma_i^a(x) + c_i^a(x^-, x)\,, \\ c_i^a(x^- \to \pm\infty, x) &\to 0\,, \end{aligned} \tag{3.13}$$

and

$$\partial^+ A_i^a = \partial^+ \tilde{A}_i^a = \delta(x^-)\gamma_i^a + \partial^+ c_i^a\,. \tag{3.14}$$

Our aim is to find the ground state of the Hamiltonian eq. (3.1) given the charge density j^a (more precisely we consider the matrix elements of the operators $j^a(x)$ on the Hilbert space of the valence modes as known).

Our first observation is that since the Hamiltonian is the integral of the positive definite Hamiltonian density over x^-, the necessary condition for finiteness of energy is vanishing of the density at $x^- \to \pm\infty$. The finiteness of the magnetic part of the Hamiltonian requires

$$G_{ij}^a(x^- \to \infty) = 0\,, \tag{3.15}$$

while the finiteness of the electric part is ensured by eq. (3.10).

We will use the gauge fixing condition and the finite energy conditions as operatorial constraints that determine γ in terms of $\tilde{A}$ (or equivalently c_i). This is equivalent to Dirac bracket quantization of the fields A which leave the canonical commutators of $\tilde{A}$ unchanged. The commutators of γ_i^a with $\tilde{A}_i^a$ and between themselves are then determined by solving the constraints.

Expressing the magnetic constraint equation (3.15) in terms of γ_i^a, we obtain

$$\partial_i\,\gamma_j^a(x) - \partial_j\,\gamma_i^a(x) - g f^{abc}\,\gamma_i^b(x)\,\gamma_j^c(x) = 0\,. \tag{3.16}$$

To express the electric constraint eq. (3.10) we use the fact that given the boundary conditions on c_i^a

$$\int dx^- \partial^+ A_i^a = \gamma_i^a\,. \tag{3.17}$$

We then find[1]

$$\partial_i\gamma_i^a - \frac{1}{2}g f^{abc}\gamma_i^b\gamma_i^c - g f^{abc}\int dx^-\,\tilde{A}_i^b(x^-)\partial^+\tilde{A}_i^c(x^-) = j^a(x)\,, \tag{3.18}$$

[1]Here we used $f^{abc}\int dx^- A_i^b(x^-)\partial^+ A_i^c(x^-) = \frac{1}{2}f^{abc}\gamma_i^b\gamma_i^c + f^{abc}\int dx^-\,\tilde{A}_i^b(x^-)\partial^+\tilde{A}_i^c(x^-)$ which follows from eq. (3.17).

or, equivalently,

$$\partial_i \gamma_i^a(x) - \frac{1}{2} g f^{abc} \gamma_i^b(x) \gamma_i^c(x) + g f^{abc} \{\gamma_i^b(x), c_i^c(x,0)\} - g f^{abc} \int dx^- c_i^b(x^-) \partial^+ c_i^c(x^-) = j^a(x)\,. \tag{3.19}$$

In this equation $c(0)$ should be understood as

$$c_i^a(x^- = 0) \;=\; \tilde{A}(x^- = 0) \;=\; \frac{1}{2}[c_i^a(x^- = 0^+) + c_i^a(x^- = 0^-)]\,, \tag{3.20}$$

where $0^+ \equiv 0 + \epsilon$; $\quad 0^- \equiv 0 - \epsilon$; $\quad \epsilon \to 0$. This is important since c is not necessarily continuous at $x^- = 0$.

3.2 Solving the constraints

Our strategy now is the following. We should solve the two constraint equations, eqs. (3.16), (3.18) and determine the commutation relations of the non dynamical field γ. Then we must substitute it back into the Hamiltonian and express the Hamiltonian in terms of the canonical degrees of freedom $\tilde{A}$.

We will do so by expanding the constraint equations and the Hamiltonian in powers of g. When doing so we must have some knowledge of the parametric dependence of the valence charge density j on the coupling constant g. The expansion in principle can be performed for any parametric dependence. In this section we take j to be of order $1/g$ as in the JIMWLK limit, and will collect all contributions to the Hamiltonian of order $1/g$ and order 1. It turns out however that this same resummation collects the leading terms in g also for any $g \le j \le 1/g$. We will discuss this point in detail in the discussion section. Thus even though in this section we treat explicitly j as being of order $1/g$ this should not be construed as limiting our calculation to the JIMWLK limit.

Thus our aim in this section is to expand γ to $O(1)$, obtain the Hamiltonian to $O(1)$ and diagonalize this $O(1)$ Hamiltonian exactly. Further corrections to this calculation are strictly perturbative (small corrections in powers of g for any parametric dependence of j on g) and will not be considered here.

To order $1/g$ the operator γ satisfies the 'classical equations' $\gamma_i^a = b_i^a$:

$$\begin{aligned} \partial_i\, b_i^a(x) &= j^a(x)\,, \\ \partial_i\, b_j^a(x) - \partial_j\, b_i^a(x) - g f^{abc}\, b_i^b(x)\, b_j^c(x) &= 0\,. \end{aligned} \tag{3.21}$$

To this order the commutation relations are calculated as

$$\begin{aligned} \left[b_i^a(x), b_j^b(y)\right] &= \int_{z,\bar{z}} \frac{\delta b_i^a(x)}{\delta j^c(z)} \left[j^c(z), j^d(\bar{z})\right] \frac{\delta b_j^b(y)}{\delta j^d(\bar{z})} \\ &= -i\, g \int_z \left[D_i \frac{1}{\partial D}\right]^{ac}(x,z)\, f^{cde}\, j^e(z) \left[\frac{1}{D\partial} D_j\right]^{db}(z,y)\,, \end{aligned} \tag{3.22}$$

where D is the transverse covariant derivative in the 'classical' background field b: $D_i^{ab} = \partial_i \delta^{ab} - g f^{acb} b_i^c$. Eq. (3.22) is the leading order result in g. Note however that it is exact in the weak field limit, where the field b is linear in the valence charge density j.

Eq. (3.22) can be further simplified, using the identity

$$g\, f^{cde}\, j^e(z) \;=\; g\, f^{cde}\, \partial_i b_i^e(z) \;=\; -\,[\partial_i, \partial_i - D_i]^{cd} = (\partial D \;-\; D\partial)^{cd}\,. \tag{3.23}$$

Thus finally, to leading order in g

$$\begin{aligned}\left[\gamma_i^a(x), \gamma_j^b(y)\right] \;&=\; \left[b_i^a(x), b_j^b(y)\right] \;=\; -\,i\, d_{ij}^{ab}(x,y) \equiv i\, \left[D_i \frac{1}{\partial D} D_j \;-\; D_i \frac{1}{D\partial} D_j\right]^{ab}(x,y)\,,\\ \left[\gamma_i^a, \tilde{A}_j^b\right] \;&=\; [b_i^a,\, \tilde{A}_j^b] \;=\; 0\,.\end{aligned} \tag{3.24}$$

Note that although γ itself is of order $1/g$, the commutator of two γ's is of order one. It is thus clear that we will not need higher order corrections to the commutator eq. (3.24) in the $O(1)$ calculation.

To order $O(1)$ we write

$$\gamma_i^a = b_i^a + \zeta_i^a\,, \tag{3.25}$$

where ζ satisfies the equations:

$$\begin{aligned}\partial_i \zeta_i^a \;&=\; -2g f^{abc} b_i^b \tilde{A}_i^c(x^- = 0)\,,\\ D_i^{ab}\zeta_j^b - D_j^{ab}\zeta_i^b \;&=\; 0\,.\end{aligned} \tag{3.26}$$

The solution to these two equations is easily found as

$$\zeta_i^a = -2\left[D_i \frac{1}{\partial D}(\partial - D)\tilde{A}(x^- = 0)\right]^a\,, \tag{3.27}$$

where the product on the right hand side is understood in the matrix sense over all indexes (including transverse coordinates). Note that the ordering of different factors of b in eq. (3.27) is irrelevant, since the covariant derivative involves gb, and the commutator of two such factors is $O(g^2)$ and is thus of higher order than the one we need to keep.

The canonical structure to $O(1)$ follows from eqs. (3.25), (3.27)

$$\begin{aligned}\left[\gamma_i^a(x), \tilde{A}_j^b(y)\right] \;&=\; [\zeta_i^a(x), \tilde{A}_j^b(y)] = -i\epsilon(y^-)\left[D_i \frac{1}{\partial D}(\partial - D)_j\right]^{ab}(x,y)\,,\\ \left[c_i^a(x), \gamma_j^b(y)\right] \;&=\; i\epsilon(x^-)\left[\partial_i \frac{1}{D\partial} D_j - D_i \frac{1}{\partial D} D_j\right]^{ab}(x,y)\,,\\ \left[c_i^a(x), c_j^b(y)\right] \;&=\; -\frac{i}{2}\epsilon(x^- - y^-)\delta_{ij}^{ab}(x-y) - \frac{i}{2}\epsilon(x^-)\epsilon(y^-) C_{ij}^{ab}(x,y)\,,\end{aligned} \tag{3.28}$$

where for future convenience we have defined

$$C_{ij}^{ab}(x,y) = \left\{2\partial_i \frac{1}{D\partial} D_j - 2D_i \frac{1}{\partial D}\partial_j\right\}^{ab}(x,y)\,. \tag{3.29}$$

3.3 The hamiltonian and the equations of motion to O(1)

Next we express the Hamiltonian to O(1) in terms of the field c_i^a.

For the magnetic piece to $O(1)$ we have:

$$G^a_{ij}(\theta(x^-)\gamma + c) = \theta(-x^-)[\partial_i c_j - \partial_j c_i] + \theta(x^-)[D_i c_j - D_j c_i] + O(g)\,. \tag{3.30}$$

For the electric piece, using the constraint and after some algebra, we obtain:

$$\Pi^-_a = -\frac{1}{\partial^+}[D_i\partial^+ A_i - j\delta(x^-)]^a = -\left[\theta(-x^-)\partial_i c_i + \theta(x^-)D_i c_i\right]^a + O(g)\,. \tag{3.31}$$

All said and done the Hamiltonian to $O(1)$ is

$$H = -\frac{1}{2}\int dx^- d^2x \left[\theta(-x^-)\; c_i^a(x^-,x)\partial^2 c_i^a(x^-,x) + \theta(x^-)\, c_i^a(x^-,x) D^{2\,ab} c_i^b(x^-,x)\right]\,. \tag{3.32}$$

This is the Hamiltonian that we have to diagonalize. The most efficient way of doing this is first to find the complete set of solutions of classical equations of motion, and then expand the quantum field operators in the canonical creation and annihilation operators with the coefficients given by the solutions of classical equations. The classical solutions have to be properly normalized in order that the quantum field operators satisfy correct commutation relations.

We start by deriving the equations of motion. Using the commutation relations eq. (3.28) we obtain

$$\begin{aligned} i\partial^+\partial^- c_i^a(x) = [H, \partial^+ c_i^a(x)] &= \int dy^- [\partial^+ c_i^a(x), c_j^b(y)]\left[\theta(-y^-)\partial^2 + \theta(y^-)D^2\right]^{bc}_{jk} c_k^c(y) \\ &= -\frac{i}{2}[\theta(-x^-)\partial^2\delta^{ab} + \theta(x^-)D^{2\,ab}]c_i^b(x) \\ &\quad - \frac{i}{4}\delta(x^-)C^{ab}_{ij}(x,y)\int dy^-\left[-\theta(-y^-)\partial^2 + \theta(y^-)D^2\right]^{bc}_{jk} c_k^c(y^-,y)\,, \end{aligned} \tag{3.33}$$

where C^{ab}_{ij} is defined in (3.29). Integrating these equations (avoiding the singularity at $y^- = 0$) gives

$$-\frac{i}{2}\int_{-\infty}^{0^-} dy^-\partial^2 c(y) = i\int_{-\infty}^{0^-} dy^-\partial^+\partial^- c(y) = i\partial^- c(0^-)\,, \tag{3.34}$$

$$-\frac{i}{2}\int_{0^+}^{\infty} dy^- D^2 c(y) = i\int_{0^+}^{\infty} dy^-\partial^+\partial^- c(y) = -i\partial^- c(0^+)\,. \tag{3.35}$$

The last term in eq. (3.33) can be rewritten as

$$-\frac{i}{4}\delta(x^-)C^{ab}_{ij}(x,y)\int dy^-\left[-\theta(-y^-)\partial^2 + \theta(y^-)D^2\right]^{bc}_{jk} c_k^c(y) = -i\delta(x^-)C^{ab}_{ij}(x,y)\partial^- c_j^b(0)\,, \tag{3.36}$$

so that finally the equations of motion are

$$i[\partial^+ + \delta(x^-)C]^{ab}_{ij}(x,y)\partial^- c_j^b(y) = -\frac{i}{2}[\theta(-x^-)\partial^2 + \theta(x^-)D^2]^{ab}(x,y)c_i^b(y)\,. \tag{3.37}$$

Matching the discontinuity across $x^- = 0$ gives the relation

$$c_i^a(0^+, x) - c_i^a(0^-, x) = -\frac{1}{2} C_{ij}^{ab}(x, y)[c_i^b(0^+, y) + c_i^b(0^-, y)] \,. \tag{3.38}$$

The solution to the equations of motion can be written down explicitly. At negative x^- this is just a free equation, and thus the solution is a superposition of plane waves. At positive x^- the solution is again a superposition of gauge rotated plane waves. This can be written as

$$c_{i,p^-}^a(x) = \exp\{ip^- x^+\} \int d^2q \Big[\Theta(-x^-) \exp\left\{ i\frac{\partial^2}{2p^-} x^- \right\} v_{p^-q}^{i-}(x) + \Theta(x^-) \exp\left\{ i\frac{D^2}{2p^-} x^- \right\} v_{p^-q}^{i+}(x) \Big] \,. \tag{3.39}$$

Except at $x^- = 0$ this solves the equations of motion with given p^- for arbitrary $v_q^{i,\pm}$. Here q is the degeneracy index. In the free theory the index q would stand collectively for transverse momentum k, polarization index i and color "polarization index" a. In the present case q also stands for i and a as well as some continuous degeneracy. For simplicity of notation we will not differentiate between discrete and continuous parts of q. In the following, integral over q stands both for the integral over continuous part with appropriate measure as well as for summation over the rotational and color 'polarizations'.

Eq. (3.38) imposes the condition

$$v_i^{a+}(x) - v_i^{a-}(x) = -\frac{1}{2} C_{ij}^{ab}(x, y)[v_i^{b+}(y) + v_i^{b-}(y)] \,. \tag{3.40}$$

This equation can be equivalently rewritten as

$$v_i^+ = [T - L]^{ij}(t - l)^{jk} v_k^- \,, \tag{3.41}$$

where the projectors $T,\ L, t,\ l$ are defined as

$$L_{ij}^{ab} = \left[D_i \frac{1}{D^2} D_j \right]^{ab}, \qquad T_{ij}^{ab} = \delta_{ij}^{ab} - L_{ij}^{ab}; \quad l_{ij} = \partial_i \frac{1}{\partial^2} \partial_j; \quad t_{ij} = \delta_{ij} - l_{ij} \,. \tag{3.42}$$

Eq. (3.41) is solved by

$$v_i^+ = [T - L]^{ij} v_j; \qquad v_i^- = [t - l]^{ij} v_j \tag{3.43}$$

for arbitrary v_j. Thus we can write the solution eq. (3.39) in terms of one set of functions $v_{p^-q}^{ai}(x)$ as

$$c_{i,p^-}^a(x) = \exp\{ip^- x^+\} \int d^2q \left[\Theta(-x^-) \exp\left\{ i\frac{\partial^2}{2p^-} x^- \right\} [t - l]_{ij} v_{p^-q}^j(x) + \Theta(x^-) \exp\left\{ i\frac{D^2}{2p^-} x^- \right\} [T - L]_{ij} v_{p^-q}^j(x) \right] . \tag{3.44}$$

On the level of the classical solution, the normalization of the functions $v_{p^-q}^{ai}(x)$ is arbitrary. However, in order to use eq. (3.44) as the basis for expansion of the operators c in terms

of canonical creation and annihilation operators the normalization of $v^{ai}_{p^-q}(x)$ has to be determined. This will be done in the following subsection.

As a corollary to this subsection we note that the classical field b does not commute with the Hamiltonian and is therefore not constant in time. Calculating the commutator we obtain

$$\begin{aligned} i\partial^- b^a_i(x) = [H, b^a_i] &= \int dy^- [b^a_i(x), c^b_j(y)] \left[\theta(-y^-)\partial^2 + \theta(y^-)D^2\right]^{bc}_{jk} c^c_k(y) \\ &= \frac{1}{2}\, d^{ba}_{ji}(y,x) \int dy^- \left[-\theta(-y^-)\partial^2 + \theta(y^-)D^2\right]^{bc}_{jk} c^c_k(y)\,. \end{aligned} \tag{3.45}$$

Using eqs. (3.34), (3.35) this can be written as

$$i\partial^- b^a_i(x) = 2i\partial^- \left\{ D_i \frac{1}{D\partial} D_j - D_i \frac{1}{\partial D} D_j \right\}^{ab} c^b_j(0)\,. \tag{3.46}$$

This can be interpreted in the following way. Let us define the operator $\bar{b}$, so that it has the same exact matrix elements on the valence part of the Hilbert space as b, but commutes with the operators c. Then we can write

$$b^a_i = \bar{b}^a_i + 2 \left\{ D_i \frac{1}{D\partial} D_j - D_i \frac{1}{\partial D} D_j \right\}^{ab} c^b_j(0)\,, \tag{3.47}$$

and

$$\gamma^a_i = \bar{b}^a_i + 2 \left\{ D_i \frac{1}{D\partial} D_j - D_i \frac{1}{\partial D} \partial_j \right\}^{ab} c^b_j(0)\,. \tag{3.48}$$

This form will be convenient for calculating correlators of γ in the vacuum state.

3.4 Normalization of the eigenfunctions and the vacuum state

Given that the $O(1)$ Hamiltonian is quadratic, and having found the complete set of solutions of the classical equations of motion, we can find the quantum vacuum state.

The vacuum state of the Hamiltonian eq. (3.32) is the Fock vacuum of the canonical annihilation operators $\beta_{p^-,q}$ defined in terms of c by

$$\begin{aligned} c^a_i(x) = \int_0^\infty \frac{dp^-}{2\pi} \int d^2q \Big[&\Theta(-x^-) e^{i\frac{\partial^2}{2p^-}x^-} [t-l]_{ij}(x,y) v^{aj}_{p^-,q}(y) \\ &+ \Theta(x^-) e^{i\frac{D^2}{2p^-}x^-} [T-L]^{ab}_{ij}(x,y) v^{bj}_{p^-,q}(y) \Big] \beta_{p^-,q} + h.c.\,, \end{aligned} \tag{3.49}$$

where the integral over the transverse coordinate y is understood but not written explicitly. The operators β satisfy canonical commutation relations

$$[\beta_{p^-,q},\, \beta^\dagger_{p'^-,q'}] \;=\; (2\,\pi)\, \delta(p^- - p'^-)\, \delta(q - q')\,. \tag{3.50}$$

Existence of such a set of canonical operators is guaranteed if the set of solutions of the classical equation is complete and the functions v entering eq. (3.49) are properly normalized. To find the correct normalization of these functions we require that c satisfy eq. (3.28).

We concentrate on negative x^- and y^- first, so that only the first term in the sum in eq. (3.49) is important. For simplicity we suppress the color indexes and also the factor $t-l$, thus we are working in terms of v^- rather than v. Consider the commutator

$$[c^i(x), c^j(y)] = \int_0^\infty \frac{dp^-}{2\pi}\left[e^{i\left\{\frac{\partial_x^2}{2p^-}x^- - \frac{\partial_y^2}{2p^-}y^-\right\}}\int_q v^{-i}_{p^-,q}(x)v^{*-j}_{p^-,q}(y)\right. \tag{3.51}$$
$$\left. - e^{-i\left\{\frac{\partial_x^2}{2p^-}x^- - \frac{\partial_y^2}{2p^-}y^-\right\}}\int_q v^{*-i}_{p^-,q}(x)v^{-j}_{p^-,q}(y)\right]$$
$$= \int_0^\infty \frac{dp^-}{2\pi}\left[e^{i\left\{\frac{\partial_x^2}{2p^-}x^- - \frac{\partial_y^2}{2p^-}y^-\right\}}\mathcal{W}^{ij}_{p^-}(x,y) - e^{-i\left\{\frac{\partial_x^2}{2p^-}x^- - \frac{\partial_y^2}{2p^-}y^-\right\}}\mathcal{W}^{*ij}_{p^-}(x,y)\right].$$

We have defined the 'correlator matrix'

$$\mathcal{W}^{ij}_{p^-}(x,y) \;=\; \int d^2q\, v^{-i}_{p^-,q}(x)\, v^{*-j}_{p^-,q}(y)\,. \tag{3.52}$$

Note that this matrix fully determines the commutators of c, and there is no need to find the individual functions $v_{p^-,q}$. Different choices of the functions v which give the same $\mathcal{W}$ correspond to unitary rotations of the set of the canonical operators β.

To determine the correct normalization we first note that taking

$$\mathcal{W}^{ij}_{p^-}(x,y) \;=\; \delta^{ij}\,\delta^2(x-y)\,\frac{1}{2\,p^-} \tag{3.53}$$

would give canonical commutation relations for the fields c. With this expression for $\mathcal{W}$ we can change variables $p^- \to -p^-$ in the second term of eq. (3.51) to get

$$[c^i(x), c^j(y)] = \int_{-\infty}^\infty \frac{dp^-}{4\,\pi\,p^-}e^{i\{\frac{\partial^2}{2p^-}x^- - \frac{\partial^2}{2p^-}y^-\}}\delta^2(x-y)\delta^{ij} = -\frac{i}{2}\delta^{ij}\delta^2(x-y)\epsilon(x^- - y^-)\,, \tag{3.54}$$

where the last line follows by change of variables $p^- \to \partial^2/2p^-$. To get the ϵ-function in the commutator we have to regulate the singularity in $1/p^-$ in the symmetric way

$$\frac{1}{p^-} \to \left(\frac{1}{p^-}\right)^2\left[\frac{1}{\frac{1}{p^-}+i\epsilon} + \frac{1}{\frac{1}{p^-}-i\epsilon}\right]. \tag{3.55}$$

To reproduce the extra term in the commutator of c^i (the second term in the last line of eq. (3.28)) we modify the matrix $\mathcal{W}$ in the following way

$$\mathcal{W}^{ij}_{p^-}(x,y) \;=\; \frac{1}{2}\,\left(\frac{1}{p^-}\right)^2\left\{\frac{1}{\frac{1}{p^-}+i\epsilon}\left[\delta^{ij}\delta^2(x-y) + \frac{1}{2}C^{ij}(x,y)\right]\right. \tag{3.56}$$
$$\left. + \frac{1}{\frac{1}{p^-}-i\epsilon}\left[\delta^{ij}\delta^2(x-y) - \frac{1}{2}C^{ij}(x,y)\right]\right\}.$$

The new term we have added is imaginary and even with respect to $p^- \to -p^-$. Thus it is still true that the two terms in eq. (3.51) are equal. The extra term under the change of variables $p^- \to 1/p^-$ gives

$$\int d\left(\frac{1}{p^-}\right)\delta\left(\frac{1}{p^-}\right), \tag{3.57}$$

and thus generates the term in the commutator independent of x^- and y^-. The result is precisely the last term of eq. (3.28). It is a matter of some straightforward algebra to check that with $\mathcal{W}$ defined in eq. (3.56) the correct commutator of the fields c is reproduced also for other values of x^- and y^-. The following identities come handy in this calculation

$$\begin{aligned} 1-\frac{1}{2}C &= \left[1+\frac{1}{2}C\right](T-L)(t-l); \\ \left[1-\frac{1}{2}C\right](t-l) &= \left[1+\frac{1}{2}C\right](T-L)\,; \\ (t-l)C(t-l) &= -C; \\ (T-L)C(T-L) &= -C\,. \end{aligned} \tag{3.58}$$

Returning from v^- to v we conclude that the operators β, $\beta^\dagger$ in the representation eq. (3.49) have canonical commutation relations when (we use eq. (3.58))

$$\begin{aligned} \int d^2q\, v^i_{p^- q}(x)\, v^{*j}_{p^- q}(y) = \frac{1}{2}\left(\frac{1}{p^-}\right)^2 \Bigg\{ &\left[\frac{1}{\frac{1}{p^-}+i\epsilon}+\frac{1}{\frac{1}{p^-}-i\epsilon}\right]\delta^{ij}\delta^2(x-y) \\ &-\frac{1}{2}\left[\frac{1}{\frac{1}{p^-}+i\epsilon}-\frac{1}{\frac{1}{p^-}-i\epsilon}\right]C^{ij}(x,y)\Bigg\}\,. \end{aligned} \tag{3.59}$$

We thus conclude that the vacuum of the Hamiltonian eq. (3.1) to $O(1)$ is the Fock vacuum of the annihilation operators β related to the original gluon field operators through

$$\begin{aligned} \tilde{A}^a_i(x^-,x) &= \epsilon(x^-)\left[b^a_i(x) - 2D_i\frac{1}{\partial D}(\partial - D)(x,y)c(0,y)\right] + c^a_i(x^-,x) \\ &= \epsilon(x^-)\left[\bar{b}^a_i(x) + 2\left\{D_i\frac{1}{D\partial}D_j - D_i\frac{1}{\partial D}\partial_j\right\}^{ab}(x,y)c^b_j(0,y)\right] + c^a_i(x^-,x) \end{aligned} \tag{3.60}$$

with the field $c^a_i(x^-,x)$ expressed in term of β and $\beta^\dagger$ in eq. (3.49) with the normalization eq. (3.59).

This completes the diagonalization of the light cone Hamiltonian to $O(1)$.

4. The Bogoliubov operator

The calculation of the previous section can be viewed as the diagonalization of the light cone Hamiltonian. Although we have only found the vacuum state, quite generally the diagonalization is affected by the action of some unitary operator Ω. Namely for the case of a quadratic operator H

$$\Omega^\dagger H \Omega = \int_{p^-,q} p^- \beta^\dagger_{p^-,q}\beta_{p^-,q}\,. \tag{4.1}$$

The explicit knowledge of the operator Ω, or alternatively the knowledge of its action on all the degrees of freedom of the theory furnishes much more information than just the vacuum wave function, as it also in principle can give us the wave functions of excited states, which are necessary to calculate more exclusive properties than the forward scattering amplitude.

The aim of this section is to find explicitly the action of Ω on the degrees of freedom of the theory.

Part of the answer to this question is already furnished by eq. (3.60) which can be viewed as the transformation of the vector potential if we read the left hand side as $\Omega^\dagger \tilde{A} \Omega$ and the canonical operators β and $\beta^\dagger$ in c on the right hand side as the original gluon creation and annihilation operators a and $a^\dagger$. The missing piece of information is the transformation of the valence charge density. This is the question we address now.

First, it is clear from eq. (3.60) that the transformation is of the Bogoliubov form, namely

$$\Omega \equiv \mathcal{C}\,\mathcal{B} = \exp\left[E\,\tilde{A}\right]\ \exp\left[\frac{1}{2}\,\tilde{A}\,M\,\tilde{A}\right]\,, \tag{4.2}$$

where E and M are operators which depend on the charge density j but do not depend on the soft fields A. We do not indicate explicitly the indexes and coordinate dependences of E and M for simplicity. Those should be clear from the context. Here $\mathcal{C}$ is a purely coherent state operator - exponent of an operator linear in $\tilde{A}$, while $\mathcal{B}$ has no linear term in the exponent. The coherent operator is easy to find by inspection, since it is the only one that induces the shift of the soft field (the very first term in eq. (3.60)):

$$\mathcal{C} = \exp\left[2\,i\int d^2x\, b_i^a(x)\,\tilde{A}_i^a(x^- = 0, x)\right]\,. \tag{4.3}$$

The Bogoliubov part of the transformation, the operator $\mathcal{B}$ is more difficult to determine. Rather than looking for the explicit form of the operator $\mathcal{B}$ in terms of j, we will find its action on the degrees of freedom of the theory by considering sequential action of $\mathcal{C}$ and $\mathcal{B}$ on $\tilde{A}$ and matching it onto eq. (3.60).

It is important to remember that we need to know the transformation of the color charge density only to $O(g)$. Only this order contributes to the JIMWLK evolution as explained in detail in [10]. Thus we will determine the action of $\mathcal{B}$ on the fields to this order only.

We first note the following 'combinatorial' identity. For any operators O and L

$$e^{-L} O e^{L} = O + [O, L] + \frac{1}{2}[[O, L], L] + \frac{1}{3!}[[[O, L], L], L] + \dots \tag{4.4}$$

Using eq. (4.4), we have for $\mathcal{C}$ of eq. (4.3)

$$\begin{aligned}\mathcal{C}^\dagger\,\tilde{A}_i^a(x)\,\mathcal{C} \,=\, & \tilde{A}_i^a(x) \;+\; \epsilon(x^-)\,b_i^a(x) \;+\; \epsilon(x^-)\int_y d_{ij}^{ab}(x,y)\,\tilde{A}_j^b(y^- = 0, y) \\ & + \frac{2\,i}{3}\,\epsilon(x^-)\int_{y,z}[d_{ij}^{ab}(x,y),\, b_k^c(z)]\ \tilde{A}_j^b(y^- = 0, y)\ \tilde{A}_k^c(z^- = 0, z)\,,\end{aligned} \tag{4.5}$$

$$\begin{aligned}\mathcal{C}^\dagger\, j^a(x)\,\mathcal{C} \,=\, & j^a(x) \;+\; 2\int_y \left\{\left(\partial D\frac{1}{D\partial} - 1\right) D_j\right\}^{ab}(x,y)\ \tilde{A}_j^b(y^- = 0, y) \\ & + 2\,i\int_{y,z}\left[\left\{\left(\partial D\frac{1}{D\partial} - 1\right) D_j\right\}^{ab}(x,y), b_k^c(z)\right]\tilde{A}_j^b(y^- = 0, y)\tilde{A}_k^c(z^- = 0, z)\,.\end{aligned} \tag{4.6}$$

To find the action of the Bogoliubov operator, we imagine diagonalizing the Hamiltonian first by acting with $\mathcal{C}$ and then subsequently acting with $\mathcal{B}$. Transforming the Hamiltonian eq. (3.1) with $\mathcal{C}$ obviously leads to

$$\mathcal{C}^\dagger\, H[\tilde{A},\, j]\, \mathcal{C} \;\equiv\; H'[\tilde{A},\, j] \;=\; H[\mathcal{C}^\dagger\, \tilde{A}\, \mathcal{C},\, \mathcal{C}^\dagger\, j\, \mathcal{C}]\,. \tag{4.7}$$

It is straightforward to see using the expression for the transformed fields eqs. (4.5), (4.6), that if we substitute for $\tilde{A}$ in the function H' the following expression

$$\tilde{A}^a_i(x) \to c^a_i(x) \;+\; \epsilon(x^-)\, \Delta^{ab}_{ij}(x,y)\; c^b_j(y^- = 0, y)\,, \tag{4.8}$$

with

$$\Delta^{ab}_{ij}(x,y) = \left\{ D_i \frac{1}{\partial D} D_j + D_i \frac{1}{D\partial} D_j - 2 D_i \frac{1}{\partial D} \partial_j \right\}^{ab} (x,y)\,, \tag{4.9}$$

we obtain to $O(1)$ precisely eq. (3.32). This substitution should be equivalent to the action of the Bogoliubov operator

$$\mathcal{B}^\dagger\, H'[\tilde{A},\, j]\, \mathcal{B} \;=\; H'[\mathcal{B}^\dagger\, \tilde{A}\, \mathcal{B},\, \mathcal{B}^\dagger\, j\, \mathcal{B}] \;\equiv\; H''[\tilde{A},\, j]\,. \tag{4.10}$$

In other words, up to (and including) $O(g)$ terms the action of the Bogoliubov operator $\mathcal{B}$ on the field $\tilde{A}$ is

$$\mathcal{A}^a_i(x,j) \;\equiv\; \mathcal{B}^\dagger\, \tilde{A}^a_i(x)\, \mathcal{B} \;=\; c^a_i(x) \;+\; \epsilon(x^-)\, \Delta^{ab}_{ij}(x,y)\; c^b_j(y^- = 0, y)\,, \tag{4.11}$$

where the field c on the r.h.s. is understood as expressed in terms of the canonical creation and annihilation operators a and $a^\dagger$ (rather than β and $\beta^\dagger$)[2].

Our aim is now to find the transformation of the color charge density j^a under the Bogoliubov transformation which induces eq. (4.11). This is indeed possible, even though we do not know the explicit form of the operator $\mathcal{B}$ itself in terms of the fundamental fields. The key is given by the following chain of arguments.

Consider a general Bogoliubov operator of the form

$$\mathcal{B} = \exp\left[\frac{1}{2} \tilde{A}_i\, M_{ij}\, \tilde{A}_j \right]\,. \tag{4.12}$$

Here we denote all indexes/coordinates of the field A by a single index i. The fields A are assumed to satisfy the commutation relation

$$[\tilde{A}_i,\, \tilde{A}_j] \;=\; P_{ij} \tag{4.13}$$

with some matrix P. Quite generally the matrix M is symmetric and anti hermitian, while P is antisymmetric. The matrix M depends on the charge density and the coupling constant only through the combination $g\, j$.

[2]We note that strictly speaking to make this identification we should also substitute into H' the transformed expression for j in eq. (4.10), which we do not know at this point. However as we will see below and is simple to understand by straightforward counting of powers of g, the operator B induces transformation of j only to order g. Since we only need the Hamiltonian to $O(1)$ it is therefore perfectly consistent to keep j unchanged in H' eq. (4.10) for the purpose of the identification of the Bogoliubov transformation of $\tilde{A}$.

Consider the transformation

$$\mathcal{A}_k \equiv \mathcal{B}^\dagger \, \tilde{A}_k \, \mathcal{B} = \tilde{A}_k + (PM\tilde{A})_k + \frac{1}{2}(PMPM\tilde{A})_k + \frac{1}{3!}(PMPMPM\tilde{A})_k + \dots$$
$$= [e^{PM}]_{kl} \, \tilde{A}_l \, . \tag{4.14}$$

Here we have used the identity eq. (4.4). Also, consistently with our counting of powers of the coupling constant we have neglected all and any terms involving commutators of gj which enter into M, since each such commutator brings a power g^2.

Now to order $O(g)$ we have

$$[j^a, M_{ij}] = igf^{abc} j^c \frac{\partial M_{ij}}{\delta j^b} \, . \tag{4.15}$$

Thus consider the transformation of $j^a(x)$ induced by the action of $\mathcal{B}$ in eq. (4.12):

$$\mathcal{B}^\dagger j^a \mathcal{B} = j^a + \frac{i}{2} g f^{abc} j^c \Bigg\{ \tilde{A} \frac{\delta M}{\delta j^b} \tilde{A} + \frac{1}{2} \tilde{A} \left(\frac{\delta M}{\delta j^b} PM - MP \frac{\delta M}{\delta j^b} \right) \tilde{A} \tag{4.16}$$
$$+ \frac{1}{3!} \tilde{A} \left(\frac{\delta M}{\delta j^b} PMPM + MPMP \frac{\delta M}{\delta j^b} - 2MP \frac{\delta M}{\delta j^b} PM \right) \tilde{A} + \dots \Bigg\} \, .$$

Here again we neglected all commutators of gj in M beyond the first term, as they are all higher order in g. The negative signs come from transposing the antisymmetric matrix P. We can now check explicitly that eq. (4.16) is expansion in powers of M of the following expression

$$j^a + \frac{i}{2} g f^{abc} j^c \mathcal{A} P^{-1} \frac{\delta \mathcal{A}}{\delta j^b} \tag{4.17}$$

with $\mathcal{A}$ defined in eq. (4.14). Remembering that in our case $P = \frac{i}{2}\epsilon(x^- - y^-)$ whose inverse is $-2i\partial^+$, and restoring all the indexes and coordinate dependences we obtain

$$\bar{j}^a(x) \equiv \mathcal{B}^\dagger j^a(x) \mathcal{B} = j^a(x) + g f^{acd} j^d(x) \int dy^- d^2y \, \partial^+ \mathcal{A}^b_j(y^-, y) \frac{\delta \mathcal{A}^b_j(y^-, y)}{\delta j^c(x)} \tag{4.18}$$

with $\mathcal{A}$ given in eq. (4.11).

An equivalent way of obtaining this result is to require that the transformed fields satisfy the same commutation relations as the non transformed ones, the transformation being unitary. Using the explicitly known commutator of the field c one can easily show that

$$\left[\mathcal{A}^a_i(x^-, x), \mathcal{A}^b_j(y^-, y) \right] = -\frac{i}{2} \epsilon(x^- - y^-) \delta^{ab}_{ij}(x - y) \, , \tag{4.19}$$
$$\left[j^a(x), \mathcal{A}^b_j(y^-, y) \right] = \int_z [j^a(x), j^c(z)] \frac{\delta \mathcal{A}^b_j(y^-, y)}{\delta j^c(z)} = igf^{acd} j^d(x) \frac{\delta \mathcal{A}^b_j(y^-, y)}{\delta j^c(x)} \, .$$

In this expression we should understand $\mathcal{A}$ as a function of j at fixed a. It is easy to check that with the transformation eq. (4.18) to $O(g)$

$$[\bar{j}^a(x), \mathcal{A}^b_j(y^-, y)] = 0 \, . \tag{4.20}$$

This is straightforward after noticing that the last term in eq. (4.18) can be written as

$$g\, f^{acd}\, j^d(x) \int dy^- d^2y \partial^+ \mathcal{A}^b_j(y^-, y) \frac{\delta \mathcal{A}^b_j(y^-, y)}{\delta j^c(x)} = \int dy^- d^2y\, d^2z\, \mathcal{A}^b_j(y^-, y)[j^a(x), j^c(z)] \frac{\delta \mathcal{A}^b_j(y^-, y)}{\delta j^c(z)}\,. \tag{4.21}$$

Therefore we conclude that the transformation eqs. (4.11), (4.18) does indeed preserve canonical commutation relations of the fields.

We can now put all the elements together and write down the transformation that the operator Ω induces on the fields:

$$\begin{aligned}
\Omega^\dagger \tilde{A}^a_i(x^-, x)\Omega = {} & c^a_i(x^-, x) + \epsilon(x^-) \left[b^a_i(x) + 2 \int_y \left\{ D_i \frac{1}{D\partial} D_j - D_i \frac{1}{\partial D} \partial_j \right\}^{ab} (x, y) c^b_j(0, y) \right] \\
& + \epsilon(x^-) \int_{y,z} \left\{ g \left[D_i \frac{1}{\partial D} \right]^{ab} (xz) f^{bcd} j^d(z) \int dy^- \partial^+ \mathcal{A}^e_j(y^-, y) \frac{\delta \mathcal{A}^e_j(y^-, y)}{\delta j^c(z)} \right. \\
& \left. + \frac{2i}{3} [d^{ab}_{ij}(x, y), b^c_k(z)] c^b_j(0, y) c^c_k(0, z) \right\}\,, \qquad (4.22) \\
\Omega^\dagger j^a(x)\Omega = {} & j^a(x) + 2 \int_y \left\{ \left(\partial D \frac{1}{D\partial} - 1 \right) D_j \right\}^{ab} (x, y) c^b_j(0, y) \\
& + g f^{acd} j^d(x) \int dy^- d^2y \partial^+ \mathcal{A}^b_j(y^-, y) \frac{\delta \mathcal{A}^b_j(y^-, y)}{\delta j^c(x)} \\
& + 2i \int_{y,z} \left[\left\{ \left(\partial D \frac{1}{D\partial} - 1 \right) D_j \right\}^{ab} (x, y), b^c_k(z) \right] c^b_j(0, y) c^c_k(0, z)\,. \qquad (4.23)
\end{aligned}$$

Here $\mathcal{A}$ is given by eq. (4.11) and the field c is understood as expressed in terms the canonical creation and annihilation operators a and $a^\dagger$ as in eq. (3.49). The first line of eq. (4.22) coincides with eq. (3.60). The second and third lines are the $O(g)$ terms. They are given here for completeness even though they do not contribute in the calculation of the previous section and also do not contribute to the transformation of the soft color charge density eq. (4.25).

Eqs. (4.22), (4.23) are the main result of this section. They give the explicit action of the diagonalizing operator Ω on the fundamental degrees of freedom of the theory.

Finally, for completeness we give the expression for the transformation of the total charge density. This is the observable directly relevant for the calculation of the scattering amplitude. It includes the contribution of the valence and the soft modes

$$J^a(x) = j^a(x) + g f^{abc} \int dx^- \tilde{A}^b_i(x) \partial^+ \tilde{A}^c_i(x)\,. \tag{4.24}$$

Collecting the formulae given above we find

$$\Omega^\dagger J^a(x)\Omega = j^a(x) + \delta_1 j^a(x) + \delta_2 j^a(x)\,, \tag{4.25}$$

with

$$\delta_1 j^a(x) = 2\left[\partial D\frac{1}{D\partial}D_j - \partial_j\right]^{ab}(x,y)c_j^b(0,y)\,, \tag{4.26}$$

$$\begin{aligned}\delta_2 j^a(x) = {} & 2gi\left[f^{aed}j^d(x)e_j^{be}(y,x), b_k^c(z)\right]c_j^b(0,y)c_k^c(0,z)\\ & +gf^{abc}\int dx^-\mathcal{A}_i^b(x)\partial^+\mathcal{A}_i^c(x)\\ & +gf^{acd}j^d(x)\int dy^-\partial^+\mathcal{A}_j^b(y^-,y)\frac{\delta\mathcal{A}_j^b(y^-,y)}{\delta j^c(x)}\,,\end{aligned} \tag{4.27}$$

with $\mathcal{A}$ given by eq. (4.11). Here

$$e_i^{ab}(x,y) = \frac{\delta b_i^a(x)}{\delta j^b(y)} = \left[D_i\frac{1}{\partial D}\right]^{ab}(x,y)\,. \tag{4.28}$$

As a consistency check with the calculation of the previous section we note that eq. (4.26) coincides with the divergence of eq. (3.48).

5. Reproducing JIMWLK/KLWMIJ

As a cross check on our derivation we reproduce in this section the two known limits of the high energy evolution - the JIMWLK evolution equation (the high density limit) and the KLWMIJ evolution equation (the low density limit) .

5.1 The JIMWLK kernel

Under boost the color charge density j transforms into J of eq. (4.25). To derive the evolution of the functional W we have to calculate the correlation functions of J over the soft gluon vacuum, that is over the Fock vacuum of operators β. In the JIMWLK limit it is only necessary to know two correlators,

$$\chi^{ab}(x,y) \equiv \lim_{\Delta Y\to 0}\frac{\langle 0_\beta|\,\delta_1 j^a(x)\delta_1 j^b(y)\,|0_\beta\rangle}{\Delta Y}, \qquad \sigma^a(x) = \lim_{\Delta y\to 0}\frac{\langle 0_\beta|\,\delta j_2^a(x)\,|0_\beta\rangle}{\Delta Y}\,, \tag{5.1}$$

since $\delta_1 j \sim gj$ and $\delta_2 j \sim g^2 j$, and so only these two correlators contribute to the evolution of $\langle j(x_1)\dots j(x_n)\rangle$ to relative order g^2. In fact our task is somewhat easier, since we can avoid the calculation of $\langle\delta j_2^a(x)\rangle$ using the following argument. In terms of χ and σ the evolution kernel has the form

$$H^{JIMWLK} = \frac{1}{2}\chi^{ab}(x,y)\frac{\delta}{\delta j^a(x)}\frac{\delta}{\delta j^b(y)} + \sigma^a(x)\frac{\delta}{\delta j^a(x)}\,. \tag{5.2}$$

However it was proved in [14] that the evolution kernel has to be a Hermitian operator (on the space of functions of j). In conjunction with the fact that $\sigma^a(x)$ is real, since it is a diagonal matrix element of an Hermitian operator (on the QCD Hilbert space), it means that σ is rigidly related to χ so that the evolution kernel is

$$H^{JIMWLK} = \frac{1}{2}\frac{\delta}{\delta j^a(x)}\chi^{ab}(x,y)\frac{\delta}{\delta j^b(y)}\,. \tag{5.3}$$

This property of the JIMWLK kernel is of course well known and has been first noted by Weigert in the last reference in [10]. Thus our task is first to calculate $\chi^{ab}(x,y)$ and then to show that the resulting evolution equation is equivalent to the standard form of JIMWLK which involves derivatives with respect to the unitary matrices U rather than with respect to the charge density j.

We start with the calculation of χ, defined as eq. (5.1). In preparation we calculate

$$\begin{aligned}\frac{1}{\Delta Y}\,\langle 0_\beta|\, c_i^a(0,x)c_j^b(0,y)\,|0_\beta\rangle &= \frac{1}{8\,\Delta Y}\int\frac{dp^-}{2\pi p^-}[t-l+T-L][t-l+T-L]_{ij}^{ab}(x,y)\\ &= \frac{1}{4\,\pi}\,[1\,-\,l\,-\,L\,+\,l\,L\,+\,L\,l]_{ij}^{ab}(x,y)\,. \end{aligned}\tag{5.4}$$

Using eq. (4.26) we then find

$$\begin{aligned}\chi^{ab}(x,y) &\equiv \frac{\langle\delta_1 j^a(x)\delta_1 j^b(y)\rangle}{\Delta Y}\\ &= \frac{4}{\Delta Y}\left[\partial D\frac{1}{D\partial}D_i-\partial_i\right]^{ac}(x,u)\langle c_i^c(0,u)c_j^d(0,v)\rangle\left[\partial_j-D_j\frac{1}{\partial D}D\partial\right]^{db}(v,y)\\ &= \frac{1}{\pi}\,\left\{\partial D\left[\frac{1}{\partial^2}+\frac{1}{D^2}-\frac{1}{\partial^2}\partial D\frac{1}{D^2}-\frac{1}{D^2}D\partial\frac{1}{\partial^2}\right]D\partial\right\}^{ab}(x,y)\,. \end{aligned}\tag{5.5}$$

5.2 From j to U

To get the evolution equation in the familiar JIMWLK form we need to change variables from j to the single gluon scattering matrix U. The matrix U is defined as the matrix of the two dimensional gauge transformation which transforms the 'classical field' b to zero value [10]

$$U^{ab}(x)=\left\{\mathcal{P}\exp\left[ig\int_C dy_i T^c b_i^c(y)\right]\right\}^{ab}\,,\tag{5.6}$$

where the contour C starts at some fixed point at infinity in the transverse plane and ends at the point x. The matrix U does not depend on the curve C but only on its end point, since the field b is two dimensionally a pure gauge. Using this definition we have

$$\begin{aligned}\frac{\delta U^{ab}(x)}{\delta j^c(z)} &= g\int_C dy_i\left[U(x)U^\dagger(y)T^d\frac{\delta b_i^d(y)}{\delta j^c(z)}U(y)\right]^{ab}\\ &= g\int_C dy_i\left[U(x)U^\dagger(y)T^dU(y)\right]^{ab}\left[D_i\frac{1}{\partial D}\right]^{dc}(y,z)\,. \end{aligned}\tag{5.7}$$

Now we use the identity

$$[U^\dagger(y)T^dU(y)]^{ab}=T^c_{ab}U^{cd}(y)\,.\tag{5.8}$$

Substituting this into eq. (5.7), and using the fact that $\int_c dy_i\partial_i F(y)=F(x)$ we find

$$\frac{\delta U^{ab}(x)}{\delta j^c(z)}=g\left[UT^b\frac{1}{\partial D}\right]^{ac}(x,z)\,.\tag{5.9}$$

This makes it possible to rewrite the real part of the JIMWLK kernel in the following form

$$\int_{x,y} \chi^{ab}(x,y)\frac{\delta}{\delta j^a(x)}\frac{\delta}{\delta j^b(y)} = \frac{g^2}{\pi}\int_{x,y}\frac{\delta}{\delta U^{ab}(x)}\frac{\delta}{\delta U^{cd}(y)}[U(x)T^b]^{al}[U(y)T^d]^{cm} \times \left[\frac{1}{\partial^2}+\frac{1}{D^2}-\frac{1}{\partial^2}\partial D\frac{1}{D^2}-\frac{1}{D^2}D\partial\frac{1}{\partial^2}\right]^{lm}(x,y)\,. \tag{5.10}$$

Now remember that

$$\frac{\delta}{\delta U^{ab}(x)}[U(x)T^b]^{al} = -\mathrm{Tr}\left[\frac{\delta}{\delta U^\dagger(x)}U(x)T^l\right] = -J^l_R\,, \tag{5.11}$$

where J_R is the operator of right rotation on matrix U. We also note that

$$\partial_i\frac{1}{\partial^2}(x,y) = \frac{1}{2\pi}\frac{x_i-y_i}{(x-y)^2}; \qquad D_i\frac{1}{D^2}(x,y) = \frac{1}{2\pi}U^\dagger(x)\frac{x_i-y_i}{(x-y)^2}U(y)\,. \tag{5.12}$$

Now, using eq. (5.3) we can write the complete kernel as

$$H^{JIMWLK} = -\frac{\alpha_s}{2\pi^2}\int_{x,y,z}\frac{(x-z)_i(y-z)_i}{(x-z)^2(y-z)^2}\Big[J^a_L(x)J^a_L(y)+J^a_R(x)J^a_R(y) -2J^a_L(x)U^{ab}(z)J^b_R(y)\Big] \tag{5.13}$$

with $J^a_L(x) = U^{ab}(x)J^b_R(x)$. This is by now one of the standard forms of the JIMWLK kernel, see [31].

5.3 The KLWMIJ evolution

Although our derivation has been formally in the high density limit, as we noted in the introduction and as we explain in the next section the result eqs. (4.25), (4.26), (4.27) is in fact valid for all physically interesting situations, including the low density case $j = O(g)$. For the low density case we have to reproduce the KLWMIJ evolution equation [24, 25]. It is easy to see that this is indeed the case. Examining the action of the Bogoliubov operator $\mathcal{B}$ on the fields, we see that in the weak field limit they are sub leading. The shift of the vector potential affected by the coherent part of the operator $\mathcal{C}$ is of order $b \sim j$, while any correction introduced by $\mathcal{B}$ is of order $gb \sim gj$. This is also true in the strong field case, however for $j \sim 1/g$ the corrections due to $\mathcal{B}$ are $O(1)$ and therefore could not be neglected. In the weak field case these are not only sub leading but also genuinely perturbative! We can therefore neglect the action of $\mathcal{B}$ altogether. Thus in this limit the operator Ω reduces to the coherent operator $\mathcal{C}$ with the 'classical field' b given by the leading order perturbative expression. This is precisely the operator that was used in [24] to derive KLWMIJ evolution equation. Obviously, repeating the same derivation we obtain the same result.

One important thing to be noted here is, that in order to derive KLWMIJ we are not allowed to expand the correlators of the transformed charge density to first order in $\delta_2 j$ eq. (4.27) as is done to derive JIMWLK equation. The reason is very simple. When $j \sim O(g)$, the second term on the r.h.s. of eq. (4.27) is of the same order as j itself . Therefore its contribution to the evolved correlators of J has to be resumed to all orders.

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This is indeed what is done in the derivation of [24]. It is the resummation to all orders in $f^{abc}\int dx^- \tilde{A}^b(x)\partial^+\tilde{A}^c(x)$ that is responsible for the appearance of the 'dual Wilson line factor'

$$R(x) \equiv \exp\left[T^a \frac{\delta}{\delta j^a(x)}\right]$$

in the KLWMIJ evolution equation [24].

6. Discussion

In this paper we have carried through the diagonalization of the QCD light cone Hamiltonian in the presence of a valence charge density j. We found that for large valence charge density, to $O(1)$ the vacuum is the Bogoliubov transform of the free gluon vacuum. We have also found the action of the Bogoliubov operator on the dynamical variables of QCD including the valence color charge density. The evolution of hadronic wave function to high energy increases the longitudinal momentum of the gluons in this state. Thus more energetic gluons scatter on the target leading to the evolution of the hadronic scattering matrix.

6.1 The calculation is valid for any j

We have shown that when the valence charge density is large $j \sim O(1/g)$ the wave function we found leads to the JIMWLK evolution equation. However, our calculation itself is valid beyond the high density limit, and does in fact give the leading solution of the light cone Hamiltonian for all physically interesting magnitudes of the color charge density $j \sim O(g^n)$; $-1 \leq n \leq 1$. The precise statement is that relative corrections to the solution we have given here are proportional to a positive power of g at any interesting value of the valence charge density.

To see this, recall that the basis of our approach was the perturbative solution of eqs. (3.16), (3.19). We have solved eq. (3.16) exactly, while eq. (3.19) was solved treating the second, third and fourth terms on the left hand side (l.h.s.) as perturbations. The solution of this pair of equations to leading order in the coupling constant is always of order $\gamma = O(j) = O(g^n)$ for $-1 \leq n \leq 1$. The magnitude of corrections is easy to estimate. Since, by definition the field c is $O(1)$, we have

$$\begin{aligned} gf^{abc}[\gamma_i^b(x), \gamma_i^c(x)] \sim g\left(\frac{\delta\gamma}{\delta j}\right)^2 [j,j] \sim g^2 j &= O(g^{2+n})\,, \\ gf^{abc}\{\gamma_i^b(x), c_i^c(x,0)\} &= O(g^{1+n})\,, \\ gf^{abc}\int dx^- c_i^b(x^-)\partial^+ c_i^c(x^-) &= O(g)\,. \end{aligned} \tag{6.1}$$

The first term is always smaller than the second. It always scales as a positive power of g and therefore can always be treated perturbatively.

The second term is also small as long as $n \neq -1$. It is a factor g smaller than the zeroth order solution and thus again can be safely treated perturbatively. For $n \neq -1$ it can be neglected since its magnitude is a positive power of g. The case $n = -1$ is a bit

different, since then this term is $O(1)$ and so has to be taken into account, which is what we did above.

Finally the third term is always $O(g)$. It can be neglected for all $n \neq 1$. For $n = 1$ this term is of the same magnitude as j and thus it may seem that it has to be taken into account already in the leading order. However this is not the case for the following reason. The vacuum of the Hamiltonian of the field c at $j \sim O(g)$ is a free vacuum. This state is annihilated by the 'soft' color charge density operator $f^{abc} \int dx^- c^b(x) \partial^+ c^c(x)$. Thus this operator only gives non vanishing contribution to γ in the sub leading order in g, where the vacuum is not a free vacuum anymore. Thus we see that for all $-1 \leq n \leq 1$ our solution of eqs. (3.16), (3.19) keeps the leading terms and for $n = -1$ also the important sub leading term of $O(1)$. The terms that we omit are not only suppressed by a positive power of g relative to the terms we keep, but also vanish in the limit $g \to 0$ at any j.

To reiterate, our procedure keeps all the terms that are important for physically interesting values of the color charge density. This is not to say that our solution can be considered as a leading order of some expansion which has the same expansion parameter for all n. The corrections to the leading term may have different magnitude for different values of n, and thus the properties of the expansion are different at different values of n. At this point however we are not interested in the sub dominant corrections and will not discuss this issue any further.

6.2 What JIMWLK misses?

We want now to return to the point briefly mentioned at the end of the previous section. Even though our diagonalization procedure and the solution for the vacuum wave function is valid for any j, the derivation of the evolution equation *for the scattering amplitude* involves one extra step, and that is adding the charge density of the soft gluons to the valence charge density. For $n \neq 1$ this is a perturbative proposition, since the soft gluon charge density is parametrically smaller than j itself. Thus for the derivation of the JIMWLK evolution equation one expands to first order in the soft gluon color charge density, the second term on the r.h.s. of eq. (4.27). For the KLWMIJ evolution on the other hand all powers of the soft gluon color charge density are resummed. The addition of the soft gluon charge density is achieved by acting on any observable function of j by the shift operator of the form

$$\hat{R}_a \, = \, \exp\left[\int d^2x \, j^a_{\rm soft}(x) \, \frac{\delta}{\delta j^a(x)}\right] . \tag{6.2}$$

In the KLWMIJ limit only one gluon is produced at one step of the evolution with probability of order α_s, and thus $j^a_{\rm soft}(x) = T^a$ when acting on the component of the wave function which contains this extra gluon. The phase factor of eq. (6.2) therefore simply becomes the dual Wilson loop R. In the general case however the action of the Bogoliubov operator $\mathcal{B}$ produces an arbitrary number of gluons. For $j \sim 1/g$ the number of gluons of order $O(1)$ is produced with probability of $O(1)$, while with probability $O(g)$ one can produce $O(1/g)$ extra gluons. The phase factor becomes a product of dual Wilson loops $R(x_1) \dots R(x_n)$ when acting on a component of the wave function with n extra gluons. Now the JIMWLK equation is valid when a large dense target scatters off a small perturbative projectile.

In this situation each gluon in the target wave function undergoes only a small number of scatterings on the projectile. In fact the leading order scattering on a small target is only due to two gluon exchange. This corresponds to expansion of each dual Wilson loop factor R to second order in $\delta/\delta j$. It is also true that in this situation it is unlikely that two or more produced gluons scatter simultaneously. Indeed the expansion of eq. (6.2) is equivalent to approximating the scattering amplitude of the configuration of n produced gluons by the sum of the individual scattering amplitudes.

Recall that in calculating the evolution of any correlation function of j in the JIMWLK approximation, we only keep terms of the first order in $\delta_2 j$ and of the second order in $\delta_1 j$. Thus the correction to the scattering matrix $S = \exp\{i\, j\, \alpha_T\}$ due to the evolution is at most of second order in the target field α_T. This is another way of saying that the whole system of soft gluons produced in one step of the evolution scatters on the target only via the two gluon exchange. To be a little more precise we have to remember that while calculating the evolution of the scattering matrix, a factor R accompanies not only each soft emitted gluon but also every factor of j in the operator Ω. The eikonal scattering matrix of the projectile wave function on the target field α_T is given by (we drop the transverse coordinate dependence to simplify the notations)

$$\begin{aligned}\Sigma^P &= \langle\Psi[j]|\Omega^\dagger[j,a,a^\dagger]e^{i(j^a+j^a_{\rm soft})\alpha^a_T}\Omega[j,a,a^\dagger]|\Psi[j]\rangle \\ &= \langle\Psi[j]|\Omega^\dagger[j,a]e^{ij^a\alpha^a_T}\Omega[j,Ra,Ra^\dagger]|\Psi[j]\rangle \\ &= \langle\Psi[j]|\Omega^\dagger[j,a]\Omega[Rj,Ra,Ra^\dagger]e^{ij^a\alpha^a_T}|\Psi[j]\rangle\,, \end{aligned} \tag{6.3}$$

where $|\Psi[j]\rangle$ is the valence wave function and the functional derivatives in R act only on the eikonal factor $e^{i\,j^a\,\alpha^a_T}$. The first equality is the reflection of the fact that multiplying every soft gluon creation operator by R is equivalent to shifting the charge density j by the charge density of this soft gluon. The second equality follows from commuting of the operators j in Ω with those in the eikonal factor as explained in detail in [32]. Since every R becomes an eikonal factor after acting on $e^{i\,j^a\,\alpha^a_T}$, multiplication of j by R in the second line in eq. (6.3) physically corresponds to the effect of scattering of the valence charges involved in the emission of soft gluons. Thus the expansion of all the factors of R to second order in $\frac{\delta}{\delta j}$ approximates the interaction of the whole system of soft gluons emitted in one step of the evolution plus the valence charges involved in their emission (in the following we will refer to this system as "soft gluons" to avoid lengthy and wordy descriptions), with the target by a two gluon exchange.

The JIMWLK evolution therefore does not take into account multiple scattering corrections to the amplitude due to simultaneous scattering of two soft gluons emitted in the same step of the evolution. This is not to say that the JIMWLK evolution does not allow any multiple scattering corrections at all. In particular the probability that a soft gluon scatters simultaneously with some of the valence gluons not participating in its emission, is accounted for. We will refer to these multiple scattering events as "long range multiple scatterings" to emphasize the fact that the two objects that scatter simultaneously have vastly different rapidities. This as opposed to "short range multiple scatterings" where both objects have similar rapidity, which are taken into account by the KLWMIJ evolution.

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It is these long range multiple scattering corrections that unitarize the scattering amplitude in the JIMWLK approximation. If no multiple scattering corrections where included at all, the amplitude would not unitarize even though the coherent effects in the wave function are taken into account exactly. Recall that the charge density itself does not saturate even in the dense regime, although its growth with rapidity is much slower than in the BFKL approximation [27]. In particular in the BFKL (or equivalently KLWMIJ) limit the color charge density grows exponentially with rapidity

$$j^2(Y) \propto j(0)^2 e^{\omega Y} \,, \tag{6.4}$$

while in the "saturated regime" the growth is a random walk process and thus [27]

$$j^2 \propto j(0)^2 + kY \,. \tag{6.5}$$

Since the charge density does not stop growing even in the saturated regime, the scattering amplitude would not saturate if no multiple scattering corrections are taken into account. It is thus precisely the long range multiple scattering corrections that stop the scattering amplitude from growing beyond one in the JIMWLK approximation.

Eqs. (6.4) and (6.5) in fact clearly indicate that the short range multiple scatterings are dominant in the KLWMIJ regime while the long range multiple scatterings are dominant in the JIMWLK regime. Consider first the evolution of a dilute projectile (KLWMIJ evolution). According to eq. (6.4) the color charge density grows exponentially fast and is always (at large enough rapidity) dominated by gluons created in the last rapidity interval of the size $\Delta Y \approx \frac{1}{\omega}$. Thus the dominant multiple scattering effects indeed are due to the simultaneous scattering of two or more gluons at approximately the same rapidity - the "short range multiple scatterings". On the other hand in the JIMWLK regime where eq. (6.5) is valid, the color charge density is uniformly distributed in rapidity. Thus clearly the dominant multiple scattering corrections are due to simultaneous scatterings of gluons at far away rapidities - the "'long range multiple scatterings"'.

We thus see explicitly that while the KLWMIJ evolution takes into account all multiple scattering effects but does not include nonlinearities in the evolution of the wave function, the JIMWLK evolution fails to take account of the short range multiple scattering corrections to the amplitude.

6.3 Short range multiple scattering and the dipole-dipole amplitude

In relation to the preceding discussion we want to comment briefly on one aspect of the Pomeron loop correction to the JIMWLK evolution. In particular recently much attention has been devoted to scattering of two unequal size dipoles. In this context there has been much discussion of the effects of discreteness and fluctuations in the target (taken to be the larger of the two dipoles) wave function [22]. Although our derivation does not indicate any reason to expect that discreteness and/or fluctuations are particularly important, it does indeed show that the application of the JIMWLK or KLWMIJ evolution to the target wave function in the dipole-dipole scattering is flawed. The reason KLWMIJ evolution fails is obvious. Starting with a dilute single dipole target initial stages of the evolution are indeed

well described by the KLWMIJ equation. However when the density in the target wave function reaches large value $j \propto 1/g$ neglecting high density effects in the evolution of the wave function is not permissible. This density is parametrically the same as that for which the scattering amplitude becomes of order one, and it is therefore also the same density at which the effect of the multiple scattering corrections in KLWMIJ evolution becomes significant. This has been recognized in the literature for a long time, see for example fifth paper in [10].

On the other hand the reason for the failure of JIMWLK is somewhat more subtle. Again starting with the dilute target one can initially evolve it with the JIMWLK equation. The multiple scattering effects are not important as long as the density is small, and thus the use of JIMWLK in the dilute regime is as good as the use of KLWMIJ. When the density is parametrically large again the JIMWLK evolution is valid, since the evolution of the wave function is accounted for appropriately and the long range multiple scattering corrections dominate at high density. It might therefore seem that JIMWLK equation can be used all the way through in this situation. This is however not the case. The reason it fails is that there is a range of rapidities in the evolution when the density is already not very small but the rate of growth is still large. This happens just before the saturation is reached. Since the density in this range of rapidities still grows exponentially, the short range multiple scattering effects dominate. Those are not included in JIMWLK evolution, and thus the rate of growth of the amplitude is overestimated. Note that if already at the initial rapidity the density in the target wave function is large (e.g. for a heavy nucleus) there is no rapidity window in which the short range multiple scatterings dominate, and thus JIMWLK evolution is valid.

We close the discussion by stressing that the calculation of the wave function given in the present paper is the correct starting point for derivation of the complete evolution equation which takes into account all relevant Pomeron loop effects. The validity of such equation will not be limited to the process of collision of two small objects, but more interestingly to the situation where two colliding objects are large. The use of JIMWLK evolution in this case is not justified since the soft gluons produced in the wave function can multiply rescatter on the large target field.

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