Dynamic Systems and Subadditive Functionals

by

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Abstract

Consider a problem where a number of dynamic systems are required to travel between points in minimum time. The study of this problem is traditionally divided into two parts: A combinatorial part that assigns points to every dynamic system and assigns the order of the traversal of the points, and a path planning part that produces the appropriate control for the dynamic systems to allow them to travel between the points. The first part of the problem is usually studied without consideration for the dynamic constraints of the systems, and this is usually compensated for in the second part. Ignoring the dynamics of the system in the combinatorial part of the problem can significantly compromise performance. In this work, we introduce a framework that allows us to tackle both of these parts at the same time. To that order, we introduce a class of functionals we call the Quasi-Euclidean functionals, and use them to study such problems for dynamic systems. We determine the asymptotic behavior of the costs of these problems, when the points are randomly distributed and their number tends to infinity. We show the applicability of our framework by producing results for the Traveling Salesperson Problem (TSP) and Minimum Bipartite Matching Problem (MBMP) for dynamic systems.

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Chapter 1

Introduction

In this thesis, we study combinatorial problems under dynamic constraints, that is, combinatorial problems where the cost depends on the evolution of the output of a dynamic system. We aim to create a framework that allows the study of the asymptotic behavior of a class of such problems for dynamic systems. We also seek to show the applicability of the framework by producing results on some interesting combinatorial problems for dynamic systems. We start here by motivating the problem we study.

1.1 Motivation

The main motivation for the problems we study are applications where a given set of dynamic systems are required to travel as quickly as possible between a set of points. One example of such applications is a surveillance mission where a given UAV equipped with sensors has to visit a number of checkpoints as quickly as possible. Another one is the vehicle-target matching problem, where a team of n UAV's are spread over a bounded area, and there are n targets that are also randomly distributed in the same area. Each target must be visited by a UAV while minimizing the average time it takes for the targets to be visited. The dynamic system and the targets in such problems are usually modeled as point masses. Such problems have been studied by first solving a combinatorial problem that concentrates on assigning points to each dynamic system and/or determining the order of traversal of the points while ignoring the dynamic constraints on the system [60, 61, 62]. An optimal control problem aiming at making the dynamic systems follow the solution of the combinatorial problem in the minimum time is then solved. In general, solving the combinatorial part of the problem while ignoring the dynamics of the system can lead to bad performance. In [14], the authors study the TSP for a dynamic system that moves with a constant velocity and has bounded curvature (the Dubins Vehicle). They prove that getting the optimal order for the Euclidean TSP and using it for the TSP for the Dubins vehicle produces (in certain situations) an error that grows at least as a constant times the number of points. Such deterioration in performance makes it important to include the dynamics of the system in the combinatorial part of the problem, and is the main motivation for our work.

Problems similar to the ones introduced above are becoming more interesting with the increase of our use of UAV's and autonomous robots for different kinds of applications. These problems range from vehicles traveling for pickup or delivery, to surveillance and search-and-rescue missions [60, 61, 12, 55]. Although certain aspects of these problems have been studied before, we tackle these problems while accounting for the dynamics of the system(s) that are required to perform the mission and traverse the resulting path. We study how the dynamics of the robot or the UAV affect the asymptotic behavior of the cost of the problem. This gives a more accurate understanding of the problem and insight on how to minimize the associated cost, and provides fundamental limits on performance. Additionally, we provide algorithms that produce order optimal paths that the dynamic systems can trace, and thus they can be used for the applications for UAV's and robots.

1.2 Previous Work

Because of the importance of studying combinatorial problems under dynamic constraints, some interesting combinatorial problems have been recently studied for specific dynamic systems. The Traveling Salesperson Problem has been recently studied for the Dubins vehicle [1, 28], the double integrator [4], the Reeds-Shepp car, the differential drive robot [22]. Additionally, in our previous work [6, 7, 8], we studied the TSP and some related problems for a general dynamic system having a state space representation that is affine in control. Our work is a natural extension of that work to a larger class of problems for dynamic systems.

This work is a generalization of some previous work in a different sense. In this work, we introduce a framework for studying general combinatorial problems where dynamic systems travel through a given set of points. Using this framework, we concentrate on studying the behavior of the cost of the problem as the number of points tends to infinity. This is inspired by the similar study of the costs of the Euclidean versions of such combinatorial problems [1, 2]. This direction of studying such problems has proven effective in many ways, producing important convergence results for the stochastic versions of the problems and bounds on the worst-case results [11]. Additionally, this way of studying the costs can be used to produce approximation algorithms for the combinatorial problems. We therefore use the same approach in our work here and direct our efforts at studying the asymptotic properties of the costs of the problems when the points are randomly distributed and their number tends to infinity. Thus the problem we deal with here can be considered a generalization of the one studied in [1, 2] to account for the dynamic constraints of the system.

The rest of this Thesis is organized as follows: Chapter 2 has the problem formulation and introduces some background on dynamic systems and subadditive Euclidean functionals. In Chapter 3 contains a study of the local behavior of dynamic systems and establish results on the time needed for a dynamic system to move locally between two points. In Chapter 4, Subadditive Quasi-Euclidean functionals are defined and their properties are studied, these will be of utmost importance for our results. Chapter 5 studies the relationship between problems for small-time locally controllable dynamic systems and Quasi-Euclidean functionals and determines the asymptotic behavior of the costs of those problems. The applicability of the formalism with some specific examples of problems for dynamic systems and their corresponding costs under dynamic constraints. Chapter 6 relaxes the initial assumptions on the dynamic systems, and produce an algorithm for the TSP for a general dynamic system. Chapter 7 has a detailed study of the Dynamic Traveling Repairperson Problem, with algorithms that perform within a constant factor of the optimal for low intensity and high intensity cases. Chapter 8 has the conclusions and discussion.

Chapter 2

Problem formulation and Background

In this chapter, we formulate the problem of interest and present some important background from the literature. The background needed can be divided into two parts: The first is about dynamic systems and their local behavior, and the second is about subadditive Euclidean functionals. These two parts represent the two areas that we are merging in this work. Since we are injecting dynamics into traditionally combinatorial problems, it is essential that we build some background in both of those areas. We start by introducing the models of the dynamic systems that we use in our study.

2.1 Problem Formulation

The goal of this work is to produce a framework that allows the study of combinatorial problems under dynamic constraints, that is, combinatorial problems where the cost depends on the evolution of a controlled dynamic system. The framework should allow us to study a class of interesting problems under dynamic constraints.

2.1.1 Problems with Subadditive Cost for Dynamic Systems

Since our aim is to incorporate dynamic constraints to classical combinatorial problems, we start by reviewing the classical formulation of the combinatorial problems that we generalize in this work. These are combinatorial problems on a given graph. A graph is defined by its nodes and edges. Thus a graph G is defined as $(\mathcal{Y}, E)$, where $\mathcal{Y} = \{y_1, ..., y_n\}, y_i \in \mathbb{R}^d$ is a set of points and $E = \{(i,j)|i,j \in \{1,...,n\}\}$ is a set of ordered pairs each of which corresponds to a directed edge of G. This means that $(i_0, j_0) \in E$ if and only if there is a directed edge between y_{i_0} and y_{j_0} in G. A weighted graph is endowed with a set of weights $\{w(y_i, y_j) : E \to \mathbb{R}^+\}$ for all edges in the graph. The combinatorial problems we study choose a subset of the edges of the given graph. To denote the chosen edges, an $n \times n$ binary matrix Z is used, where n is the number of vertices in the graph. Thus the variables that we use for the optimization are $z_{i,j}$, and the classical versions of the problems can be formulated as follows:

$$L(\mathcal{Y}) = \min_{Z \in \Upsilon} \sum_{i,j} w(y_i, y_j) z_{i,j}, \tag{2.1}$$

where $\Upsilon \subset \{0,1\}^{n \times n}$ is a set that enforces a certain structure on Z. We study a generalization of the previous problems where the weights are produced from a dynamic system. Consider an example where a dynamic system (UAV or a car) is required to travel between a number of points in minimum time. In this example, the cost of an edge connecting two points y^i and y^j doesn't only depend on y^i and y^j, but it also depends on the state of the dynamic system at y^i and y^j. Thus we consider that the points in $\mathcal{Y} = \{y_1, ..., y_n\}$ are in the output space of m dynamic systems whose states we denote by x and whose output equation is $y = h(x)$. Without loss of generality, we assume that $n > m$ and that $y_1, ..., y_m$ are the output points corresponding to the initial states $x_1, ..., x_m$ of the dynamic systems ($y_i = h(x_i)$, for $i = 1, ..., m$.) This gives the following formulation:

$$
\begin{aligned}
&L_S(x_1, ..., x_m, \mathcal{Y}) = \min_{Z \in \Upsilon} \min_{x_i : h(x_i) = y_i, i = m+1, ..., n} \sum_{i,j} T_S(x_i, x_j) z_{i,j}, \\
&y^i \equiv h(x_i), \quad \forall i = 1, ..., m,
\end{aligned}
\tag{2.2}
$$

where T_S is a function that depends on the evolution of the dynamic system. Thus the minimization is over the states that correspond to points in $\mathcal{Y}$ *other than the initial states.* In particular, we are interested in the case where $T_S(x_i, x_j)$ is the minimum time needed for a the state of a dynamic system to move from x_i to x_j with bounded input:

$$
\begin{aligned}
&T_S(x_i, x_j) = \min_{u_1(\cdot), ..., u_m(\cdot) \in \mathbb{U}, T} \int_0^T 1 dt, \\
&\frac{dx}{dt} = g_0(x) + \sum_{i=1}^{m} g_i(x) u_i, \\
&\mathbb{U} = \{u(.) : \text{measurable } \mathbb{R}^+ \to [-M, M]\}, \\
&x(0) = x_i, \\
&x(T) = x_j.
\end{aligned}
\tag{2.3}
$$

In this equation, the dynamic system is modeled with a state space representation that is affine in control. We describe systems that are affine in control and study some of their properties later in this section. Different costs $T_S(x_i, x_j)$ might be studied in a similar fashion,

The weights in the combinatorial problem are the minimum time needed for the dynamic system to move its output between pairs of points in y. Of course, this time depends on the state of the system and not only on its output. Thus when the dynamics are inserted into the problem, both minimization over the control in (2.3) and over the state in (2.2) are needed.

What we study here is the asymptotic behavior of the costs ($L_S : \mathbb{R}^d \to \mathbb{R}$) of such problems when the points are sampled from a random variable Y_i and their number tends to infinity. This is a generalization of the study for the classical Euclidean case (where the cost $w(y_i, y_j)$ is the Euclidean distance between y_i and y_j) that was done by Beardwood, Halton and Hammersley [2] and later generalized by Steele [1]. Thus our study can be considered to be a generalization of their results to account for the dynamics of the system traversing the paths. We aim to introduce a framework for the study of such problems, and determine some sufficient properties of L_S that allow us to directly determine the asymptotic behavior of L_S as $n \to \infty$. We will start by studying these problems when the dynamic system has no drift ($g_0 = 0$) and $Y_1, ..., Y_n$ are uniformly, independently and identically distributed in $[0,1]^d$, and then extend the results to non-uniform distributions and systems with drift.

Two functionals that we will use to show the applicability of our framework are the costs of the Dynamic System Traveling Salesperson Problem and the Dynamic System Minimum Bipartite Matching Problem. These are special cases of the functionals described in equations (2.2) and (2.3) that are given by:

1. DyTSP: (Dynamic system TSP) Find the minimum cost Hamiltonian circuit over $\mathcal{Y}$ where the edges are output curves of the dynamic system S, and let $L_S^0(\mathcal{Y})$ be its cost. Thus

$$L_S^0(x_1, \mathcal{Y}) = \min_Z \min_{\substack{x_2, ..., x_n \\ h(x_k) = y_k, \\ k = 1, ..., n}} \sum_{i,j} T_S(x_i, x_j) z_{i,j},$$

$$\sum_j z_{i,j} = 1 \quad \forall\, i \in \{1, ..., n\}.$$

$$\sum_j z_{j,i} = 1 \quad \forall\, i \in \{1, ..., n\}.$$

$$\sum_{i \in V, j \notin V} z_{i,j} \geq 2 \qquad \forall V \subset \mathcal{Y}, 2 \leq |V| \leq n-1.$$

The requirement on Z means that there should be one incoming edge and one outgoing edge for every node in the graph, and that the graph should be connected.

2. DyMBMP: (Dynamic system MBMP) Find the graph that connects pairs of

points in $\mathcal{Y}$ while minimizing the average (or total) travel time:

$$L^1_S(x_1, ..., x^0_n, \mathcal{Y}) = \min_{Z} \min_{\substack{x_{n+1}, ..., x_{2n} \\ h(x_k) = y_k, \\ k = n+1, ..., 2n}} \sum_{i,j} T_S(x_i, x_j) z_{i,j},$$

$$\sum_j z_{i,j} = 1, \quad \sum_j z_{i,j} = 0 \quad \forall\, i \in \{1, ..., n\},$$
$$\sum_j z_{i,j} = 0, \quad \sum_j z_{i,j} = 1 \quad \forall\, i \in \{n+1, ..., 2n\}.$$

The requirement here on Z means that $\forall\, i, j \in \{1, ..., n\}$, there should be an outgoing edge from each y^1_i and an incoming edge into each y^2_j.

In both of these problems, $T_S(x_i, x_j)$ is the minimum time needed for a dynamic system to move its state from x_i to x_j as in (2.3). We study these examples as important applications of our results on the general class of problems. We study the properties they satisfy and their asymptotic behavior when $y_1, \ldots, y_n$ are randomly, independently and identically distributed in $[0, 1]^d$ (in the output space of system S). We also seek algorithms for the DyTSP as a practical application of the framework. Finally, we study the DTRP (Dynamic Traveling Repairperson Problem) for dynamic systems.

DTRP for Dynamic Systems

Given a dynamic system that is modeled as in (2.4), let R be compact region in the output space of the system (R is assumed to be a d−dimensional cuboid with dimensions $W_1, W_2, ..., W_q$). We study the DTRP, where "customer service requests" are arising according to a Poisson process with rate λ and, once a request arrives, it is randomly assigned a position in R uniformly and independently.

The repairperson is modeled as in (2.4) and is required to visit the customers and service their requests. At each customer's location, the repairperson spends a service time s which is a random variable with finite mean $\overline{s}$ and second moment $\overline{s^2}$. We study the expected waiting time a customer has to wait between the request arrival time and the time of service, and we mainly focus on how that quantity scales in terms of the traffic intensity $\lambda\bar{s}$ for low traffic ($\lambda\bar{s} \to 0$) and high traffic ($\lambda\bar{s} \to 1$). We also study the stability of the queuing system, namely whether the necessary condition for stability ($\lambda < \frac{1}{\bar{s}}$) is also sufficient.

2.2 Dynamic System Models that are Affine in Control

In the problem formulation, the costs $T_S(x_i, x_j)$ are functions of the evolution of a dynamic system. We model the dynamic systems with state space models that are affine in control,have an output in $\mathbb{R}^d$ and bounded input. Thus they are described as follows:

$$\dot{x} = g_0(x) + \sum_{i=1}^{m} g_i(x)u_i, \tag{2.4}$$

$$y = h(x),$$

$$x(0) = x_0,$$

$$x \in \mathbb{R}^b,\ y \in \mathbb{R}^d,\ u_i \in \mathbb{U},$$

$$\mathbb{U} = \{u(.) : \text{measurable } \mathbb{R} \to [-M, M]\}.$$

We will use Assumption 3.3 in most of this work, and then in Chapter 6 generalize the results for the cases where the dynamic system has drift. This class of systems is very general and descriptive. We use this class of models for the dynamic systems because even though it is general, studying its local behavior properties using differential geometric methods is mathematically elegant and established in the literature [25, 26]. The boundedness on the input is assumed because unbounded inputs can make minimum time problems trivial.

We now introduce two examples of dynamic systems that are affine in control.

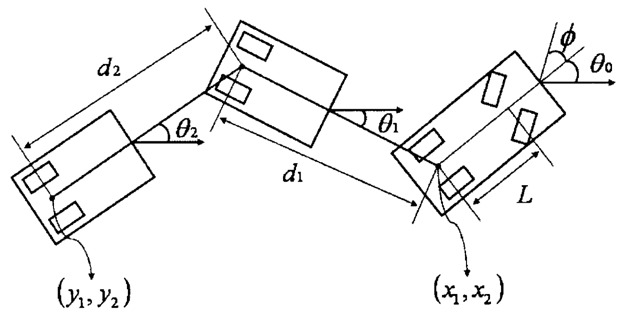

Figure 2-1: Parameters for a car pulling k-trailers

We use these examples throughout this thesis to clarify certain concepts about local reachability of dynamic systems and the behavior of the TSP and similar problems for dynamic systems.

2.2.1 Examples of Dynamic Systems of Interest

The first model we use is that of a linear time-invariant system with its output in $\mathbb{R}^3$. The state space model of that system is as follows:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{2.5}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.6}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.7}$$

and

$$\mathbb{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, u_i(\cdot) : \mathbb{R} \to [-1, 1]. \tag{2.8}$$

Here, g_0, g_1, g_2 are given by:

$$g_0 = Ax = \begin{bmatrix} x_2 \\ 3x_1 \\ 6x_5 \\ x_3 \\ x_4 \end{bmatrix}, g_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{2.9}$$

The second example is a simplified model of a car pulling k trailers (from [31]). The first two states in that model are the location of the first car in the plane and the rest are the angles at the axles of the trailers; the output is the location of the last trailer (Fig. 2-1). The state space model for the car is therefore [31]:

$$
\begin{aligned}
\dot{x}_1 &= \cos(\theta_0) \\
\dot{x}_2 &= \sin(\theta_0) \\
\dot{\theta}_0 &= \frac{u}{L} \\
\dot{\theta}_1 &= \frac{1}{d_1}\sin(\theta_0 - \theta_1) \\
&\vdots \\
\dot{\theta}_i &= \frac{1}{d_i}\left(\prod_{j=1}^{i-1}\cos(\theta_{j-1} - \theta_j)\right)\sin(\theta_{i-1} - \theta_i) \\
&\vdots \\
\dot{\theta}_k &= \frac{1}{d_k}\left(\prod_{j=1}^{k-1}\cos(\theta_{j-1} - \theta_j)\right)\sin(\theta_{k-1} - \theta_k) \\
y &= \begin{bmatrix} x_1 - \sum_{i=1}^{k} d_i \cos(\theta_i) \\ x_2 - \sum_{i=1}^{k} d_i \sin(\theta_i) \end{bmatrix} \\
\mathbb{U} &= \{u(\cdot) = \tan(\phi(\cdot)), \phi(\cdot) : \mathbb{R} \to [-\phi_0, \phi_0]\}.
\end{aligned}
\tag{2.10}
$$

The car is assumed to have a constant speed forward, and the control we have on the car is the steering angle ϕ (actually $\tan(\phi)$). It is easy to see that this is a special case of our general dynamic system (2.4), where g_0 and g_1 are given by:

$$g_0 = \begin{bmatrix} \cos(\theta_0) \\ \sin(\theta_0) \\ 0 \\ \frac{1}{d_1}\sin(\theta_0 - \theta_1) \\ \vdots \\ \frac{1}{d_i}\left(\prod_{j=1}^{i-1}\cos(\theta_{j-1} - \theta_j)\right)\sin(\theta_{i-1} - \theta_i) \\ \vdots \\ \frac{1}{d_k}\left(\prod_{j=1}^{k-1}\cos(\theta_{j-1} - \theta_j)\right)\sin(\theta_{k-1} - \theta_k) \end{bmatrix}, \qquad g_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.11}$$

We will follow these systems throughout this work to clarify some concepts. Additionally, in our detailed study of the TSP and DTRP for dynamic systems, we will generate the asymptotic solutions of the TSP/DTRP for these examples to demonstrate our results for some specific dynamic systems.

2.3 Dynamic Systems Background

We first introduce some terminology and definitions for systems that are affine in control; most of these definitions are standard in the literature [25, 26]. We will in general use subscripts to indicate components of a vector, and superscripts to label individual instances. Thus x^i_j will indicate the j^{th} component of vector x^i. A related piece of notation that we will use is the function $x_j(x)$ which extracts the j^{th} component of x.

We start by introducing the most basic object we need, the reachable set of a dynamic system.

Definition 2.1 (Reachable set)**.** Given $T \geq 0$, the reachable set from state x^0 for a dynamic system is the set $R_T(x^0)$ of states x such that $\forall\, x^1 \in R_T$, $\exists\, u_1^*, u_2^*, ..., u_m^* \in \mathbb{U}$ such that:

$$x(0) = x^0,\ x(T) = x^1,$$

This is the set of states that are reachable in exactly time T. We define the set of states reachable in time less than or equal to T by:

$$R_{\leq T}(x^0) = \cup_{0 \leq t \leq T} R_t(x^0).$$

We extend the previous definition to the output space, and so we define the output-reachable set from a state x^0 to be the set $O_T(x^0)$ of points

$$y = h(x), x \in R_T(x^0),$$

and

$$O_{\leq T}(x^0) = \cup_{0 \leq t \leq T} O_t(x^0).$$

We indicate by $A_{\leq T}(x^0)$ the volume of $O_{\leq T}(x^0)$.

We turn to some important properties of some systems that are affine in control.

Definition 2.2 (Small-time Locally Controllable Systems)**.** A system is small-time locally controllable at $x^0 \in \mathbb{R}^p$ if $\exists\, T > 0$ such that

$$x^0 \in \text{Interior}(R_{\leq t}(x^0))\ \forall\, t \text{ such that } 0 < t \leq T.$$

We call a system small-time locally controllable if it is small-time locally controllable at all $x \in \mathbb{R}^p$.

We also extend the previous definition to the output space, and say that a system is output small-time locally controllable at x^0 if $\exists\, T > 0$ such that

$$h(x^0) \in \text{Interior}(O_{\leq t}(x^0))\ \forall\, t \text{ such that } 0 < t \leq T.$$

Definition 2.3 (Output small-time locally controllable systems). A system is called "output small-time locally controllable" if it is output small-time locally controllable for every x in $\mathbb{R}^p$.

Definition 2.4 (Vector Fields). For all the purposes of this work, a vector field $f(x)$ is a smooth mapping from $\mathbb{R}^p$ to $\mathbb{R}^p$.

Given a vector field f and a function $w : \mathbb{R}^p \to \mathbb{R}$, we denote the derivative of w along f by :

$$\mathcal{L}\left(f(x), w(x)\right) = \sum_{i=1}^{p} \frac{\partial w(x)}{\partial x_i} f_i(x).$$

Note that

$$\mathcal{L}\left(f(x), x_j(x)\right) = f_j(x).$$

Given a vector field f and $g : \mathbb{R}^p \to \mathbb{R}^q$, we call the derivative of g along f the new $\mathbb{R}^p \to \mathbb{R}^q$ function:

$$\mathcal{L}\left(f(x), g(x)\right) = \frac{\partial g(x)}{\partial x} f(x),$$

where $\frac{\partial g}{\partial x}$ is the Jacobian matrix of g.

Note that the i^{th} component of $\mathcal{L}(f, g)$ is the derivative of the function g_i along f. Thus the use of similar notation should not be confusing.

Definition 2.5 (Lie Brackets). Given two vector fields f and g, the Lie bracket (or product) of f and g is another vector field denoted by $[f, g]$ and is given by:

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x),$$

where $\frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x}$ are the Jacobian matrices of g and f.

Lie brackets can be iterated since the result of a Lie bracket is itself a vector field. A notion that is related to the iteration of vector fields of a dynamic system that is affine in control is the "order" of a Lie bracket.

Definition 2.6 (Order of Lie Brackets). The orders of Lie brackets of a dynamic system as in (2.4) are defined iteratively as follows:

1. The order of $g_i, i \in \{0, ..., m\}$ is defined to be 1.

2. The order of $[g^i, g^j]$ is the sum of the order of g^i and the order of g^j, where g^i and g^j are themselves in $g_0, ..., g_m$ or iterated Lie brackets of $g_0, ..., g_m$.

Definition 2.7 (Indices of an iterated Lie Bracket). Given vector fields $g_1, ..., g_m$, note that an iterated Lie bracket V of $g_1, ..., g_m$ that has order r is determined by the following:

1. An ordered set of indices , $I_V = (i_1; ...; i_r), i_k \in \{1, ..., m\}$, that determine the vector fields from $\{g_1, ..., g_m\}$ that are in V, and their order in V. This set of indices is defined iteratively as follows:

$$I_{g_i} = i \quad i \in \{1, ...m\},$$

$$I_{[g^1,g^2]} = I_{g^1} \cup I_{g^2},$$

with the order conserved.

2. An ordered set $B_V = ((i_1; j_1); ...; (i_{r-1}, j_{r-1}))$ of $r-1$ pairs designating the order in which the iterated Lie brackets are applied. This set of indices is also defined iteratively:

$$B_{g_i} = \phi \quad i \in \{1, ..., m\},$$

$$B_{[g^1,g^2]} = B_{g^1} \cup \{B_{g^2} + r_{g^1}\} \cup \{(1, r_{g^1} + r_{g^2})\},$$

where g^1 and g^2 are iterated Lie brackets of $g^1, ..., g^m$, r_{g^i} is the order of g^i, and the addition $B_{g^2} + r_{g^1}$ is defined as follows:

$$\{B_{g^2} + r_{g^1}\} = \begin{cases} \phi & \text{if } B_{g^2} = \phi \\ \{(i + r_{g^1}, j + r_{g^1}) : (i,j) \in B_{g^2}\} & \text{otherwise.} \end{cases} \tag{2.12}$$

Definition 2.8 ($\{Ap(u^i, t)\}$)**.** We introduce a family of inputs, denoted by $\{Ap(u_i, t)\}$, that is related to $g_1, ..., g_m$ and their Lie brackets. We use the notation $Ap(u_i, t)$ (apply u_i for time t), which is defined as follows:

1. $Ap(u_i, t)$ means to set the input $u_i = 1$ and $u_j = 0, j \neq i$ for a time duration equal to t. $Ap(-u_i, t)$ means to set the input $u_i = -1$ and $u_j = 0, j \neq i$ for a time duration equal to t.

2. $Ap([u^i, u^j], t) = Ap(u_i, t) \circ Ap(u_j, t) \circ Ap(-u_i, t) \circ Ap(-u_j, t)$, where $\circ$ denotes concatenation and u^i and u^j might be brackets themselves.

Definition 2.9 (Order of $Ap(u^i, t)$)**.** The "order" of an input $Ap(u^i, t)$ is defined similarly to the order of Lie brackets of the dynamic system:

1. The order of $Ap(u_i, t), i \in \{1, ..., m\}$ is defined to be 1.

2. The order of $Ap([u^i, u^j], t)$ is the sum of the order of $Ap(u^i, t)$ and the order of $Ap(u^j, t)$.

We introduce the notation for the "projection" of state space Lie brackets onto the output space.

Definition 2.10 (Domain Space Lie Brackets)**.** Given analytic vector fields $f_1, ..., f_s : \mathbb{R}^p \to \mathbb{R}^p$, and an analytic function $k : \mathbb{R}^p \to \mathbb{R}^d$, we define the Lie brackets of $f_1, ..., f_s$ in the domain space of k as follows:

$$[f_i]_k \equiv \mathcal{L}(f_i, k), \quad \forall i \in \{1, ..., s\},$$

$$[f^1, f^2]_k = \mathcal{L}(f^1, [f^2]_k) - \mathcal{L}(f^2, [f^1]_k),$$

where f_1 and $f_2 \in \{g_1, ..., g_m\}$ or are lie brackets of $g_1, ..., g_m$. Note that the operator $[]_k$ takes vector fields in $\mathbb{R}^p$ to vector fields in $\mathbb{R}^d$. Thus $[f_1]_k = \mathcal{L}(f_1, k)$, $[f_1, f_2]_k = \mathcal{L}(f_1, \mathcal{L}(f_2, k)) - \mathcal{L}(f_2, \mathcal{L}(f_1, k))$.

The order and indices of domain space Lie brackets are defined similarly to those of Lie Brackets in Definitions 2.6-2.7.

To denote a generic Lie Bracket of $g_1, ..., g_m$, we will use the notation $[g^i]$. To denote an output space Lie bracket of $g_1, ..., g_m$ we will use the notation $[g^i]_h$ and to denote a generic input in the family $\{Ap(u^i, t)\}$, we will use the notation $Ap([u^i], t)$. All of these quantities are defined by their indices as in Definition 2.7.

Definition 2.11 (Nilpotent Systems). A dynamic system that is affine in control is called nilpotent if the Lie brackets of $\{g_0, g_1, ..., g_m\}$ vanish after a certain order. This happens for example when the vector fields are polynomials.

Some additional notions that we use in our study for both the local behavior of dynamic systems and Quasi-Euclidean functionals are the following:

Definition 2.12 (Dilation function (k^r)). Given an ordered set of positive real numbers $r = (r_1, ..., r_d)$, define the function $k^r : \mathbb{R} \to \mathbb{R}^d$ componentwise, by setting $k_i^r(\alpha) = \alpha^{r_i}$. Note that the ordering in r affects k^r.

Definition 2.13 (Asymptotic Notation). Finally, we use the standard asymptotic notation for the scaling of functions, and thus we say a function $f(n)$ is $O(g(n))$ if

$$\exists c, N > 0 \text{ such that } f(n) \leq cg(n) \; \forall \; n > N.$$

We say $f(n)$ is $\Omega(g(n))$ if $g(n)$ is $O(f(n))$. $f(n)$ is $\Theta(g(n))$ if $f(n)$ is $O(g(n))$ and $\Omega(g(n))$. Finally, $f(l)$ is $o(g(l))$ if $\lim_{l \to 0} \frac{f(l)}{g(l)} = 0$ (for functions) or $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$ (for sequences).

2.3.1 Evolution of the Output under Inputs from the family $Ap(u^i, t)$

We start with the following theorem about the evolution of the output of dynamic systems when inputs from the family $\{Ap(u^i, t)\}$ are applied. First note that the time needed to apply the input $Ap([u^i], t)$ is equal to $\alpha_{[u^i]}t$, where $\alpha_{[u^i]}$ is a constant that only depends on the indices of u^i.

Theorem 2.1. *Consider the dynamic system given in* (2.4)*. If the input $Ap([u^i], t)$ (whose order is r) is applied, then*

$$y(\alpha_{u^i}t) = y(0) + t^r [g^i]_h|_{x^0} + t^{r+1} \sum_{i=0}^{\infty} t^i \sum_k G_i^k(g_1, ..., g_m, h(x^0)),$$

where $[g^i]_h|_{x^0}$ is the output space Lie bracket with the same indices as $[u^i]$, that is, $I_{[g^i]_h|_{x^0}} = I_{[u^i]}$ and $B_{[g^i]_h|_{x^0}} = B_{[u^i]}$. $G_i^k(g_1, ..., g_m, h(x^0)$ is a derivative of h with respect to $g_1, ..., g_m$ whose order is higher than r and is evaluated at x^0.

Proof. The proof is by mathematical induction. If the input $Ap(u_i, t)$ is applied, then by theorem 2.2, the state and output of the dynamic system evolve as:

$$x(t) = x^0 + \sum_{k=1}^{\infty} \mathcal{L}^k(g_i, x)\frac{t^k}{k!}$$

$$y(t) = y(0) + \sum_{k=1}^{\infty} \mathcal{L}^k(g_i, h(x^0))\frac{t^k}{k!}.$$

If the input $Ap([u_i, u_j], t) = Ap(u^i, t) \circ Ap(u^j, t) \circ Ap(-u^i, t) \circ Ap(-u^j, t)$ is applied, then the state and output of the dynamic system evolve as:

$$x^1 \equiv x(t) = x^0 + \sum_{k=1}^{\infty} \mathcal{L}^k(g_i, x(x^0))\frac{t^k}{k!}$$

$$y(t) = y(0) + \sum_{k=1}^{\infty} \mathcal{L}^k(g_i, h(x^0))\frac{t^k}{k!}.$$

$$x^2 \equiv x(2t) = x^0 + \sum_{k=1}^{\infty} \mathcal{L}^k(g_i, x(x^0))\frac{t^k}{k!} + \sum_{k=1}^{\infty} \mathcal{L}^k(g_j, x(x^1))\frac{t^k}{k!}$$

$$= x^0 + \sum_{k=1}^{\infty} \mathcal{L}^k(g_i, x(x^0))\frac{t^k}{k!} + \sum_{k=1}^{\infty} \mathcal{L}^k(g_j, x(x^0))\frac{t^k}{k!} + \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \mathcal{L}^{k_1}(g_j, \mathcal{L}^{k_2}(g_i, x(x^0)))\frac{t^{k_1+k_2}}{k_1!k_2!}. \tag{2.13}$$

$$y(2t) = y(0) + \sum_{k=1}^{\infty} \mathcal{L}^k(g_i, h(x^0))\frac{t^k}{k!} + \sum_{k=1}^{\infty} \mathcal{L}^k(g_j, h(x^1))\frac{t^k}{k!}$$

$$= y(0) + \sum_{k=1}^{\infty} \mathcal{L}^k(g_i, h(x^0))\frac{t^k}{k!} + \sum_{k=1}^{\infty} \mathcal{L}^k(g_j, h(x^0))\frac{t^k}{k!} + \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \mathcal{L}^{k_1}(g_j, \mathcal{L}^{k_2}(g_i, h(x^0)))\frac{t^{k_1+k_2}}{k_1!k_2!}. \tag{2.14}$$

$$x^3 \equiv x(3t) = x^0 + \sum_{k=1}^{\infty} \mathcal{L}^k(g_i, x(x^0))\frac{t^k}{k!} + \sum_{k=1}^{\infty} \mathcal{L}^k(g_j, x(x^1))\frac{t^k}{k!} + \sum_{k=1}^{\infty} \mathcal{L}^k(g_i, x(x^2))\frac{(-t)^k}{k!}$$

$$= x^0 + \sum_{k=1}^{\infty} \mathcal{L}^{2k}(g_i, x(x^0))\frac{2t^{2k}}{k!} + \sum_{k=1}^{\infty} \mathcal{L}^k(g_j, x(x^0))\frac{t^k}{k!} + \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \mathcal{L}^{k_1}(g_j, \mathcal{L}^{k_2}(g_i, x(x^0)))\frac{t^{k_1+k_2}}{k_1!k_2!}$$

$$+ \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \mathcal{L}^{k_1}(g_i, \mathcal{L}^{k_2}(g_i, x(x^0)))\frac{(-1)^{k_1}t^{k_1+k_2}}{k_1!k_2!} + \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \mathcal{L}^{k_1}(g_i, \mathcal{L}^{k_2}(g_j, x(x^0)))\frac{(-1)^{k_1}t^{k_1+k_2}}{k_1!k_2!}$$

$$+ \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty} \mathcal{L}^{k_1}(g_i, \mathcal{L}^{k_2}(g_j, \mathcal{L}^{k_3}(g_i, x(x^0))))\frac{(-1)^{k_1}t^{k_1+k_2+k_3}}{k_1!k_2!k_3!}. \tag{2.15}$$

$$y(3t) = y(0) + \sum_{k=1}^{\infty} \mathcal{L}^k(g_i, h(x^0))\frac{t^k}{k!} + \sum_{k=1}^{\infty} \mathcal{L}^k(g_j, h(x^1))\frac{t^k}{k!} + \sum_{k=1}^{\infty} \mathcal{L}^k(g_i, h(x^2))\frac{(-1)^k t^k}{k!}$$

$$= y(0) + \sum_{k=1}^{\infty} \mathcal{L}^{2k}(g_i, h(x^0))\frac{2t^{2k}}{k!} + \sum_{k=1}^{\infty} \mathcal{L}^k(g_j, h(x^0))\frac{t^k}{k!} + \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \mathcal{L}^{k_1}(g_j, \mathcal{L}^{k_2}(g_i, h(x^0)))\frac{t^{k_1+k_2}}{k_1!k_2!}$$

$$+ \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \mathcal{L}^{k_1}(g_i, \mathcal{L}^{k_2}(g_i, h(x^0)))\frac{(-1)^{k_1}t^{k_1+k_2}}{k_1!k_2!} + \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \mathcal{L}^{k_1}(g_i, \mathcal{L}^{k_2}(g_j, h(x^0)))\frac{(-1)^{k_1}t^{k_1+k_2}}{k_1!k_2!}$$

$$+ \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty} \mathcal{L}^{k_1}(g_i, \mathcal{L}^{k_2}(g_j, \mathcal{L}^{k_3}(g_i, h(x^0))))\frac{(-1)^{k_1}t^{k_1+k_2+k_3}}{k_1!k_2!k_3!}. \tag{2.16}$$

$$
\begin{aligned}
y(4t) &= y(0) + \sum_{k=1}^{\infty} \mathcal{L}^k(g_i, h(x^0))\frac{t^k}{k!} + \sum_{k=1}^{\infty} \mathcal{L}^k(g_j, h(x^1))\frac{t^k}{k!} \\
&+ \sum_{k=1}^{\infty} \mathcal{L}^k(g_i, h(x^2))\frac{(-1)^k t^k}{k!} + \sum_{k=1}^{\infty} \mathcal{L}^k(g_j, h(x^3))\frac{(-1)^k t^k}{k!} \\
&= y(0) + \sum_{k=1}^{\infty} \mathcal{L}^{2k}(g_i, h(x^0))\frac{2t^{2k}}{(2k)!} + \sum_{k=1}^{\infty} \mathcal{L}^{2k}(g_j, h(x^0))\frac{2t^{2k}}{(2k)!} + \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \mathcal{L}^{k_1}(g_j, \mathcal{L}^{k_2}(g_i, h(x^0)))\frac{t^{k_1}}{k_1!} \\
&+ \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \mathcal{L}^{k_1}(g_i, \mathcal{L}^{k_2}(g_i, h(x^0)))\frac{(-1)^{k_1} t^{k_1+k_2}}{k_1!k_2!} + \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \mathcal{L}^{k_1}(g_i, \mathcal{L}^{k_2}(g_j, h(x^0)))\frac{(-1)^{k_1} t^{k_1+k_2}}{k_1!k_2!} \\
&+ \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty} \mathcal{L}^{k_1}(g_i, \mathcal{L}^{k_2}(g_j, \mathcal{L}^{k_3}(g_i, h(x^0))))\frac{(-1)^{k_1} t^{k_1+k_2+k_3}}{k_1!k_2!k_3!} \\
&+ \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \mathcal{L}^{k_1}(g_j, \mathcal{L}^{2k_2}(g_i, h(x^0)))\frac{(-1)^{k_1} t^{k_1} 2t^{2k_2}}{k_1!k_2!} + \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} \mathcal{L}^{k_1}(g_j, \mathcal{L}^{k_2}(g_j, h(x^0)))\frac{(-1)^{k_1} t^{k_1+k_2}}{k_1!k_2!} \\
&+ \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty} \mathcal{L}^{k_1}(g_j, \mathcal{L}^{k_2}(g_j, \mathcal{L}^{k_3}(g_i, h(x^0))))\frac{(-1)^{k_1} t^{k_1+k_2+k_3}}{k_1!k_2!k_3!} \\
&+ \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty} \mathcal{L}^{k_1}(g_j, \mathcal{L}^{k_2}(g_i, \mathcal{L}^{k_3}(g_i, h(x^0))))\frac{(-1)^{k_1+k_2} t^{k_1+k_2+k_3}}{k_1!k_2!k_3!} \\
&+ \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty} \mathcal{L}^{k_1}(g_j, \mathcal{L}^{k_2}(g_i, \mathcal{L}^{k_3}(g_j, h(x^0))))\frac{(-1)^{k_1+k_2} t^{k_1+k_2+k_3}}{k_1!k_2!k_3!} \\
&+ \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\sum_{k_4=1}^{\infty} \mathcal{L}^{k_1}(g_j, \mathcal{L}^{k_2}(g_i, \mathcal{L}^{k_3}(g_j, \mathcal{L}^{k_4}(g_i, h(x^0)))))\frac{(-1)^{k_1+k_2} t^{k_1+k_2+k_3+k_4}}{k_1!k_2!k_3!k_4!}.
\end{aligned}
\tag{2.17}
$$

After collecting the coefficients of t^2:

$$
\begin{aligned}
y(4t) &= y(0) + t^2\left(\mathcal{L}^2(g_i, h(x^0)) + \mathcal{L}^2(g_j, h(x^0)) - \mathcal{L}(g_i, \mathcal{L}(g_i, h(x^0))) - \mathcal{L}(g_j, \mathcal{L}(g_j, h(x^0)))\right) \\
&+ t^2\left(\mathcal{L}(g_j, \mathcal{L}(g_i, h(x^0))) - \mathcal{L}(g_i, \mathcal{L}(g_j, h(x^0)))\right) + t^3\sum_{i=0}^{\infty} t^i \sum_k F_i^k(g_i, g_j, h(x^0)) \\
&= y(0) + t^2[g_i, g_j]_h|_{x^0} + t^3\sum_{i=0}^{\infty} t^i \sum_k F_i^k(g_i, g_j, h(x^0)),
\end{aligned}
\tag{2.18}
$$

where $F_i^k(g_i, g_j, h(x^0))$ is a derivative of h with respect to g_i and g_j whose order is higher than 2 and is evaluated at x^0. Similarly,

$$x(4t) = x(0) + t^2[g_i, g_j]|_{x^0} + t^3 \sum_{i=0}^{\infty} t^i \sum_k F_i^k(g_i, g_j, x(x^0)),$$

where $F_i^k(g_i, g_j)$ is a derivative of x with respect to g_i and g_j whose order is higher than 2 and is evaluated at x^0.

Given an input $Ap([u^i], t)$, let $r(u^i)$ be the order of u^i. Assume that if $r(u^i) \in \{1, ..., s-1\}$ and the input $Ap([u^i], t)$ is applied, then

1. $$y(\alpha_{[u^i]}t) = y(0) + t^{r(u^i)}[g^i]_h|_{x^0} + t^{r(u^i)+1} \sum_{i=0}^{\infty} t^i \sum_k F_i^k(g_1, ..., g_m, h(x^0)),$$

   where $[g^i]_h$ is the output Lie bracket that has the same indices as $[u^i]$, and $F_i^k(g_1, ..., g_m, h(x^0))$ is a derivative of h with respect to $g_1, ..., g_m$ whose order is higher than $r(u^i)$ and is evaluated at x^0.

2. $$x(\alpha_{[u^i]}t) = x(0) + t^{r(u^i)}[g^i]|_{x^0} + t^{r(u^i)+1} \sum_{i=0}^{\infty} t^i \sum_k F_i^k(g_1, ..., g_m, x(x^0)),$$

   where $[g^i]$ is the Lie bracket that has the same indices as $[u^i]$, and $F_i^k(g_1, ..., g_m, x(x^0))$ is a derivative of x with respect to $g_1, ..., g_m$ whose order is higher than $r(u^i)$ and is evaluated at x^0.

We aim to prove that if $Ap([u^i], t)$ is applied, where $r([u^i]) = s$, then

$$y(\alpha_{[u^i]}t) = y(0) + t^s[g^i]_h + t^{s+1} \sum_{i=0}^{\infty} t^i \sum_k F_i^k(g_1, ..., g_m, h(x^0),$$

where $[g^i]_h$ is the output Lie bracket that has the same indices as $[u^i]$, and $F_i^k(g_1, ..., g_m, h(x^0))$ is a derivative of h with respect to $g_1, ..., g_m$ whose order is higher than s and is evaluated at x^0.

This can be done by using the above assumption and the fact that given $[u^i]$ such that $r([u^i]) = s$, then $[u^i] = [u_1^i, u_2^i]$, where $r([u_1^i]) < s$ and $r([u_2^i]) < s$ and using the same calculations and evaluations as before. □

We now turn to an important series representation for dynamic systems. One side of the importance of this representation is that it gives a lot of insight about the local behavior of the dynamic system. We therefore use it in Chapter 3 to study the scaling of the area of the small-time reachable set of the system in terms of time.

2.3.2 Chen-Fliess Series for Nonlinear Control Systems

The formal power series property of the Chen-Fliess series allows a representation of the evolution of nonlinear control systems that are described as in (2.4). This representation is given by the following theorem:

Theorem 2.2. *[25]*

If $g_0, ..., g_m, h$ are analytic functions, then $\exists T > 0$ such that $\forall t \leq T$, the j^{th} component of the output of system (2.4) *evolves as follows:*

$$y_j(t) = h_j(x^0) \tag{2.19}$$

$$+\sum_{k=0}^{\infty}\sum_{i_0=0}^{m}\cdots\sum_{i_k=0}^{m}\mathcal{L}\left(g_{i_0}, ..., \mathcal{L}\left(g_{i_k}, h_j(x^0)\right)\right)\int_0^t d\xi_{i_k}...d\xi_{i_0},$$

Where the integral in (2.19) *is defined iteratively by:*

$$\xi_0(t) = t,$$

$$\xi_j(t) = \int_0^t u_j(\tau)d\tau,$$

$$\int_0^t d\xi_{i_k}...d\xi_{i_0} = \int_0^t d\xi_{i_k}(\tau)\int_0^\tau d\xi_{i_{k-1}}...d\xi_{i_0}.$$

An important consequence of this theorem is a set of necessary and sufficient conditions for an output y_j not to be affected by an input u_i [25]. Specifically, y_j is unaffected by u_i if

$$\mathcal{L}(g_i, h_j) = 0$$

$$\mathcal{L}\left(g_i, \left(g_{i_1}, ..., \mathcal{L}\left(g_{i_k}, h_j(x^0)\right)\right)\right) = 0,$$

$$\forall i_1, ..., i_k \in \{0, ..., m\}.$$

This result hints at the usefulness of the Chen-Fliess series in small-time optimal control problems, and is indirectly used in our study of the small-time behavior of dynamic systems in Chapter 3. The proof of the theorem is in [25] and is repeated in the appendix for easy reference. We finally note that this theorem directly implies that

$$x_j(t) = x_j(x^0) \tag{2.20}$$

$$+\sum_{k=0}^{\infty}\sum_{i_0=0}^{m}\cdots\sum_{i_k=0}^{m}\mathcal{L}\left(g_{i_0},\ldots,\mathcal{L}\left(g_{i_k},x_j(x^0)\right)\right)\int_0^t d\xi_{i_k}\ldots d\xi_{i_0}.$$

This equation describes the evolution of the state of a dynamic system in small-time as a series. The integrals in the end can be used to describe the time scaling effect of each set of Lie derivatives. When using constant inputs, a single integral scales like t, a double integral scales as t^2. Therefore it hints that lower order Lie derivatives dominate in the small-time setting.

We now turn to some background on the other side of the problems we are addressing, namely the cost of the combinatorial problem. The background here deals with the costs in the case where the distance between every two points is the Euclidean distance between them. We modify this work later to allow the use of an approximation of the time a dynamic system needs to move between points in its output space.

2.4 Subadditive Euclidean Functionals

In the case where the weights of the edges in the our problem formulation in Section 2.1 are given by the Euclidean distances between the points, the costs of the TSP is known to belong to a class of functionals called subadditive Quasi-Euclidean functionals [1]. Additionally, it can be shown that the cost of the Euclidean version of MBMP is also similar to subadditive Euclidean functionals when $d \geq 3$ [12]. We therefore turn to introduce subadditive Euclidean functionals, study their properties and produce some of the known results on them. Subadditive Euclidean Functionals were introduced by Steele in [1], and are defined as follows:

Denote by L a real valued function of the finite subsets of $\mathbb{R}^d$ $(d \geq 2)$. L is a Euclidean functional if it satisfies the two following properties[1]:

Property 1 (Homogeneity). $L(\{\alpha x_1, ..., \alpha x_n\}) = \alpha L(\{x_1, ..., x_n\})\ \forall \alpha \in \mathbb{R}^+, x \in \mathbb{R}^d$.

Property 2 (Translation Invariance). $L(\{x_1 + x, ..., x_n + x\}) = L(\{x_1, ..., x_n\})\ \forall x \in \mathbb{R}^d$.

A functional L is called *bounded, monotone* and *subadditive* if is satisfies the following three properties:

Property 3 (Boundedness). $Var(L(\{X_1, ..., X_n\})) < \infty$ *if the* X_i*'s are independently and uniformly distributed in* $[0, 1]^d$.

Property 4 (Monotonicity). $L(x \cup A) \geq L(A)\ \forall x \in \mathbb{R}^d$ *and finite subset* A *of* $\mathbb{R}^d$.

Let $Q_i,\ 1 \leq i \leq m^d$, be a partition of the d-cube $[0, 1]^d$ into cubes with sides that are parallel to the axis and have length $\frac{1}{m}$ and let $tQ_i = \{x | x = ty, y \in Q_i\}$.

Property 5. *(Subadditivity)*

$$\exists C \in \mathbb{R} \text{ such that } \forall m \in \mathbb{N} \text{ and } t \in \mathbb{R}^+,\ x_1, ..., x_n \in \mathbb{R}^d,$$

$$L(\{x_1, ..., x_n\} \cap [0, t]^d) \leq \sum_{i=1}^{m^d} L(\{x_1, ..., x_n\} \cap tQ_i) + Ctm^{d-1}. \tag{2.21}$$

Subadditive Euclidean functionals have been studied in [1]. The first result produced in that work and is relevant to our work here is the following theorem:

Theorem 2.3. *If L is a functional that satisfies properties 1-5, and $X_i; 1 \leq i < \infty$ are independent and uniformly distributed in $[0, 1]^d$, then there exists a constant $\beta(L)$ such that:*

$$\lim_{n \to \infty} \frac{L(\{X_1, ..., X_n\})}{n^{1-\frac{1}{d}}} = \beta(L)\ a.s. \tag{2.22}$$

This theorem signifies that a monotone subadditive Euclidean functional is asymptotically sub-linear in the number of variables, or, more interestingly, that it behaves as $\beta(L)n^{1-\frac{1}{d}}$ when n is large. This result was used to prove that the length of the Euclidean TSP, the cost of Euclidean minimum spanning tree and the cost of the Euclidean minimum matching problem all scale as $n^{1-\frac{1}{d}}$.

Some additional properties can be used to generalize the previous theorem to the case where the variables are not uniformly distributed. These properties are *simple subadditivity, scale boundedness, and upper linearity.*

A functional L is *simply subadditive* if it satisfies the following property:

Property 6 (Simple Subadditivity). *$\exists B$ such that:*

$$L(A_1 \cup A_2) \leq L(A_1) + L(A_2) + tB, \tag{2.23}$$

for all finite subsets A_1 and A_2 of $[0,t]^d$, $\forall\, t > 0$.

It is is called *scale bounded* if it satisfies the following property:

Property 7 (Scale Boundedness). *$\exists B$ such that:*

$$\frac{L(\{x_1, ..., x_n\})}{tn^{1-\frac{1}{d}}} \leq B,\ \forall n \geq 1, t \geq 1, \tag{2.24}$$

and $\{x_1, ..., x_n\} \subset [0,t]^d$.

Finally, a functional L is *upper linear* if it satisfies the following property:

Property 8 (Upper Linearity). *For any finite collection of cubes Q_i, $1 \leq i \leq s$ (Q_i defined above), and any infinite sequence x_i, $1 \leq i < \infty$, in $\mathbb{R}^d$, L satisfies:*

$$\sum_{i=1}^{s} L(\{x_1, ..., x_n\} \cap Q_i) \leq L(\{x_1, ..., x_n\} \cap \cup_{i=1}^{s} Q_i) + o(n^{1-\frac{1}{d}}). \tag{2.25}$$

Using these properties allows the following theorem to hold:

Theorem 2.4. *If L is a functional that satisfies properties 1-8, and $\{X_i\}$ are i.i.d. random variables with bounded support such that the absolutely continuous part of* *their distribution is* $f(x)$, *then* $\exists\beta(L)$ *such that:*

$$\lim_{n\to\infty}\frac{L(\{X_1,...,X_n\})}{n^{1-\frac{1}{d}}}=\beta(L)\int_{\mathbb{R}^d}f(x)^{1-\frac{1}{d}}dx\ a.s. \tag{2.26}$$

An interesting part of this theorem is that $\beta(L)$ is the same from the uniform distribution case, the only difference between the the asymptotic behavior of L under different distributions of $X_1,...,X_n$ is the factor $\int_{\mathbb{R}^d}f(x)^{1-\frac{1}{d}}dx$. This means that $\beta(L)$ can be calculated from the case of the uniform distribution (for example), and then the asymptotic behavior of L can be determined under any distribution $f(x)$ of $X_1,...,X_n$.

Now that we have introduced all of the background we need from the literature to tackle the problems at hand, we start by dealing with the time a dynamic system needs to move its output between two points in its output space. We will mainly focus on the case where the two points are close to each other, in the sense that the system can travel between them in a small amount of time. We also introduce the notion of the r-warped distance between two points. The idea of r-warped distance is very central to bringing dynamic constraints to combinatorial problems.

Once we have studied the small-time behavior of dynamic systems, we can deal with the first two of our main problems. Namely, we study the TSP and DTRP for dynamic systems in detail. We produce lower bounds on the costs of the problems and algorithms that scale optimally.

Next, we introduce and study a class of functionals that we call subadditive Quasi-Euclidean functionals. The properties of these functionals are inspired by the small-time properties of the minimum time curves of dynamic systems. We introduce their properties, and produce results paralleling the results we have here for subadditive Euclidean functionals. We also show that if the weights in the graphs (as described in Section 2.1.1) are the r-warped distances between the points, then all of the costs of interest are subadditive Quasi-Euclidean functionals. This allows us to determine the asymptotic behavior of the costs of problems of interest when the weights are the r-warped distances. Since the notion of the r-warped distance was inspired by the local behavior of dynamic systems, these results bring us one step closer to the desired results on combinatorial problems for dynamic systems.

Finally, we use subadditive Quasi-Euclidean functionals to generalize the results for the TSP and DTRP of dynamic systems to a larger class of problems. We link problems for dynamic systems with the problems for the r-warped distance and deduce results about the asymptotic behavior of the costs of the given combinatorial problems under dynamic constraints.

Chapter 3

Local Behavior of Dynamic Systems

In this chapter, we study the local behavior of dynamic systems. As we will see later in this work, the local behavior of the system governs the global results in the problems that we are interested in. This is mainly because of the subadditivity property of the functionals we deal with, as we will see later in chapter 5. We will specifically study the distance a given dynamic system S can move its output in a certain direction f when it is given a small time t. Of course, the scaling (in terms of t) of the volume of the small-time reachable set will follow. For most of the thesis, we have the following assumptions on the dynamic system in (2.4):

Assumption 3.1. *$h, g_0, g_1, ..., g_m$ are analytic functions.*

Assumption 3.2. *The integral curves of $g_0, g_1, ..., g_m$ are defined.*

Assumption 3.3. *The system is small-time locally controllable, so without loss of generality, $g_0 = 0$.*

This chapter is divided into four sections: In the first section, we introduce some entities that are very important for our study of the small-time reachability properties of dynamic systems. These are the *elementary output vector fields* and their corresponding *orders*. In the second section, we bound how the area of the output-reachable set $O_{\leq T}(x^0)$ (definition 2.1) of system (2.4) scales in terms of T as $T \to 0$.

In the third section, we study how to steer system (2.4) between two points that are close together. This provides a bound on performance, and makes our results more applicable. In the final section we introduce the notion of the r-warped distance between points. We relate the idea of the r-warped distance between points in the output space of a dynamic system to the time the dynamic systems needs to move its output between those points. This notion will be very useful to us when we insert dynamics into the combinatorial problems of interest, as we will see in chapters 4 and 5.

3.1 Elementary Output Vector Fields of a Dynamic System

Given an initial state x^0, we construct a basis $\{f_1(x^0), ..., f_d(x^0)\}$ for the output space at $y^0 = h(x^0)$ as follows:

Definition 3.14 (Elementary Output Vector Fields). Let $r_1(x^0)$ be the smallest natural number such that there are r_1 vector fields in the set $\{g_1, ..., g_m\}$ (denoted $g_0^1, ..., g_{r_1-1}^1$) that satisfy:

$$\mathcal{L}\left(g_0^1, ..., \mathcal{L}\left(g_{r_1-1}^1, h(x^0)\right)\right) \neq 0. \tag{3.1}$$

Assumption 3.4. *There exists a non-zero iterated Lie bracket in the output space, designated $\mu_1(x^0)$, such that the order of $\mu_1(x^0)$ is equal to r_1.*

For $j = 2, ..., d$, let $r_j(x^0)$ be the least natural number such that $\exists\ g_0^j, ..., g_{r_j-1}^j$ with

$$\mathcal{L}\left(g_0^j, ..., \mathcal{L}\left(g_{r_j-1}^j, h(x^0)\right)\right) \tag{3.2}$$

being linearly independent of $\mu^1(x^0), ..., \mu^{j-1}(x^0)$.

Assumption 3.5. *If*

$$\mathcal{L}\left(g_0^j, ..., \mathcal{L}\left(g_{r_j-1}^j, h(x^0)\right)\right) \notin \ span\{\mu^1, ..., \mu^{j-1}\},$$

then there exists a non-zero iterated Lie bracket in the output space, designated μ_j *such that* $\mu_j \notin span\{\mu^1, ..., \mu^{j-1}\}$, *and the order of* μ_j *is equal to* r_j.

We call $\mu^1, ..., \mu^d$ the elementary output vector fields and $r^1, ..., r^d$ their corresponding *orders.*

Note that the elementary output vector fields are not necessarily orthonormal, although they form a basis for the output space (locally). To get an orthonormal basis for the output space that is more suitable for our study, note that $r_1 \leq r_2 \leq ... \leq r_d$, and define the new coordinates $f^1(x^0), f^2(x^0), ..., f^d(x^0)$ by using the Gram-Schmidt procedure:

1. $f^1(x^0)$ is a unit vector in the output space that is parallel to $\mu^1(x^0)$.

2. $f^j(x^0)$ is a unit vector in the output space that is parallel to the component of $\mu^j(x^0)$ that is orthogonal to $\mu^1(x^0), \mu^2(x^0), ..., \mu^{j-1}(x^0)$.

Definition 3.15 (Coordinate Transformation Θ). We apply a change of coordinates transformation

$$\Theta : \mathbb{R}^d \to \mathbb{R}^d, f^i \to e_i,$$

where $e_i \in \mathbb{R}^d$ is the vector whose i^{th} component is 1 and all other components are 0.

This is a change of coordinates transformation in the output space of the dynamic system, where the new coordinates of any vector $V \in \mathbb{R}^d$ are the projections of V on $f^1, ..., f^d$. We let the transform of $h(x^0)$ be $h^*(x^0)$, that is, $h^*(x^0) = \Theta h(x^0)$ and $h^*_k(x^0)$ is its k^{th} component, i.e. the projection of $h(x^0)$ on f^k.

In particular, we have the following lemma:

Lemma 3.1. *Given a dynamic system as in* (2.4)*, after applying the transformation* Θ*, we have:*

1.

$$\mathcal{L}\left(g_{i_0}, ..., \mathcal{L}\left(g_{i_r}, h^*_k(x^0)\right)\right) = 0, \forall r < r_k, i_0, ..., i_r \in \{1, ..., m\}.$$

2.

$$\mu^i_k \equiv (\Theta\mu^i)_k = 0, \quad \forall i < k, \tag{3.3}$$

where μ^i_k is the k^{th} coordinate of the vector $\mu^i(x^0)$ after the change of coordinates Θ is applied (Definition 3.15).

Proof. Note that if for a given $j \in \{1, ..., d\}$ and $i_0, ..., i_r \in \{1, ..., m\}$,

$$\mathcal{L}\left(g^i_0, ..., \mathcal{L}\left(g^i_r, h(x^0)\right)\right) \in \text{ span } \{\mu^1, ..., \mu^j\},$$

then

$$\mathcal{L}\left(g^i_0, ..., \mathcal{L}\left(g^i_r, h^*_k(x^0)\right)\right) = 0 \quad \forall j < k \leq d.$$

This is because

$$\mathcal{L}\left(g^i_0, ..., \mathcal{L}\left(g^i_r, h^*_k(x^0)\right)\right) = \mathcal{L}\left(g^i_0, ..., \mathcal{L}\left(g^i_r, h^*(x^0)\right)\right)_k = \mathcal{L}\left(g^i_0, ..., \mathcal{L}\left(g^i_r, h(x^0)\right)\right)^\top f^k(x^0) = 0.$$

Both parts of the lemma follow as special cases. □

3.1.1 W

e now consider the examples of the dynamic systems that we have from Section 2.2.1 to see how elementary output vector fields are constructed. For the linear system model we have,

$$\mu_1(x^0) = \mathcal{L}\left(g_0, h(x^0)\right) = \begin{bmatrix} x_2^0 \\ x_3^0 \\ x_4^0 \end{bmatrix},$$

$$\mu_2(x^0) = \mathcal{L}\left(g_1, h(x^0)\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{3.4}$$

$$\mu_3(x^0) = \mathcal{L}\left(g_2, \mathcal{L}\left(g_0, h(x^0)\right)\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$r_1 = 1, r_2 = 1, r_3 = 2 \text{ when } x_4^0 \neq 0.$$

For the car pulling k-trailers, let

$$P_i = \prod_{j=1}^{i} \cos(\theta_{j-1} - \theta_j),$$

then

$$\begin{aligned} \mu_1(x^0) &= \mathcal{L}\left(g_0, h(x^0)\right) \\ &= \begin{bmatrix} \cos(\theta_0) + \sum_{i=1}^{k} d_i \sin(\theta_i) \sin(\theta_{i-1} - \theta_i) P_i \\ \sin(\theta_0) - \sum_{i=1}^{k} d_i \cos(\theta_i) \sin(\theta_{i-1} - \theta_i) P_i \end{bmatrix}, \\ \mu_2(x^0) &= \mathcal{L}\left(g_1, \mathcal{L}\left(g_0, h(x^0)\right)\right) = \begin{bmatrix} -\frac{\sin(\theta_0)}{L} \\ \frac{\cos(\theta_0)}{L} \end{bmatrix}, \\ r_1 &= 1, r_2 = 2. \end{aligned} \tag{3.5}$$

3.1.2 T

o get a deeper understanding of the elementary output vector fields and their orders, consider the case of a general LTI system more closely. For an LTI system with states in $\mathbb{R}^b$ and output in $\mathbb{R}^d$ at $x^0 = 0$, the elementary output vector fields $\mu_1, \ldots, \mu_d$ are the first d vectors of

$$C\left[B|AB|A^2B|\ldots|A^kB\right],$$

such that $[\mu_1|\ldots|\mu_d]$ is full rank, and if

$$\mu_i = CA^jB, 1 \leq i \leq d, 0 \leq j \leq k,$$

then the corresponding $r_i = j + 1$.

Thus given the minimum k such that

$$C\left[B|AB|A^2B|\ldots|A^kB\right]$$

has rank d, then r_d is equal to $k+1$. It is obvious that the matrix on the right is the controllability matrix of the LTI system. This means that the orders of the elementary output vector fields (and our index k here) are indicators of "output controllability". The inputs don't have to be able to steer the state arbitrarily (like in the classical controllability condition), they just have to move it in directions that will affect the output. Of course, if the controllability matrix is full rank (and C is full rank) then the system is "output controllable." This means that the orders of the elementary output vector fields are upper bounded by the controllability index of the system.

The orders of elementary output vector fields for nonlinear systems carry the same interpretation, and thus they can represent a similar index (like the k here). Thus a local version of the controllability index can be defined for nonlinear systems that are affine in control, and its relation to the orders of the elementary output vector fields is the same as in the LTI case.

After applying the change of coordinates 3.15, we are ready to introduce our first result:

Theorem 3.5. *Given a system* (2.4) *at state* x^0 *and* $r_i(x^0)$ *as above, let*

$$||r||_1(x^0) = \sum_{j=1}^{d} r_j(x^0).$$

$\exists\ C_U(x^0), C_L(x^0) > 0$ *such that:*

$$C_L(x^0) \le \lim_{T\to 0} \frac{A_{\le T}}{T^{||r||_1(x^0)}} \le C_U(x^0),$$

where $A_{\le T}$ *is the volume of the reachable set in the output space (Definition 2.1 .)*

More specifically, we will prove the stronger version:

Theorem 3.6. *Given a dynamic system as in* (2.4) *that satisfies Assumptions 3.1-3.5 at an initial state* x^0*, after applying the transformation* Θ *in Definition 3.15,* $\exists C_l(x^0), C_U(x^0), \delta(x^0) > 0$ *such that* $\forall y^f$ *such that* $|y^f - y^0| < \delta(x^0)$ $(y^0 = h(x^0))$*, the solution of the minimum time problem:*

$$\begin{aligned}
&T_S(y^0, y^f) = \min_{u_1(\cdot),\ldots,u_m(\cdot)\in\mathbb{U},T} \int_0^T 1dt,\\
&\frac{dx}{dt} = \sum_{i=1}^m g_i(x)u_i,\\
&y = h^*(x),\\
&\mathbb{U} = \{u(.) : \text{measurable } \mathbb{R}^+ \to [-1,1]\},\\
&y(T) = y^f,
\end{aligned} \tag{3.6}$$

satisfies

$$\begin{aligned}
C_L(x^0)\max\{|y_1^f - y_1^0|^{\frac{1}{r_1(x^0)}}, \ldots, |y_d^f - y_d^0|^{\frac{1}{r_d(x^0)}}\} &\le T_S(y^0, y^f)\\
&\le C_U(x^0)\max\{|y_1^f - y_1^0|^{\frac{1}{r_1(x^0)}}, \ldots, |y_d^f - y_d^0|^{\frac{1}{r_d(x^0)}}\}.
\end{aligned} \tag{3.7}$$

To prove theorem 3.5, we will first prove that given system (2.4) at state x^0, and for $s \in \{1, 2, ..., d\}$, the *maximum distance* $d_s(t)$ that the output of system (2.4) can move in direction f_s in time t satisfies the following:

Proposition 3.1. $\exists T > 0, C_{U_s}(x^0) > 0$ *such that* $\forall t \le T$

$$\frac{d_s(t)}{t^{r_s(x^0)}} \le C_{U_s}(x^0). \tag{3.8}$$

We then prove that

Proposition 3.2. *$\exists T > 0, C_{L_s}(x^0) > 0$ such that $\forall t \leq T$*

$$\frac{d_s(t)}{t^{r_s}} \geq C_{L_s}(x^0).$$

In our detailed study of the TSP for a dynamic system in Chapter 6, we will show that the upper bound on the area of the reachable set is important for the lower bound on the expected time the system needs to trace $\mathcal{C}_P$. The lower bound is useful for the steering algorithm we will use as a sub-algorithm in Section 6.2.1.

The following two sections present the proofs of propositions (3.1) and (3.2). We start by recalling theorem 2.2, which is central to our proof:

Theorem. *The j^{th} component of the output of system (2.4) evolves as follows:*

$$y_j(t) = h_j(x^0) \tag{3.9}$$

$$+\sum_{k=0}^{\infty}\sum_{i_0=0}^{m}\cdots\sum_{i_k=0}^{m}\mathcal{L}\left(g_{i_0},...,\mathcal{L}\left(g_{i_k},h_j(x^0)\right)\right)\int_0^t d\xi_{i_k}...d\xi_{i_0},$$

Where the integral is defined iteratively by:

$$\xi_0(t) = t,$$

$$\xi_j(t) = \int_0^t u_j(\tau)d\tau,$$

$$\int_0^t d\xi_{i_k}...d\xi_{i_0} = \int_0^t d\xi_{i_k}(\tau)\int_0^\tau d\xi_{i_{k-1}}...d\xi_{i_0}.$$

We introduced this theorem in Chapter 2, analyzed it, and looked at its implications. We now use it as the basis for the proofs of propositions (3.1) and (3.2). The proof of this theorem can be found in [25] and is in the appendix for easy reference.

3.2 Bounds on the Area of the Reachable Set

3.2.1 Upper Bound on the Volume of the Small-Time Reachable Set

This upper bound is given by the following lemma:

Lemma 3.2. *Given system (2.4) at state x^0, for all $s \in \{1, 2, ..., d\}$ and $r_s(x^0)$ as in Definition 3.14, (after applying the transformation Θ as described in Definition 3.15) there exists T, and $C_{U_s} > 0$ such that for all $t < T$ and $u(\cdot) \in \mathbb{U}$,*

$$|y_s(t) - y_s(0)| < C_{U_s(x^0)} t^{r_s(x^0)}.$$

Proof. From lemma 3.1, we have that $\forall k$ such that $0 \leq k < r_s - 1$,

$$\mathcal{L}\left(g_{i_0}, ..., \mathcal{L}\left(g_{i_k}, h_s^*(x^0)\right)\right) = 0.$$

Plugging this in (2.19) produces:

$$y_s(t) = y_s(0)$$
$$+ \sum_{k=r_s-1}^{\infty} \sum_{i_0=0}^{m} ... \sum_{i_k=0}^{m} \mathcal{L}\left(g_{i_0}, ..., \mathcal{L}\left(g_{i_k}, h_s^*(x^0)\right)\right) \int_0^t d\xi_{i_k} ... d\xi_{i_0}.$$

To establish the bound, we use the following two facts:

1. Since $g_i(\cdot)$, $x_j(\cdot)$, and their partial derivatives of any order with respect to x are bounded around x^0, $\exists M_1$ such that:

$$\mathcal{L}\left(g_{i_0}, ..., \mathcal{L}\left(g_{i_k}, h_j^*(x^0)\right)\right) \leq M_1^{k+1}. \tag{3.10}$$

2. Since $|u_j(\cdot)| \leq 1$,

$$\int_0^t d\xi_{i_k} ... d\xi_{i_0} \leq \frac{(t)^{k+1}}{(k+1)!}. \tag{3.11}$$

Plugging in the bounds from (3.10) and (3.11), we get:

$$|y_j(x(t)) - y_j(x^0)| \leq \sum_{k=r_s-1}^{\infty} \frac{(mM_1t)^{k+1}}{(k+1)!}.$$

Therefore, $\exists T$ small enough such that the sum is convergent, and the lemma is proven. The lower bound in theorem 3.6 follows directly. □

3.2.2 Lower Bound on the Volume of the Reachable Set

To prove the upper bound on T_S, consider the truncated system S_T that is formed by annihilating

$$\mathcal{L}\left(g_{i_0}, ..., \mathcal{L}\left(g_{i_k}, h_s^*(x^0)\right)\right)$$

of the original system for $k > r_s$ (after Θ was applied to the output of the original system.)

The output of this system is given by

$$\tilde{y}_j = y_j - \sum_{k=r_j}^{\infty} \sum_{i_0=0}^{m} ... \sum_{i_k=0}^{m} \mathcal{L}\left(g_{i_0}, ..., \mathcal{L}\left(g_{i_k}, h_j(x^0)\right)\right) \int_0^t d\xi_{i_k} ... d\xi_{i_0}. \tag{3.12}$$

By theorem 2.2, $\exists T > 0$ such that $\forall t \leq T$, the j^{th} component of the output of this system $\tilde{y}_j$ evolves as:

$$\begin{aligned} &\tilde{y}_j(t) = h_j^*(x^0) \\ &+ \sum_{i_0=0}^{m} ... \sum_{i_{r_j-1}=0}^{m} \mathcal{L}\left(g_{i_0}, ..., \mathcal{L}\left(g_{i_{r_j-1}}, h_j^*(x^0)\right)\right) \int_0^t d\xi_{i_{r_j-1}} ... d\xi_{i_0}. \end{aligned} \tag{3.13}$$

Steering Algorithm For the Truncated System (The Component Steering Algorithm)

The first step in the proof is to design a steering algorithm for the truncated system such that starting from an initial state x^0, $\exists C_U(x^0), \delta(x^0) > 0$ such that $\forall y^f$ such that $|y^f - y^0| < \delta(x^0)$, the algorithm steers the output of the truncated system from y^0 to y^f in time $T_{ST}(y^0, y^f)$ that satisfies:

$$T_{ST}(y^0, y^f) \leq C_U(x^0) \max\{|y_1^f - y_1^0|^{\frac{1}{r_1(x^0)}}, ..., |y_d^f - y_d^0|^{\frac{1}{r_d(x^0)}}\}.$$

We start with a lemma

Lemma 3.3. *This family of inputs $\{Ap(u_i, t)\}$ has the following properties when applied to the truncated system:*

1. *If the control input $Ap([u^i], t)$ (with order r) is applied, then*

$$\tilde{y_s}(rt) = \tilde{y_s}(0), \quad \forall s \text{ such that } r_s < r.$$

2. *If $[u^i]$ has the same indices as μ^i, and the control input $Ap([u^i], t)$ is applied, then*

$$\tilde{y_i}(rt) = \tilde{y_i}(0) + ct^{r_i},$$

   where r_i is the order of μ^i. Similarly, if $Ap(-[u^i], t)$ is applied, then

$$\tilde{y_i}(rt) = \tilde{y_i}(0) - ct^{r_i}.$$

3. *If $[u^i]$ has the same indices as μ^i, and the control input $Ap([u^i], t)$ is applied, then*

$$\tilde{y_s}(rt) = \tilde{y_s}(0), \quad \forall s \text{ such that } s > i, r_s = r_i.$$

Proof. The first two claims result from plugging in the property of the truncated system

$$\mathcal{L}\left(g_{i_1}, ..., \mathcal{L}\left(g_{i_k}, h_s^*(x^0)\right)\right) = 0, \quad \forall k > r_s,$$

in Theorem 2.1.

The third claim follows again from plugging in the property of the truncated system

$$\mathcal{L}\left(g_{i_1}, ..., \mathcal{L}\left(g_{i_k}, h_s^*(x^0)\right)\right) = 0, \quad \forall k > r_s,$$

in Theorem 2.1. This results in

$$\tilde{y_k}(rt) = \tilde{y_k}(0) + ct^{r_i},$$

where c is the projection of μ^i on f_k. By using the second claim of lemma 3.1, it follows that $c = 0$.

□

These facts about the evolution of the output when inputs from $\{Ap(u^i, t)\}$ are applied are used in the following steering algorithm for the truncated system:

Algorithm Description

To simplify the description of the algorithm, we rearrange the coordinates in the output space of the dynamic system as follows:

Definition 3.16 (Transformation σ). For all $i, i+1, ..., i+c$ such that $r(\mu^i) = r(\mu^{i+1}) = \cdots = r(\mu^{i+c})$, $r(\mu^{i-1}) \neq r(\mu^i)$, and $r(\mu^{i+c}) \neq r(\mu^{i+c+1})$ (for completeness, we define $r(\mu^0) = 0$ and $r(\mu^{d+1}) = \infty$), then $i, ..., i+c$ are reversed. This means that the rearrangement $\sigma(j)$ is defined as

$$\sigma(j) = (2i + c - j), \quad j \in \{i, ..., i+c\}.$$

This rearrangement just switches the components in the output space that have the same order. This is done so that lemma 3.3 after the transformation would be:

Lemma 3.4. *After the transformations Θ and σ (Definitions 3.15 and 3.16) are applied in the output space of the dynamic system described as in* (2.4), *we have:*

1. *If $[u^i]$ has the same indices as μ^i, and the control input $Ap([u^i], t)$ is applied, then*

$$\tilde{y}_i(rt) = \tilde{y}_i(0) + ct^{r_i},$$

where r_i is the order of μ^i. Similarly, if $Ap(-[u^i],t)$ is applied, then

$$\tilde{y}_i(rt) = \tilde{y}_i(0) - ct^{r_i}.$$

2. If $[u^i]$ has the same indices as μ^i, and the control input $Ap([u^i],t)$ is applied, then

$$\tilde{y}_s(rt) = \tilde{y}_s(0), \quad \forall s < i.$$

Here, $\tilde{y}$ is the output of the truncated dynamic system given by equation (3.12), *$\mu_i, r_i \quad i \in \{1,...,d\}$ are as in Definition 3.14.*

Thus the algorithm after the transformations are applied steers the components of the output in ascending order using the inputs $Ap([u^i],t)$, where u^i has the same indices as μ^i. This is because lemma 3.4 guarantees that steering $\tilde{y}_i$ this way doesn't affect $\tilde{y}_j$ if j is less than i, and therefore after the d^{th} component is steered, the output of the truncated system reaches the final point exactly. The algorithm is described as follows:

1. Set the counter $i = 1$. For $j = 1,...,d$, let $T_l^j = 0$, for $j \in \{1,...,d\}$ (T_l^j is the time the steering algorithm takes before steering the j^{th} component of $\tilde{y}$.) Let $\epsilon_j = 0$, for $j \in \{1,...,d\}$ (ϵ_j is the drift in the j^{th} component of $\tilde{y}$ that was caused by steering $\tilde{y}_1,...,\tilde{y}_{j-1}$.

2. If $\sigma_i > 0$, $Ap([u^i], C_i^A \sigma_i^{\frac{1}{r_i}})$, otherwise $Ap(-[u^i], C_i^A \sigma_i^{\frac{1}{r_i}})$. Here, $[u^i]$ is the input with the same indices as μ^i, and $C_i^A = c_i^{\frac{-1}{r_i}}$, where c_i is the projection of μ_i on f_i.

3. If $y_j^f - y_j^0 > 0$, $Ap([u^i], C_i^A(|y_i^f - y_i^0|)^{\frac{1}{r_i}})$, otherwise $Ap([u^i], C_i^A(|y_i^f - y_i^0|)^{\frac{1}{r_i}})$. Here,$[u^i]$ is the input with the same indices as μ^i, and $C_i^A = c_i^{\frac{-1}{r_i}}$, where c_i is the projection of μ_i on f_i.

4. Let $T_l^{i+1} = T_l^i + C_i^A(|y_i^f - y_i^0|)^{\frac{1}{r_i}}) + C_i^A \sigma_i^{\frac{1}{r_i}})$, $\epsilon_{i+1} = \tilde{(y)}_{i+1}(T_l^{i+1}) - \tilde{y}(0)$. Increment i by 1 and go to Step 2.

Lemma 3.5. *Given the truncated system whose output is described as in equation (3.12) at an initial state x^0, $\exists C_U(x^0)$ and $\delta(x^0) > 0$ such that $\forall y^f$ such that $|y^f - y^0| < \delta(x^0)$, the algorithm steers the output of the truncated system from y^0 to y^f in time $T_{ST}(y^0, y^f)$ that satisfies:*

$$T_{ST}(y^0, y^f) \leq C_U(x^0) \max\{|y_1^f - y_1^0|^{\frac{1}{r_1(x^0)}}, ..., |y_d^f - y_d^0|^{\frac{1}{r_d(x^0)}}\}.$$

Proof. □

For every j, we let T_1^j be the time needed by the algorithm to steer the j^{th} component in Step 2 of the algorithm and let T_2^j be the time needed by the algorithm to steer the j^{th} component in Step 3 of the algorithm. Note that T_l^j was the total time needed to steer the components lower than j.

By lemma 3.2, $\exists T_1 > 0$, and $C_{U_j}(x^0) > 0$ such that if the total time for steering lower order components was T_l^j, then $\epsilon_j = |y_j(T_l^j) - y_j(0)| \leq C_{U_j}(x^0)(T_l^j)^r$ (if $T_l^j < T_1$). Therefore $\exists T_1, C_1^j(x^0) > 0$ such that if $T_l^j < T_1$, the time needed to steer the j^{th} component in Step 2 T_1^j satisfies

$$T_1^j \leq C_1^j(x^0) T_l^j.$$

Similarly, $\exists C_2^j > 0$ such that the time needed to steer the j^{th} component in Step 3 similarly satisfies

$$T_2^j \leq C_2^j 2|y_j^f - y_j^0|^{\frac{1}{r_j}}.$$

Thus this algorithm steers the output of the truncated system from y^0 to y^f in time

$$\begin{aligned} T &\leq C_2^d |y_d^f - y_d^0|^{\frac{1}{r_d}} + (C_1^d + 1) T_l^d \\ &\leq C_2^d |y_d^f - y_d^0|^{\frac{1}{r_d}} + (C_1^d + 1)\left(C_2^{d-1}|y_{d-1}^f - y_{d-1}^0|^{\frac{1}{r_{d-1}}} + (C_1^{d-1} + 1) T_l^{d-1}\right) \\ &\leq ... \leq C \sum_{i=1}^{d} |y_i^f - y_i^0|^{\frac{1}{r_i}}, \end{aligned} \tag{3.14}$$

for some $C > 0$. Therefore $\exists C_U > 0$ such that

$$T_{ST}(y^0, y^f) \leq C_U \max_i(|y_i^f - y_i^0|^{\frac{1}{r_i}}).$$

Steering the Original System

We finally use the following algorithm to steer the original system:

1. Set the counter $i = 1$, and let

$$T_{ST} \leq C_U \max_i(|y_i^f - y_i^0|^{\frac{1}{r_i}})$$

   be the time to steer the output of the truncated system from y^0 to y^f using the Component Steering Algorithm.

2. Steer the system by applying the control $u_1^*, ..., u_m^*$, which steers the truncated system from y^{i-1} to y^f. Let y^i be the resulting point, and x_i be the resulting state.

3. If $x_i \in R_{T_{ST}}(x^f)$, for some x^f such that $h(x^f) = y^f$, stop. Otherwise, increment i by 1 and return to step 2.

This algorithm doesn't steer the output of the system to y^f, but it guarantees that it can be steered to y^f quickly enough. To finish the proof, note that $\exists T_2 > 0, C_3 > 0$ such that if $T_{ST} < T_2$, then

$$|y_j(T_{ST}) - y_j^{ST}(T_{ST})| < CT_{ST}^{r_j+1}.$$

This is seen by following a proof similar to the proof of lemma 3.2. Since

$$T_{ST} \leq C_U \max_i(|y_i^f - y_i^0|^{\frac{1}{r_i}}),$$

it follows that $\exists \delta_2 > 0$ such that

$$|y_j(T_{ST}) - y_j^{ST}(T_{ST})| \leq \frac{1}{2^{r_d}}|y_j(0) - y_j^{ST}(0)|, \quad \text{if } \max_i(|y_i^f - y_i^0|^{\frac{1}{r_i}}) < \delta_2.$$

Thus if

$$\max_i(|y_i^f - y_i^0|^{\frac{1}{r_i}}) < \delta_2,$$

the i^{th} step of the algorithm will steer the output in a time duration not more than $\frac{T_{ST}}{2^{i-1}}$. Thus y^i will be arbitrarily close to y^f in time that is less than $2T_{ST}$. More specifically, the state at the end of step i, x^i, will be arbitrarily close to the set $\{x : h(x) = y^f\}$. This means that in a finite number of steps (and steering time less than $2T_{ST}$), the state will be in $R_{T_{ST}}(x^f)$ for some x^f such that $h(x^f) = y^f$.

Finally, since $\exists T_3 > 0$ such that $\forall T_{ST} < T_3$, if $x_i \in R_{T_{ST}}(x^f)$, then $x^f \in R_{T_{ST}}(x^i)$, then $\exists \delta > 0$ such that if $|y^f - y^0| < \delta$, the output can be steered from y^0 to y^f in time that is not more than

$$3T_{ST} \leq 3C_U \max_i(|y_i^f - y_i^0|^{\frac{1}{r_i}}).$$

3.3 Locally Steering Dynamic Systems

In this section, we study the implications of proposition (3.2) on the small-time maneuverability of system (2.4) that might possibly have drift. To steer system (2.4) between two close points, we will steer it along the individual f_i directions ($i = 1, 2, ..., d$) after the transformations Θ and σ are applied. This steering algorithm is used in the algorithms that we produce for the TSP and MBMP. We will prove the following property:

Lemma 3.6. *Given two points y^i and y^f in the output space of system* (2.4), *such that:*

$$\left(\frac{|y_j^i - y_j^f|}{C_{L_j}}\right)^{\frac{1}{r_j}} \leq \frac{1}{(q-1)(q-j+1)} \left(\frac{|y_1^i - y_1^f|}{C_{U_1}}\right)^{\frac{1}{r_1}}, \tag{3.15}$$

for $j = 2, ..., d$.

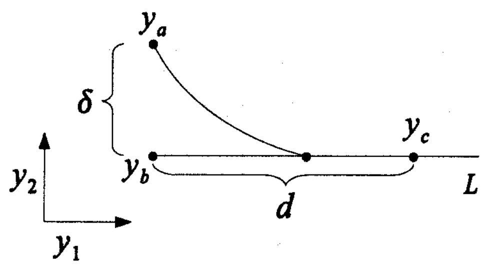

Figure 3-1: Steering the output locally

Then system (2.4) *can be steered from* y^i *to* y^f *in time that is less than*

$$\left(qC_{L_1}^{-\frac{1}{r_1}} + C_{U_1}^{-\frac{1}{r_1}}\right)|y_1^i - y_1^f|^{\frac{1}{r_1}}.$$

The inequalities in (3.15) mean that the distances in the directions of $f_2, ..., f_d$ are "small" compared to the distance between the points in the f_1 direction. The result here means that if system (2.4) is required to move between two close points in the output space that satisfy the relation (3.15), then it can do so in a time that is proportional to time needed to move from y_1^i to y_1^f (Fig. 3-1).

Consider the situation in figure 3-1, where the system's output is steered from y^a to y^c, and we have $\delta = \frac{|(y^a - y^c).f_j|}{|f_j|}$ and $\epsilon = \frac{|(y^a - y^c).f_1|}{|f_1|}$.

To prove lemma (3.6), system (2.4) will be steered along $f_1, ..., f_d$ (in that order), if the system under study is locally controllable, and $f_2, ..., f_d$ if it has drift.

In both cases, we will prove that the time for steering the system in the $f_2, ..., f_d$ directions takes less time than

$$|y_1^i - y_1^f|^{\frac{1}{r_1}}.$$

Note that if there is drift, this still guarantees that the distance traveled by the system in the drift direction (f_1) is less than $|y_1^i - y_1^f|$ and so the system can still move in small time to y^f.

Proof. Lemma (3.6)

The proof of this lemma uses the fact that $f_1, ..., f_d$ were designed such that f_j

has an additional degree of freedom over $f_1, ..., f_{j-1}$. This means that given a small time t, the system's output can be steered in the f_j direction a distance larger than $C_{L_j} t^{r_j}$ without moving in the $f_1, ..., f_{j-1}$ direction.

Additionally, if the system's output have moved in the $f_1, ..., f_{j-1}$ directions for a small time τ, the drift these motions caused in the f_j direction can be countered also in time τ.

Therefore, to move from y^i to y^f, the system's output needs to move in the f_2 direction for time $t_1 + t_2 \leq t_1 \left(\frac{|y_2^i - y_2^f|}{C_{L_2}}\right)^{\frac{1}{r_2}}$. The time it needs to move in the f_3 direction is less than $t_1 + t_2 + \left(\frac{|y_3^i - y_3^f|}{C_{L_3}}\right)^{\frac{1}{r_3}}$, and so on.

Therefore, the time to move from y^i to y^f along all coordinates other than f_1 is

$$t_{-1} \leq \sum_{j=2}^{q} (q - j + 1) \left(\frac{|y_j^i - y_j^f|}{C_{L_j}}\right)^{\frac{1}{r_j}} + (q-1)t_1,$$

which by using equation (3.15) is not more than

$$\left(\frac{|y_1^i - y_1^f|}{C_{U_1}}\right)^{\frac{1}{r_1}}.$$

Therefore the time to steer the system's output from y^i to y^f is less than

$$\left(qC_{L_1}^{-\frac{1}{r_1}} + C_{U_1}^{-\frac{1}{r_1}}\right) |y_1^i - y_1^f|^{\frac{1}{r_1}}.$$

□

In Chapter 6, we will use motions like the ones described here to steer the output between two points. Of course, we will make sure that δ and ϵ are small and that condition (3.15) is satisfied.

3.4 r-warped Distance

In this section, we introduce the r-warped distance, which plays an important role in our study of subadditive Quasi-Euclidean functionals. The reason we are interested in the r-warped distance is that it is tightly related to the time a dynamic system needs to travel between two close points in its output space, as we saw in this section.

It directly follows from theorem 3.5 that around any point y^i in the output space of system S (where S is locally controllable (Definition 2.2) at some x such that $h(x) = y$) , there is a set $B(y^i)$ (with a nonempty interior) and constant C_L and C_U such that for every point $y^f \in B(y^i)$, the minimum time needed for the output of system s to move from y^i to y^f ($T_s(y^i, y^f)$) satisfies:

$$C_L \max\{|y_1^f - y_1^i|^{\frac{1}{r_1}}, ..., |y_d^f - y_d^i|^{\frac{1}{r_d}}\} \leq T_S(y^i, y^f) \leq C_U \max\{|y_1^f - y_1^i|^{\frac{1}{r_1}}, ..., |y_d^f - y_d^i|^{\frac{1}{r_d}}\}. \tag{3.16}$$

The previous inequality means that (locally), the time needed for the output to move between two points scales like the infinity norm of a "warped" version of the distance between them. That is, given two points y^i and y^f, we can apply a transformation that takes the j^{th} component $y_j^f - y_j^i$ to $|y_j^f - y_j^i|^{\frac{1}{r_j}}$ and take the infinity norm of the transformed difference to get a "good" estimate of the system needs to move its output from y^i to y^f. More formally, we have the following definition:

Definition 3.17 (r-warped distance)**.** Given an ordered set of positive integers $r = \{r_1, ..., r_d\}$ and two points y^i and y^f, let $l(y^i, y^f) = |y^f - y^i|$ and $r^{inv} = \{\frac{1}{r_1}, ..., \frac{1}{r_d}\}$ be the ordered set of the inverses of the elements of r. r^{inv} has the order that is inherited from the indexes in r. We define $l^r(y^i, y^f) = k^{r^{inv}}\left(l(y^i, y^f)\right)$, where k^r is the function introduced in definition (2.12). We call $l^r(y^i, y^f)$ the r-warped distance between y^i and y^f.

Since we are working in finite dimensional output spaces, all p-norms are equivalent if $p > 0$, and so we know that $\forall p > 0, \exists C_L(p), C_U(p)$ such that

$$C_L(p)||l^r(y^i, y^f)||_p \leq T_S(y^i, y^f) \leq C_U(p)||l^r(y^i, y^f)||_p. \tag{3.17}$$

This fact will be of important use for us in our later development, for it links the time the system's output needs to move between two points to the r-warped distance between those points. Thus it provides a link between the dynamic study of a problem and its static (geometric) counterpart. We are now ready to study the TSP and MBMP for dynamic systems and establish the asymptotic behavior of their costs. After that, We introduce subadditive Quasi-Euclidean functionals, which are a generalization of subadditive Euclidean functionals that have invariance properties inherited from the r-warped distance we introduced here. These will be very useful when we generalize our results for the TSP to a larger class of combinatorial problems.

Chapter 4

Quasi-Euclidean Functionals

In this chapter, we introduce a new class of functionals: Subadditive Quasi-Euclidean functionals. These functionals are similar to subadditive Euclidean functionals, but are non-isotropic. They inherit their heterogeneity structure from the r-warped distance that we defined in the previous chapter. The intuition is that since the r-warped distance behaves like the time a dynamic system needs to travel locally between two points in its output space, then we can relate subadditive Quasi-Euclidean functionals to the costs of problems for dynamic systems. We start by introducing the notation used and then the properties of this class of functionals.

4.1 Notation for Quasi-Euclidean Functionals

The Quasi-Euclidean functionals we introduce in this chapter are generalizations of Subadditive Euclidean Functionals, and we show that they have similar asymptotic properties. For our study of subadditive Quasi-Euclidean functionals, we use the following notation:

Given a set $r = \{r_1, ..., r_d\}$ such that $r_1 \leq, r_2, ..., \leq r_d$, and $r_i \in \mathbb{N}$ for $i = 1, ..., d$, we denote their sum as

$$||r||_1 = \sum_{i=1}^{d} r_i.$$

Definition 4.18 (r-cuboids). Let $Q_i(r, m)$, $1 \leq i \leq m^{||r||_1}$, be a partition of the

d-cube $[0,1]^d$ into cuboids with sides that are parallel to the axis and are such that the length of the side parallel to the i^{th} axis is $\frac{1}{m^{r_i}}$.

With no loss of generality, we let $Q_1(r,m)$ be the cuboid containing the origin. These are the cuboids that we use to establish a new subadditivity property similar to property 5 of the Subadditive Euclidean functionals.

Finally, we define the multiplication operator $*$ as the componentwise multiplication in $\mathbb{R}^d$:

$$(y * z)_i = y_i z_i,$$

and use $k^r(t) * Q_i(r,m)$ (k^r was introduced in Definition 2.12) to indicate the cuboid defined as:

$$\{z : z = k^r(t) * y, y \in Q_i(r,m)\}.$$

Note that $k^r(t) * Q_i(r,m)$ has sides that are parallel to the axis and the length of the side parallel to the i^{th} axis is $\left(\frac{t}{m}\right)^{r_i}$.

4.2 Quasi-Euclidean Functionals' Properties

We will introduce Quasi-Euclidean functionals in a way that parallels the description of Euclidean functionals in Section 2.4. Let $r = \{r_1, ..., r_d\}$ be as above, we call a functional $L_r(\{y_1, ..., y_n\}), y_i \in \mathbb{R}^d$, Quasi-Euclidean with parameter set $r = \{r_1, ..., r_d\}$ if it satisfies the following two properties:

Property 1A (Structured Heterogeneity). *$\forall \alpha \in \mathbb{R}^+$ and $y \in R^d$,*

$$L_r\left(\{k^r(\alpha) * y_1, ..., k^r(\alpha) * y_n\}\right) = \alpha L_r(\{y_1, ..., y_n\}).$$

This scaling property seems similar to the one for Euclidean functionals, except that the scalings of different dimensions of the space are warped according to r. It is evident that if $r_i = 1 \; \forall i$, then these two properties describe a Euclidean functional.

Property 2A (Translation Invariance). *$L_r(\{y_1, ..., y_n\}) = L_r(\{y_1 + y, ..., y_n + y\}) \; \forall y, y_1, ..., y_n \in \mathbb{R}^d$.*

We introduce properties that are similar to properties 3-4. Thus a functional is called *bounded* and *monotone* respectively if it satisfies the following two properties:

Property 3A (Boundedness). *If $Y_1, ..., Y_n$ are independently and uniformly distributed random variables in $\mathbb{R}^d$, $Var(L_r(\{Y_1, ..., Y_n\})) < \infty$.*

Property 4A (Monotonicity). *For every finite subset A of $\mathbb{R}^d$,*

$$L_r(A \cup y) \geq L_r(A).$$

We assume that $L_r(\phi) = 0$, and so $L_r(A) \geq 0 \quad \forall$ finite subsets A of $\mathbb{R}^d$.

Let $Q_i(r,m)$, $1 \leq i \leq m^{||r||_1}$, be a partition of the d-cube $[0,1]^d$ into cuboids that are as in Section 2.4 , Definition 4.18. A Quasi-Euclidean functional L_r with parameter set $r = r_1, ..., r_d$ is called *subadditive with respect to r* if it satisfies the following property.

Property 5A (Subadditivity).

$$\exists C \in \mathbb{R} \text{ such that } \forall m \in \mathbb{N} \text{ and } t \in \mathbb{R}^+,\ y_1, ..., y_n \in \mathbb{R}^d,$$

$$L_r(\{y_1, ..., y_n\} \cap k^r(t) * [0,1]^d) \leq \sum_{i=1}^{m^{||r||_1}} L_r(\{y_1, ..., y_n\} \cap k^r(t) * Q_i(r,m)) + Ctm^{||r||_1 - 1}. \tag{4.1}$$

The previous properties are all that is needed for the case of independent uniformly distributed variables Y_i. Some additional properties are needed for a subadditive functional for the results with non-uniform distributions to hold. These properties are the *simple subadditivity, scale boundedness with respect to r,* and *upper linearity with respect to r*:

Property 6A (Simple subadditivity). *$\exists B$ such that:*

$$L(A_1 \cup A_2) \leq L(A_1) + L(A_2) + tB, \tag{4.2}$$

for all finite subsets A_1 and A_2 of $[0,t]^d$.

Property 7A (Scale Boundedness with respect to r).

$$\frac{L(\{y_1, ..., y_n\})}{tn^{1-\frac{1}{\|r\|_1}}} \leq B, \; \forall n \geq 1, t \geq 1, \tag{4.3}$$

and $\{y_1, ..., y_n\} \subset [0, t]^d$.

Property 8A (Upper Linearity). *For any finite collection of cubes* $Q_i(r, m)$, $1 \leq i \leq m^{\|r\|_1}$ *($Q_i(r, m)$ defined above), and any infinite sequence* y_i, $1 \leq i < \infty$, *in* $\mathbb{R}^d$, *the functional* L *satisfies:*

$$\sum_{i=1}^{s} L_r(\{y_1, ..., y_n\} \cap Q_i(r, m)) \leq L_r(\{y_1, ..., y_n\} \cap \cup_{i=1}^{s} Q_i(r, m)) + o(n^{1-\frac{1}{\|r\|_1}}). \tag{4.4}$$

We now turn to establish the asymptotic properties of Quasi-Euclidean functionals. We produce a theorem for the case where the variables are uniformly and independently distributed in the $[0, 1]^d$ cube, and another for the case where the distribution is not uniform but has bounded support.

4.3 Quasi-Euclidean Functionals' Results

We present results for Quasi-Euclidean functionals that are direct parallels to the results in [1] for Euclidean functionals. The proofs will be left to the appendix, and the main focus here will be on the implications of these theorems. The first theorem is similar to theorem 2.3:

Theorem 4.7. *Given* $Y_1, ..., Y_n$ *are identically and uniformly distributed in* $[0, 1]^d$, *and a functional* L_r *that satisfies properties 1A-5A with parameter set* $r = \{r_1, ..., r_d\}$, $L_r(\{Y_1, ..., Y_n\})$ *satisfies:*

$$\lim_{n \to \infty} \frac{L_r(\{Y_1, ..., Y_n\})}{n^{1-\frac{1}{\|r\|_1}}} = \beta(L_r) \; a.s. \tag{4.5}$$

This theorem tells us that subadditive Quasi-Euclidean functionals are sub-linear asymptotically, just like subadditive Euclidean functionals. More importantly, in contrast to the $1 - \frac{1}{d}$ exponent, they have a $1 - \frac{1}{\|r\|_1}$ exponent. This means that the larger

$||r||_1$ is, the closer the behavior is to linearity. Thus the asymptotic cost of a subadditive Quasi-Euclidean functional increases when any of the $r_i, i \in \{1, .., d\}$ increases. This implies that the parameters r_i are indicators of the cost of the functional. In chapter 5, we will relate r of a subadditive Quasi-Euclidean functionals to the set of orders of elementary output vector fields of a dynamic system. This tells us that the orders of the elementary output vector fields of the system are measures of how the cost of the combinatorial problems behaves asymptotically. Namely, the larger the orders of the output vector fields of the dynamic system, the larger the cost will be.

4.3.1 Variables with General Distributions

As in the case of Euclidean functionals, more restrictions are needed to deal with random variables that are not uniformly distributed. One of the main differences between the case of the uniform and non-uniform random variables is the possibility of singular distributions. Direct generalizations of theorem 4.7 can deal with some non-uniform distributions, but not those that have a singular component.

We therefore first turn to a lemma that deals with variables in $\mathbb{R}^d$ that have bounded, singular distributions. Intuitively, since a singular distribution constrains the random variables to a subspace with less than d dimensions, then the evaluation of the Quasi-Euclidean functional L_r on points generated from that distribution would be small compared to it's evaluation on points generated from a non-singular distribution. The intuitive reason is that we take the sum of r_i over a subset of the dimensions, and thus get something less than $||r||_1$. The following lemma confirms our intuition:

Lemma 4.7. *Consider a Quasi-Euclidean functional L_r with parameter set $r = \{r_1, ..., r_d\}$ satisfying properties 6A (Scale boundedness with respect to $r_1, ..., r_d$) and 7A (simple subadditivity). Given $X_i, 1 \leq i < \infty$ are i.i.d. random variables with singular support E, then:*

$$\lim_{n \to \infty} \frac{L_r(X_1, ..., X_n)}{n^{1 - \frac{1}{||r||_1}}} = 0 \quad a.s.$$

This lemma is important for the result with general distributions, since those might include a singular part. Additionally, it gives an easy test to whether a certain functional is not a subadditive Quasi-Euclidean function: If a given functional satisfied properties 6A and 7A, and there is a configuration of points for which the functional scales as $\Omega(n^{1-\frac{1}{\|r\|_1}})$, then the functional is not a subadditive Quasi-Euclidean function.

This test is useful because it only needs *one* configuration of points. Additionally, this result combined with theorem 4.7 tells us that the worst case cost of a subadditive Quasi-Euclidean function is also $\Theta(n^{1-\frac{1}{\|r\|_1}})$.

The next result targets Quasi-Euclidean functionals on variables that are "the middle ground" between uniformly distributed and a generally distributed random variables: Variables that are uniform on a finite number of small cuboids (that have the form of $Q_i(r,m)$ described above). Given variables Y_i that are uniformly distributed on a finite number of cuboids, theorem 4.7 almost immediately implies that $L_r(Y_1,...,Y_n)$ is $\Theta(n^{1-\frac{1}{\|r\|_1}})$. Still, the interesting part of the result is what the constant is in terms of the distribution. From this information, we will be able to form results for variables with general distributions that describe both the asymptotic behavior (in terms of powers of n) and the constant of scaling in terms of $\beta(L_r)$ and the absolutely continuous part of the distribution ($f(x)$). Finally, note that by lemma 4.7 the singular part of the distribution will asymptotically have a negligible contribution to L_r.

Since we are now dealing with variables that are non-uniformly distributed, we require L_r to satisfy all the properties:

Lemma 4.8. *Let L_r be a Quasi-Euclidean functional with parameter set $r = \{r_1, ..., r_d\}$ that satisfies properties 1A-8A. Given Y_i, $1 \le i < \infty$ that are i.i.d., have a distribution whose absolutely continuous part is $g(x) = \sum_{i=1}^{s} a_i 1_{Q_i(r,m)}(x)$, where $r = \{r_1, ..., r_d\}$, $Q_i(r,m)$ are cuboids as Definition 4.18, and $1_{Q_i(r,m)}(x) = 1$ if $x \in Q_i(r,m)$ and is 0 otherwise, then*

$$\lim_{n\to\infty} \frac{L_r(Y_1,...,Y_n)}{n^{1-\frac{1}{\|r\|_1}}} = \beta(L_r) \int_{\mathbb{R}^d} g(x)^{1-\frac{1}{\|r\|_1}} dx, \quad a.s.$$

As indicated before, the interesting part of this lemma is that the constant of the scaling is changed by a factor that is $\int_{\mathbb{R}^d} g(x)^{1-\frac{1}{\|r\|_1}} dx$. This means that if $\beta(L_r)$ is known for a certain Quasi-Euclidean functional under the uniform distribution, the asymptotic behavior of that functional can be calculated for any distribution $g(x)$ as above.

Finally, we turn to a result establishing the same property as lemma 4.8 for any general distribution $f(x)$, which might have a singular part.

Theorem 4.8. *If L_r is a functional that satisfies properties 1A-8A with parameter set $r = \{r_1, ..., r_d\}$, and $\{Y_i\}$ are i.i.d. random variables with bounded support such that the absolutely continuous part of their distribution is $f(z)$, then $\exists \beta(L_r)$ such that:*

$$\lim_{n\to\infty} \frac{L_r(Y_1, ..., Y_n)}{n^{1-\frac{1}{\|r\|_1}}} = \beta(L_r) \int_{\mathbb{R}^d} f(x)^{1-\frac{1}{\|r\|_1}} dx, \tag{4.6}$$

This final theorem uses lemmas 4.7 and 4.8 to mirror theorem 2.4 for subadditive Euclidean functionals. It has all the usefulness of the previous theorem and both lemmas, but imposes more constraints on the functional.

4.3.2 Requirements Relaxations

In this section, we relax the properties of Quasi-Euclidean functionals while making sure that the results still hold. This will allow Quasi-Euclidean functionals to deal with a larger array of applications. We will give examples of such applications in Section 4.4. We first relax property 4A to accommodate a larger class of functionals; following the terminology in [1], we call a Quasi-Euclidean functional L_r with parameter set $r = \{r_1, ..., r_d\}$ *sufficiently monotone* if it satisfies the following property:

Property 9A. *(Sufficient Monotonicity)*

A functional L_r is called sufficiently monotone with respect to r if $\exists$ a sequence $t_n = o(n^{1-\frac{1}{\|r\|_1}})$ such that for any infinite sequence $y_1, y_2, ... \in \mathbb{R}^d$ and any $m > n$, we have

$$L_r(x_1, ..., x_n) \leq L_r(x_1, ..., x_m) + t_n.$$

All of the proofs follow with monotonicity substituted by *sufficient monotonicity.* Note that if a functional is monotone, then it's sufficiently monotone. Still, sufficient monotonicity is useful for some very important problems. For example, the minimum matching problem, where given points $y_1, ..., y_{2n}$ we are required to find the cost of the minimum matching between them. First, note that this problem is different from the minimum bipartite matching problem, since here, every points has $2n-1$ possible points to match to, while in the MBMP, every point has n possible matches. It might seem that this difference is negligible, but the MBMP in general has a higher cost. For example, the Euclidean MBMP in two dimensions scales as $\sqrt{n \ln(n)}$ [13], while the Euclidean Minimum matching problem has a cost that scales as $\sqrt{n}$. Still, to study the Minimum Matching problem, sufficient monotonicity is needed, since it is not monotone [1].

We now turn to some other relaxations, that allow certain functionals to behave like Quasi-Euclidean functionals under certain distributions of the the argument variables. For theorem 4.7 to hold, properties 1A-5A only need to hold almost surely when $Y_1, ..., Y_n$ are uniformly, independently and identically distributed. Additionally, if all of the properties 1A-8A hold with probability 1 when $Y_1, ..., Y_n$ are independently and identically distributed according to a distribution $f(y)$, then theorem 4.8 will hold. These relaxations are important for some of the examples we deal with in Section 4.4. We will see an example of a functional that is subadditive only when $Y_1, ..., Y_n$ are sampled from specific distributions, and thus the results we prove hold only for certain distributions for that functional.

Thus far, we have shown that Quasi-Euclidean functionals behave almost exactly like Euclidean functionals except with number of dimensions d replaced by $||r||_1$, but we still haven't dealt with path planning for dynamic systems. In the following section, we relate the Quasi-Euclidean functionals we studied here to classical problems dealing with moving the output of a dynamic system between a number of points. We first reduce the gap between Quasi-Euclidean functionals and problems for dynamic systems by showing some examples of Quasi-Euclidean functionals that involve the r-warped distance between points.

4.4 Quasi-Euclidean Functionals Applications

It is well-known that the costs of the Euclidean TSP and MBMP (when $d \geq 3$) are Euclidean functionals [1, 12]. In this section, we assume that we are given a set of integers $r = \{r_1, ..., r_n\}$ and we consider the TSP and MBMP where the weight of an edge between y^i and y^f is $||l^r(y^i, y^f)||_1$, the 1−norm of the r-warped difference between them (Definition 3.17). We now show that the costs of the "r-warped" versions of the TSP and MBMP are Quasi-Euclidean functionals with parameter set r. Denote the costs of these versions of the TSP and MBMP by $L_{G_r^0}$ and $L_{G_r^1}$.

Properties 1A, 2A, 3A and 9A are easily established for the costs of these problems, from the definition of the r-warped distance and the formulations of the problems. Property 5A is most easily proved by using the following lemma (whose proof is similar to lemma 5.12 below):

Lemma 4.9. *The cost for the TSP where the weights of the edges are warped according to $r = (r_1, ..., r_d)$ satisfies:*

$$L_{G_r^0}(y_1, ..., y_n) \leq c_r V^{\frac{1}{||r||_1}} n^{1-\frac{1}{||r||_1}},$$

$\forall y_1, ..., y_n$ in a bounded cuboid of volume V. Here, c_r is a constant that depends only on r.

With lemma 4.9, we can proceed to prove subadditivity for $L_{G_r^0}$:

Proposition 4.3. *Let $r = (r_1, ..., r_d)$ be a set of positive integers and $\mathcal{Y} = \{y_1, ..., y_n\}$, $y_i \in [0, 1]^d$. If $L_{G_r^0}$ is the cost of the TSP on $\mathcal{Y}$ when the weights $w_{i,j} = ||l^r(y_i, y_j)||_1$, then $L_{G_r^0}$ satisfies property 5A with parameter set r.*

Proof. Consider any points $y_1, ..., y_n$ in $k^r(t) * [0, 1]^d$. A candidate tour of the points is the one using the TSP paths for $\{y_1, ..., y_n\} \cap k^r(t) * Q_i(r, m)$ and connecting them. To connect them, a path between $2m^{||r||_1}$ endpoints is needed. The length of this path is less than $c_r 2^{1-\frac{1}{||r||_1}} t m^{||r||_1 - 1}$, by lemma 4.9. This gives directly:

$$L_{G_r^0}(\{y_1,...,y_n\} \cap k^r(t) * [0,1]^d) \leq \sum_{i=1}^{m^{||r||_1}} L_{G_r^0}(\{y_1,...,y_n\} \cap k^r(t) * Q_i(r,m)) + C_{G_r^0} t m^{||r||_1 - 1}. \tag{4.7}$$

□

It is obvious that $L_{G_r^1}$ is not subadditive for any set of points $y_1^1, ..., y_n^1, y_1^2, ..., y_n^2$. This can be seen by choosing $y_1^1, ..., y_n^1$ and $y_1^2, ..., y_n^2$ on opposite sides of the $[0,1]^d$ cube. Still, we have the following lemma:

Lemma 4.10. *If $d \geq 3$ and $Y_1^1, ..., Y_n^1, Y_1^2, ..., Y_n^2$ are uniformly, independently and identically distributed on $[0,1]^d$, then*

$$\begin{gathered} L_{G_r^1}(\{Y_1^1, ..., Y_n^1, Y_1^2, ..., Y_n^2\} \cap k^r(t) * [0,1]^d) \leq \\ \sum_{i=1}^{m^{||r||_1}} L_{G_r^2}(\{Y_1^1, ..., Y_n^1, Y_1^2, ..., Y_n^2\} \cap k^r(t) * Q_i(r,m)) + C_{G_r^1} t m^{||r||_1 - 1} \quad a.s. \end{gathered} \tag{4.8}$$

It is interesting that the cost function of the MBMP is subadditive only for $d \geq 3$. It is actually known that in the Euclidean case the functional is *not* subadditive for $d = 1, 2$ [12, 13]. This hints that subadditivity of a functional depends on the details of the setting, which makes it hard to predict whether a functional is subadditive or not beforehand.

Now, with all of the properties needed established, we know that $L_{G_r^0}$ and $L_{G_r^1}$(again, when $d \geq 3$) are Quasi-Euclidean functionals with parameter set r, and we have the following result by theorem 4.7:

Theorem 4.9. *Let $r = (r_1, ..., r_d)$ be a set of positive integers, and let $L_{G_r^0}$ and $L_{G_r^1}$ be the costs of the TSP and MBMP when the weights $w_{i,j}$ are given by*

$$w_{i,j} = ||l^r(y_i, y_j)||_p.$$

If $Y_1, ..., Y_n$ are uniformly, independently, and identically distributed in $[0,1]^d$, we have for $i = 0, 1$ ($d \geq 3$ when $i = 1$):

$$\lim_{n\to\infty} \frac{L_{G_r^i}(Y_1, ..., Y_n)}{n^{1-\frac{1}{||r||_1}}} = \beta_p(L_{G_r^i}) \quad a.s. \tag{4.9}$$

The examples produced here are one step closer between Quasi-Euclidean functionals and problems for dynamic systems. Since the time a dynamic system needs to travel locally between two points scales as the r-warped distance between the points, we expect the problems for dynamic systems to behave similar to the examples we have here. Still, an important difference is that the non-local behavior of the dynamic systems does not in general have any structure and doesn't satisfy any homogeneity properties. We therefore turn to proving that in problems whose cost is monotone and subadditive, local behavior determines the global scaling.

Chapter 5

Quasi-Euclidean Functionals and Small-Time Controllable Dynamic Systems

In this chapter, we study the asymptotic behavior of some problems for small-time controllable dynamic systems by connecting them to subadditive Quasi-Euclidean functionals. Still, since the connection between the time of the dynamic systems and the r-warped distance is only local, we need to ensure that the local behavior dictates the global behavior of the cost functionals. This is our first aim in this chapter

5.1 Local and Global Behavior of Monotone, Subadditive Functionals

In this section, we study the relationship between the local and global asymptotic behavior of monotone, subadditive functionals. We note that the functionals we study here are not necessarily Euclidean or Quasi-Euclidean. The relationships we establish between the local and global behavior of functionals depends only on the monotonicity and subadditivity properties, as we see in the following lemma:

Lemma 5.11. *Let $L_r : \mathbb{R}^d \to \mathbb{R}$ be a monotone functional that satisfies property 5A* *with parameter set $r = \{r_1, ..., r_d\}$, and let $\mathbf{Q} = \{Q_1, ..., Q_M\}$ be an arbitrary partition of the $[0,1]^d$ cube, and $\forall\, i$, let $L_r^i(y_1, ..., y_n) = L_r(\{y_1, ..., y_n\} \cap Q_i)$.*

If $\forall\, i$, $L_r^i(\{Y_1, ..., Y_n\})$ is $\Theta(n^{1-\frac{1}{\|r\|_1}})$ with probability 1 when $Y_1, ..., Y_n$ are independently, identically, and uniformly distributed in $[0,1]^d$, then:

$$L_r(\{Y_1, ..., Y_n\}) \text{ is } \Theta(n^{1-\frac{1}{\|r\|_1}}) \text{ a.s.} \tag{5.1}$$

This lemma simply means that since monotonicity forces L_r^i to be a lower bound of L_r and subadditivity forces a similar upper bound on L_r, the local behavior in Q_i is what governs the asymptotic behavior. More specifically, it tells us that if we can divide the $[0,1]^d$ cube into small parts such that the inside of each part L_r is a subadditive Quasi-Euclidean functional, then all we need globally from L_r is that it is monotonic and satisfies property 5A. The usefulness of this lemma might not be directly obvious, but for dynamic, systems studying properties locally is much easier than doing so globally. Thus this theorem offers a relatively easy way to study planning problems for dynamic systems, by allowing us to check all the properties locally and only monotonicity and subadditivity globally.

It is important to note that this result is not the same as that of theorem 4.7. While theorem 4.7 says that the value of the functional converges to $\beta(L_r)n^{1-\frac{1}{\|r\|_1}}$ almost surely, lemma 5.11 only says that the value of L_r scales as $n^{1-\frac{1}{\|r\|_1}}$ almost surely. Thus while theorem 4.7 guarantees that the value of L_r is the same for all instantiations of $Y_1, ..., Y_n$ if n is large enough, lemma 5.11 only guarantees that the values of L_r from different instantiations of $Y_1, ..., Y_n$ are only close together (within a constant factor which is independent of n) when n is large enough.

5.2 Applications to Problems with Locally Controllable Dynamic Systems

We now turn to the problems defined in Section 2.1, namely, the DyTSP and DyMBMP. We consider systems that are locally controllable.

From our study of locally controllable systems in Section 3.4, we know that the behavior of the minimum travel time of these systems between two points locally is similar to the behavior of the r-warped distance between the points. We therefore expect $L_{G_S^0}$ and $L_{G_S^1}$ to behave asymptotically similarly to $L_{G_r^0}$ and $L_{G_r^1}$ respectively, where $r = (r_1, ..., r_d)$ is the set of orders of elementary output vector fields of system S. More specifically, we have the following result:

Proposition 5.4. *If for every $y \in [0,1]^d$, $C_l = \min_{x:h(x)=y} C_L(x)$ and $C_u = \max_{x:h(x)=y} C_U(x)$ exist, then there exists a partition $Q_1, ..., Q_{M_S}$ of the $[0,1]^d$ cube such that for every $i \in \{1, ..., M_S\}$ and $j \in \{0,1\}$, we have:*

$$C^j_{L_r} L^i_{G^j_r} \leq L^i_{G^j_S} \leq C^j_{U_r} L^i_{G^j_r}, \tag{5.2}$$

where $L^i_{G^j_r}(y_1, ..., y_n) = L_{G^j_r}(\{y_1, ..., y_n\} \cap Q_i)$, $L^i_{G^j_S}(x_1, ..., x_m, y_1, ..., y_n) = L_{G^j_S}(x^i_1, ..., x^i_{m^i}, \{y_1, ..., y$ $Q_i)$ $(x^i_1, ..., x^i_{m^i}$ are such that $h(x^i_k) \in Q_i)$, and $C^j_{L_r}$ and $C^j_{U_r}$ are constants.

From our previous results, we know that the restrictions of $L_{G_S^0}$ and $L_{G_S^1}$ to Q_i are $\Theta(n^{1-\frac{1}{\|r\|_1}}), \forall\, i$.

Now, since we intend to use lemma 5.11, we only need to establish that $L_{G_S^0}$ and $L_{G_S^1}$ are monotone and subadditive on cuboids that form a partition of the $[0,1]^d$ cube. Monotonicity of the cost is direct from the descriptions of the three problems. The proof for subadditivity of the three functionals underhand is a bit more complicated, but is most easily seen from the following lemma:

Lemma 5.12. *Let $L_{G_S^0}$ be the cost for the TSP for a locally controllable dynamic system S, and let the elementary output vector fields of S be $f_1, ..., f_d$ and their corresponding orders be $r_1, ..., r_d$. It follows that $L_{G_S^0}$ satisfies:*

$$L_{G_S^0}(x_1^0, \{y_1, ..., y_n\}) \leq c_S V n^{1-\frac{1}{\|r\|_1}},$$

$\forall y_1, ..., y_n$ in a bounded cuboid of volume V. Here, c_S is a constant that depends only on S.

The proof of this lemma is in Section 5.3.1 and 5.3.2, where we construct a tour and prove that the time needed for the dynamic system to trace is satisfies the bound in this lemma. With this lemma, we can prove the following proposition:

Proposition 5.5. *Let S by a dynamic system as in (2.4), $r = (r_1, ..., r_d)$ be the set of orders of the elementary output vector fields corresponding to S, and $\mathcal{Y} = \{y_1, ..., y_n\}$, $y_i \in [0,1]^d$. If $L_{G_S^0}$ is the cost of the DyTSP for system S on $\mathcal{Y}$, then $L_{G_r^0}$ satisfies property 5A with parameter set r.*

Proof. Similar to the r-warped costs case in Section 4.4, the subadditivity of $L_{G_S^0}$ can be proven as follows: Consider any points $y_1, ..., y_n$ in $k^r(t) * [0,1]^d$. A candidate path for the DyTSP is the one using the DyTSP paths for $\{y_1, ..., y_n\} \cap k^r(t) * Q_i(r,m)$ and connecting them. To connect them, a path between $2m^{||r||_1}$ endpoints is needed. The length of this path is less than $c_S 2^{1-\frac{1}{||r||_1}} t m^{||r||_1 - 1}$, by lemma 5.12. This gives directly:

$$L_{G_S^0}(x_1^0, \{y_1, ..., y_n\} \cap k^r(t) * [0,1]^d) \leq \sum_{i=1}^{m^{||r||_1}} L_{G_S^0}(x_1^0, \{y_1, ..., y_n\} \cap k^r(t) * Q_i(r,m)) + Ctm^{||r||_1 - 1}. \tag{5.3}$$

□

As for $L_{G_S^1}$, we know that it is not subadditive for any choice of $y_1^1, ..., y_1^n, y_1^2, ..., y_n^2$. Still, the following lemma, which is similar to lemma 4.10 can be established:

Lemma 5.13. *Given $d \geq 3$, let $L_{G_S^1}$ be the cost for the MBMP for a locally controllable dynamic system S, and let the elementary output vector fields of S be $f_1, ..., f_d$ and their corresponding orders be $r_1, ..., r_d$. It follows that*

$$L_{G_S^1}(X_1^0, ..., X_n^0, \{Y_1^1, ..., Y_n^1, Y_1^2, ..., Y_n^2\} \cap k^r(t) * [0,1]^d) \leq$$
$$\sum_{i=1}^{m^{||r||_1}} L_{G_S^1}(X_1^0, ..., X_n^0, \{Y_1^1, ..., Y_n^1, Y_1^2, ..., Y_n^2\} \cap k^r(t) * Q_i(r,m)) + C_{G_S^1} t m^{||r||_1 - 1} \quad a.s., \tag{5.4}$$

when $Y_1^1, ..., Y_1^n, Y_1^2, ..., Y_n^2$ are uniformly, independently and identically distributed.

Thus, we have proven that $L_{G_S^i}$ is subadditive and monotone for $i \in \{0,1\}$ (with $d \geq 3$ for $L_{G_S^1}$). Using lemma 5.11, we establish that $L_{G_S^i}$ has a cost that is $\Theta(n^{1-\frac{1}{\|r\|_1}})$. This is given in the following theorem:

Theorem 5.10. *Let $Y_1, ..., Y_n$ be uniformly, independently, and identically distributed in $[0,1]^d$ in the output space of a locally controllable system S described as in (2.4), and $L_{G_S^i}, i \in \{0,1\}$. For $i = 0,1$ ($d \geq 3$ for $L_{G_S^1}$), we have: :*

$$L_{G_S^i}(X_1^0, ..., X_m^0, \{Y_1, ..., Y_n\}) \text{ is } \Theta(n^{1-\frac{1}{\|r\|_1}}) \quad a.s.,$$

where $r = (r_1, ..., r_d)$ is the set of orders or elementary output vector fields.

All of these results can be appropriately generalized to non-uniform distributions. Additional properties similar to the ones in [1] are need to be satisfied for the subadditive Quasi-Euclidean functionals. Lemma 5.11 still holds, so it is only needed to prove all of the properties locally, and only monotonicity and subadditivity has to be proven globally.

5.3 TSP Algorithm for Small-Time Controllable Dynamic Systems

In this section, we introduce an algorithm for the DyTSP that is inspired by our framework. In Section 5.2, we proved the subadditivity of the DyTSP by patching sub-tours from each of $Q_i(r,m)$ to make a completer tour over the points in $[0,1]^d$. The algorithm produced here is inspired by that proof, when the number of partitions ($m^{\|r\|_1}$) is equal to the number of points (n).

5.3.1 Algorithm Description

The algorithm for the DyTSP of a small-time controllable dynamic system is as follows (we assume that the points are in the $[0,1]^d$ cube in the output space of the dynamic system:

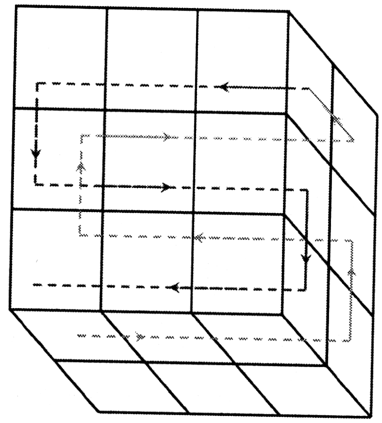

Figure 5-1: A tour visiting all of the cuboids in the partition.

1. Partition the $[0,1]^d$ cube into cuboids $Q_i(r, n^{\frac{1}{\|r\|_1}})$ as in Section 4.2. Since $n^{\frac{1}{\|r\|_1}}$ is not necessarily an integer, some additional cuboids may be needed. Thus the maximum number of cuboids in a row along the k^{th} dimension is $\lceil n^{\frac{r_k}{\|r\|_1}} \rceil$.

2. Create a tour over the partition in the following way:

   (a) Start at a corner of the $[0,1]^d$ cube.

   (b) Move along the y_1 direction, once the end of a row along the y_1 direction is reached, move one cuboid in the y_2 direction and traverse a new row along the y_1 direction (with the opposite orientation of the previous row.)

   (c) In general, when the end of a row in the y_i direction is reached, move one cuboid in the y_{i+1} direction and make another pass of y_i (figure 5-1).

3. When in a cuboid, visit every point in that cuboid.

This algorithm is a sweep of the partition of $[0,1]^d$. The interesting part is that the time for the dynamic system to travel the tour resulting from this algorithm scales as $n^{1-\frac{1}{\|r\|_1}}$. This algorithm is a generalization of the algorithm designed in [34] to the case of a general dynamic system that is small-time locally controllable.

5.3.2 Time to Trace the Tour

The first entity needed in the study of the time the output of the dynamic systems needs to trace the tour is the time it needs to move between any two points in one of the cuboids of the partition. From theorem 3.17, we have

$$T_S(y^i, y^f) \leq C_U ||l^r(y^i, y^f)||_1,$$

where $l^r(y^i, y^f)$ is the r-warped difference between y^i and y^f. Additionally, since y^i and y^f are in the same cuboid, it follows that

$$|y^i - y^f|_k \leq n^{-\frac{r_k}{||r||_1}}.$$

Thus

$$||l^r(y^i, y^f)||_1 \leq \sum_{k=1}^{d} n^{-\frac{1}{||r||_1}} = dn^{-\frac{1}{||r||_1}}.$$

It follows that the time needed to "concatenate" two sub-tours that are in consecutive cuboids is not more than $2dn^{-\frac{1}{||r||_1}}$. This is because the output of the system needs to move from the last point in the first tour to a point on the edge between the cuboids, and then to the first point in the second tour.

Finally, the total number of cuboids in the partition is not more than

$$\prod_{k=1}^{d} \lceil n^{\frac{r_k}{||r||_1}} \rceil = n + o(n).$$

Thus the time needed to concatenate all of the tours is not more than

$$(n + o(n))\, 2dn^{-\frac{1}{||r||_1}} + T,$$

where T is the maximum time needed to close the tour at the end.

The time spent in the sub-tours themselves is bounded by the total number of edges in the sub-tours times the maximum time to trace an edge in a sub-tour. The latter is less than $n^{-\frac{1}{||r||_1}}$ because all points in a sub-tour are in the same cuboid. The number of edges in every sub-tour is less than the number of points in that sub-tour, and so the total number of edges in all sub-tours is less than n. Therefore the time needed to trace all of the sub-tours is not more than

$$n^{1-\frac{1}{\|r\|_1}}.$$

Thus the total time needed for the output of the system to trace the tour produced by the algorithm can be bounded by:

$$T_{\text{alg}} \leq n^{1-\frac{1}{\|r\|_1}} + 2dn^{1-\frac{1}{\|r\|_1}} + o(n^{1-\frac{1}{\|r\|_1}}) = O(n^{1-\frac{1}{\|r\|_1}}).$$

One fact that simplifies the algorithm is that the systems is small-time locally controllable and that we assumed that the elementary output vector fields have directions that are invariant over $[0, 1]^d$. In the next chapter, we relax these assumptions to cover a larger class of dynamic systems.

Chapter 6

Problems for Dynamic Systems with Drift

In this chapter we relax the assumptions that we had in the previous chapter. Specifically, we allow the dynamic system to have drift as long as they are locally reachable, and we allow the directions of the elementary output vector fields to change over $[0,1]^d$.

Thus Assumption 3.3 is replaced by:

Assumption 6.6. *The dynamic system S is locally reachable, that is, the volume of the small-time reachable set is non-zero.*

6.1 Dynamic Systems with Drift

We now aim to relax the assumptions we imposed on the dynamic systems. Dynamic systems that have drift are not small-time locally controllable. Our framework this far can't deal with such systems because if a dynamic system is not necessarily locally controllable at a certain point y_0 in the output space, then a point $y_1(\epsilon)$ can always be found such that the (Euclidean) distance between y_0 and $y_1(\epsilon)$ is less than ϵ but property 2A (structured homogeneity) doesn't hold. This means that for a dynamic system that is not locally controllable, the cost is *not* similar to a subadditive Quasi-Euclidean functional.

Still, problems for dynamic systems that are not locally controllable can be dealt with using this framework, if the dynamic system is locally reachable. Given a dynamic system S that is described as in (2.4) but is not locally controllable, we define the companion system for S, S_c, by adding a new control u_0 to the drift component in (2.4). Thus S_c has the state space representation given by:

$$\dot{x} = u_0 g_0(x) + \sum_{i=1}^{m} g_i(x) u_i, \tag{6.1}$$

$$y = h(x),$$

$$x(0) = x_0,$$

$$x \in \mathbb{R}^b,\ y \in \mathbb{R}^d,\ u_i \in \mathbb{U},$$

$$\mathbb{U} = \{u(.) : \text{measurable } \mathbb{R} \to [-M, M]\},$$

where $g_i(x), h(x)$ and M are the ones in the state space representation of S. Note that f_i and r_i are the same for S and S_c. Now since S is locally reachable, S_c is locally controllable and the cost for a combinatorial problem under S_c can be studied using our framework. To relate the optimization problem for system S to that of system S_C, we need a property similar to property 7A:

Property 9. *L_r is called* scale bounded *with respect to set $r = \{r_1, ..., r_d\}$ and a distribution $f(y)$ if:*

$$L(Y_1, ..., Y_n) = O(n^{1-\frac{1}{||r||_1}}), \tag{6.2}$$

when $Y_1, ..., Y_n$ are i.i.d. distributed according to $f(y)$.

This property actually restricts the ratio discussed above sufficiently *for variables from a given distribution.* This is important for problems for dynamic systems with drift, since there are distributions where this will fail and so the choice of the distribution is critical. Informally, what is being checked by property 9 is that under the given distribution, S can perform asymptotically as well as S_c. This is because if the cost for the problem under S satisfies Property 9, then the cost will scale as the cost of the problem under S_c.

Theorem 6.11. *Let L_S^0 be the cost of the DyTSP as described in Section 2.1. If the dynamic system S is a locally reachable system, and $Y_1, ..., Y_n$ are uniformly, independently and identically distributed on $[0,1]^d$, then*

$$L_S^0(Y_1, ..., Y_n) \text{ is } \Theta(n^{1-\frac{1}{\|r\|_1}}) \quad a.s.$$

Proof. We start the proof by the following lemma,

Lemma 6.14. *Let L_S^0 be the cost of the DyTSP. If S is a locally reachable dynamic system, then L_S^0 is scale bounded with respect to $r = \{r_1, ..., r_d\}$ and the uniform distribution over $[0,1]^d$.*

The proof of this lemma is in the next section, where we introduce an algorithm that produces a tour that satisfies the result. Since the companion system S_c designed as above is small-time locally controllable, the cost of the DyTSP for S_c is $\Theta(n^{1-\frac{1}{\|r\|_1}})$ a.s. since

$$L_{S_c}^0(x_1^0, ..., x_m^0, y_1, ..., y_n) \leq L_S^0(x_1^0, ..., x_m^0, y_1, ..., y_n), \quad \forall y_1, ..., y_n \in [0,1]^d.$$

This fact, in addition to lemma 6.14 completes the proof. □

We now turn to the proof of lemma 6.14 and an in-depth study of the TSP for dynamic system (that possibly has drift). We initially assume that the directions of the elementary output vector fields and their orders don't change over $[0,1]^d$, and then relax this assumption later. Thus given a dynamic system as in (2.4), we study the TSP tour for such systems, and how the dynamics of the system affect the expected value of the optimal time duration.

We assume that the points $Y_1, ..., Y_n$ belong to a closed and bounded set R in the output space of the system (R is assumed to be a $d-$dimensional cuboid with dimensions $W_1, W_2, ..., W_d$). For ease of notation, let

$$V = \prod_{j=1}^{d} W_j.$$

We assume that $\exists\ T_T > 0$ such that for any two points y^1 and y^2 in R, the system can be steered from y^1 to y^2 in time less than T_T (starting at any initial state x^1 such that $h(x^1) = y^1$.) We also assume that R (the $d-$dimensional cuboid containing the n points that the system is required to visit) is such that the i^{th} side (W_i) is parallel to f_i. The proof of lemma 6.14 is constructive, that is, we will introduce an algorithm for the DyTSP (for a dynamic system that has drift).

6.2 DyTSP Upper bound

We now turn to the proof of lemma 6.14, which can be rephrased as follows:

Lemma 6.15. *Let the minimum time for the output of system* (2.4) *be* $T_{TSP}(\{y_1, ..., y_n\})$. *If the* $Y_1, ..., Y_n$ *uniformly, independently, and identically distributed, then* $T_{TSP}(\{Y_1, ..., Y_n\})$ *is* $O(n^{1-\frac{1}{\|r\|_1}})$ *a.s.*

The proof of lemma 6.15 is constructive, that is, we provide an algorithm that produces an output curve $\mathcal{C}_{LA}$ for system (2.4) such that the time needed for the system to trace $\mathcal{C}_{LA}$ is $O(n^{1-\frac{1}{\|r\|_1}})$ a.s. Note that the algorithm introduced in the previous chapter does not perform well in general for a dynamic system with drift. This is because the time to trace every sub-tour in a cuboid can possibly scale as the number of points in the cuboid (if the dynamic system has drift), and so the time to trace the whole tour can scale as n.

6.2.1 Level Algorithm

The algorithm we use to construct the upper bound is a generalization of our algorithm for the TSP for dynamic systems in [30]. We first give some intuition about the Level algorithm. In this algorithm, R is divided into a number of $d-$dimensional cuboids (called l-cuboids). l-cuboids have sides parallel to W_i, and will be formally defined shortly.

The algorithm we use is just a sweeping of R in the output space. The system output starts at a corner of R and it first sweeps a row of l-cuboids in the f_1 direction, moves one l-cuboid in the f_2 direction and then moves back along the f_1 direction. Similarly, after the system's output reaches the end of R along the f_i direction, it moves one l-cuboid in the f_{i+1} direction and repeats.

When sweeping a row of l-cuboids along the f_1 direction, the system's output can be guaranteed to visit one point in alternating cuboids. This is done by using motions as in Section 3.3. When the system is moving between points in alternating cuboids that are in the same row, it is directly guaranteed that the distance between the points along f_1 direction is more that w_1^l (the length of the side of an l-cuboid along the first dimension) and the distance along the f_i direction is less than w_i^l (the length of the side of an l-cuboid along the i^{th} dimension, where $i \geq 2$). We will construct the l-cuboids such that this fact guarantees that the distances satisfy inequality (3.15).

There are two points that make the Level algorithm a bit more complicated than the basic sweep described so far:

1. Motions described in Section 3.3 only allow us to guarantee that the output visits one point every other l-cuboid (in the f_1 direction). Therefore, the system has to do two sweeps to guarantee that it visits one point per nonempty l-cuboid.

2. The whole sweep does not visit all the points in R. Therefore the sweep has to be iterated with a smaller number of l-cuboids until the number of targets left is small (the exact number will be determined later), and the rest of the points are visited using a greedy algorithm.

Now that we introduced the intuition about the algorithm, we define it formally. The simplest version of the Level Algorithm for system (2.4) is as follows:

1. Set level counter $l = 1$ and the maximum level $l^* = \lceil \log_2(n^{\frac{||r||_1-1}{||r||_1}}) - \frac{||r||_1-1}{||r||_1} \rceil$.

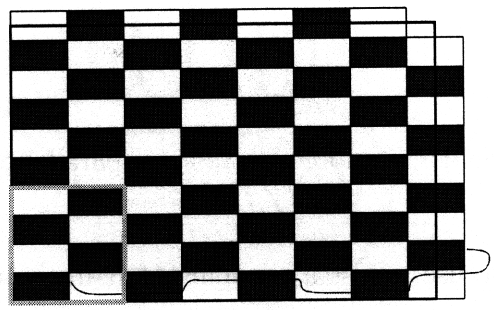

Figure 6-1: Dividing R into l-rectangles

2. Let

$$k^l = 2^{(l-2)(||r||_1)},$$

$$c_1 = \alpha \left(\frac{1}{C_{U_1}}\right)^{\frac{1}{r_1}},$$

$$\text{and } c_i = \alpha(d+1-i)\left(\frac{1}{C_{L_i}}\right)^{\frac{1}{r_i}}, i = 2, ..., d,$$

where α is a normalization constant:

$$\alpha = \left(\left(\frac{1}{C_{U_1}}\right)^{\frac{1}{r_1}} \prod_{i=2}^{d}(d+1-i)\left(\frac{1}{C_{L_i}}\right)^{\frac{1}{r_i}}\right)^{\frac{-1}{d}}.$$

Cover R with cuboids whose i^{th} side has length $w_i^l = c_i(\frac{k_l V}{n})^{\frac{r_i}{||r||_1}}$. Note that the $c_1, ..., c_d$ were chosen so that the volume of any l-cuboid at level l is $\frac{k^l V}{n}$ and that the assumption (3.15) needed in Section 3.3 is satisfied. We call these cuboids l-cuboids (Figure 3).

3. Visit at least one point in every non-empty l-cuboid by doing two passes (Pass(f_d)), where Pass(f_i) ($i = 1, ..., d$) is defined iteratively as follows: Pass(f_i)

   (a) If $i = 1$, move along the f_1 direction across W_1. Visit one point in alternating l-rectangles, using motions as in Section 3.3.

   (b) If $i \neq 1$, then do $\lfloor \frac{W_{i-1}}{w_{i-1}} \rfloor + 1$ of Pass(f_{i-1}), moving one l-cuboid along the f_i direction between every two passes.

4. If $i \leq i^*$, increment it by 1 and go to 2.

5. If $i > i^*$, use a greedy algorithm to pick up the points that are left.

To establish the results on the performance of the Level algorithm, there are several facts that need to be noted. Some of these important points about the Level Algorithm are the following:

1. The l-cuboids are aligned such that w_i is parallel to W_i of R, which are assumed to be aligned with f_i.

2. Each of the l-cuboids of a certain level l is made of $2^{||r||_1}$ l-cuboids of the previous level. Therefore each dimension w_i is 2^{r_i} times w_i in the previous level.

3. The reason we visit a point in alternating l-cuboids is that in that case we can only use motions like in Section 3.3. This will guarantee that the time traveled in each l-cuboid is less than $\left(dC_{L_1}^{-\frac{1}{r_1}} + C_{U_1}^{-\frac{1}{r_1}}\right)(w_1^l)^{\frac{1}{r_1}}$. This is also why we use two passes Pass(f_d), because in each pass, we deal with half of the l-cuboids.

4. We have the following lemma about the number of points left in R:

   Lemma 6.16. *The number of points not visited after the $\lfloor i^* \rfloor$ level is $n_l = O(n^{\frac{||r||_1 - 1}{2(||r||_1)}})$ a.s.*

We now turn to the proof that the time the system needs to trace $\mathcal{C}_{LA}$ is $O(n^{1-\frac{1}{||r||_1}})$. Note that by lemma (6.16), the time needed to clear the points left after the $\lfloor i^* \rfloor^{th}$ level will not affect the order of the time needed to trace $\mathcal{C}_{LA}$ (since it is less than $T_T(n_l + 1) = O(n^{\frac{||r||_1 - 1}{2(||r||_1)}})$. We therefore just need to prove that the time the system needs to trace $\mathcal{C}_{LA}$ over the levels is $O(n^{1-\frac{1}{||r||_1}})$ a.s.

6.2.2 Time to trace $\mathcal{C}_{LA}$

To bound the time system (2.4) needs to trace $\mathcal{C}_{LA}$ over the levels, we start by bounding the time needed to trace $\mathcal{C}_{LA}$ in one level. The bound is given by the following lemma:

Lemma 6.17. *The maximum time needed to trace one pass at level l is less than:*

$$
\begin{aligned}
&(A_1 V (w_1^l)^{\frac{1}{r_1}-1} + T_T^o) \prod_{j=2}^{d} \frac{1}{w_i^l} \\
&+ o\left((A_1 V (w_1^l)^{\frac{1}{r_1}-1} + T_T^o) \prod_{j=2}^{d} \frac{1}{w_i^l} \right),
\end{aligned}
\tag{6.3}
$$

where

$$
T_T^o = \begin{cases} T_T & \text{if } r_1 = 1, \\ 0 & \text{if } r_1 \neq 1, \end{cases}
$$

and

$$
A_1 = dC_{L_1}^{-\frac{1}{r_1}} + C_{U_1}^{-\frac{1}{r_1}},
$$

is a constant that depends only on the system's dynamics.

Proof. First, we will start by finding the time needed to trace one row of l-cuboids along the f_1 direction. We know that the number of l-cuboids along W_1 are at most $\frac{W_1}{w_1^l} + 1$.

Therefore, since in every l-cuboid the system used curves as in Section 3.3, the time traveled in traversing one row is bounded by

$$
T_r = A_1 W_1 (w_1^l)^{\frac{1}{r_1}-1} + A_1 (w_1^l)^{\frac{1}{r_1}} + o((w_1^l)^{\frac{1}{r_1}}).
$$

To turn from one row to the other, some additional time bounded by T_T is needed. The number of rows is

$$
N_r \leq \prod_{j=2}^{d} \frac{W_j}{w_j^l} + o(\prod_{j=2}^{d} \frac{W_j}{w_j^l}).
$$

Finally, to go back to the beginning of the first row, the system will also need some additional time bounded by T_T.

Thus the total time to trace one pass over R is bounded by:

$$
N_r[T_r + T_T] + T_T,
$$

which is equal to

$$
\begin{aligned}
&(A_1 V (w_1^l)^{\frac{1}{r_1}-1} + T_T) \prod_{j=2}^{d} \frac{1}{w_i^l} + A_1 \frac{V}{W_1} (w_1^l)^{\frac{1}{r_1}} \prod_{j=2}^{d} \frac{1}{w_i^l} \\
&+ o((w_1^l)^{\frac{1}{r_1}-1} \prod_{j=2}^{d} \frac{1}{w_i^l}) \qquad (6.4) \\
&= A_1 V (w_1^l)^{\frac{1}{r_1}-1} \prod_{j=2}^{d} \frac{1}{w_i^l} + T_T \prod_{j=2}^{d} \frac{1}{w_i^l} + o((w_1^l)^{\frac{1}{r_1}-1} \prod_{j=2}^{d} \frac{1}{w_i^l}),
\end{aligned}
$$

$$
= \begin{cases} (A_1 V + T_T) \prod_{j=2}^{d} \frac{1}{w_i^l} + o(\prod_{j=2}^{d} \frac{1}{w_i^l}) & \text{if } r_1 = 1, \\ A_1 V (w_1^l)^{\frac{1}{r_1}} \prod_{j=1}^{d} \frac{1}{w_i^l} + o((w_1^l)^{\frac{1}{r_1}} \prod_{j=1}^{d} \frac{1}{w_i^l}) & \text{if } r_1 \neq 1. \end{cases}
$$

$$
\begin{aligned}
&= (A_1 V (w_1^l)^{\frac{1}{r_1}-1} + T_T^o) \prod_{j=2}^{d} \frac{1}{w_i^l} \\
&+ o\left((A_1 V (w_1^l)^{\frac{1}{r_1}-1} + T_T^o) \prod_{j=2}^{d} \frac{1}{w_i^l} \right). \qquad (6.5)
\end{aligned}
$$

□

Therefore, The length of the time traveled by the system at any level is bounded by two times the maximum time traveled in any certain pass over the rows. From Lemma 6.17 and the fact that

$$
w_i = c_i (\frac{k^l V}{n})^{\frac{r_i}{||r||_1}},
$$

$$
\begin{aligned}
T_l \leq\ & 2 c_1^{\frac{1}{r_1}} A_1 (V)^{\frac{1}{||r||_1}} (\frac{n}{k^l})^{1-\frac{1}{||r||_1}} \\
&+ 2 T_T^o c_1 V^{\frac{r_1}{||r||_1}-1} (\frac{n}{k^l})^{1-\frac{r_1}{||r||_1}} + o(n^{1-\frac{1}{||r||_1}}). \qquad (6.6)
\end{aligned}
$$

The total time system (2.4) needs to trace C_{LA} over all of the levels can be bounded by:

$$
T_{LA} \leq 2 C^o n^{1-\frac{1}{||r||_1}} \sum_{i=1}^{\lfloor i^* \rfloor} 2^{(||r||_1 - 1)(2-i)},
$$

where

$$C^o = 2c_1^{\frac{1}{r_1}} A_1(V)^{\frac{1}{||r||_1}} + 2T_T^o c_1 V^{\frac{r_1}{||r||_1}-1}.$$

Thus

$$T_{LA} \leq \frac{2^{||r||_1}}{1 - 2^{1-||r||_1}} C^o n^{1-\frac{1}{||r||_1}} = C_{LA} n^{1-\frac{1}{||r||_1}} \quad \text{a.s.} \tag{6.7}$$

To complete the proof of lemma 6.15, we have to prove lemma 6.16. The proof is actually close to the one in [3], and will be left to the appendix.

For the examples we have, the Level Algorithm produces tours whose expected length is $O(n^{3/4})$ for the LTI system and $O(n^{2/3})$ for the vehicle pulling k trailers. Again, the reason for that scaling for the LTI system is that $r_1 = r_2 = 1$ and $r_3 = 2$. Thus motions in the first two directions are "easy", but motions in the f_3 directions are comparatively slow. The car pulling k trailers has only two directions, and although the corresponding $r_1 = 1$ and $r_2 = 2$, it still has the advantage (its $||r||_1 = 3$ compared to $||r||_1 = 4$ for the LTI system) because of the additional dimension. The specific constants multiplying those powers of n depend on the dimensions of R, and bounds on the inputs of the dynamic systems (which affect A_1). In the case of the LTI system, it actually also depends on the location of R in the output space, since the the elementary output vector fields are varying over the output space.

6.3 Heterogenous Dynamic systems

In this section, we relax the assumption that the elementary output vector fields and their orders are uniform over $[0,1]^d$. We begin by relaxing the constraints to a class of systems that is uniform over cuboids that form a partition of the $[0,1]^d$ cube.

6.3.1 Piece-wise Uniform Dynamic Systems

In this section we consider dynamic systems with the following property: $\exists$ a partition $\{Q_1, ..., Q_M\}$ of the $[0,1]^d$ cube such that the directions of the elementary output vector fields and their orders are uniform over Q_i, $\quad \forall i \in \{1, ..., M\}$.

By lemma 5.11, if L is the cost of a combinatorial problem for the dynamic system is monotone and subadditive, then the cost scales like L^i, the cost of the problem restricted to Q_i. If $||r||_1^i$ is not the same for all Q_i, then the L^{i^*} that corresponds to the largest $||r||_1^{i^*}$ will dominate and L is $\Theta(n^{1-\frac{1}{||r||_1^{i^*}}})$. This is given by the following lemma, which is a generalization of lemma 5.11:

Lemma 6.18. *Let $L_r : \mathbb{R}^d \to \mathbb{R}$ be a monotone functional that satisfies property 5A with parameter set $r = \{r_1, ..., r_d\}$, and let $\mathbf{Q} = \{Q_1, ..., Q_M\}$ be a partition of the $[0,1]^d$ cube, and $\forall$ i, let $L_r^i(y_1, ..., y_n) = L_r(\{y_1, ..., y_n\} \cap Q_i)$.*

If $\forall$ i, $L_r^i(Y_1, ..., Y_n)$ is $\Theta(n^{1-\frac{1}{s_r^i}})$ with probability 1 when $Y_1, ..., Y_n$ are independently, identically, and uniformly distributed in $[0,1]^d$, then:

$$L_r(Y_1, ..., Y_n) \text{ is } \Theta(n^{1-\frac{1}{s_r^{i^*}}}) \text{ a.s.,} \tag{6.8}$$

where $i^* = \text{argmax}_i \, ||r||_1^i$.

Thus our framework can deal with dynamic system that are uniform over a partition of the $[0,1]^d$ cube. This class of systems is large and can approximate any dynamic system whose vector fields have a finite number of discontinuities arbitrarily well. What is needed to be proven is that the error from the approximation doesn't add up to be too big.

Now given a general dynamic system with analytic vector fields, we aim to prove that there is a partition $\{Q_1, ..., Q_M\}$ of $[0,1]^d$ and a transformation T_i such that after applying T_i in Q_i, the directions and orders of the elementary output vector fields are uniform over Q_i.

6.3.2 Local Transformations for Dynamic Systems

A local transformation that makes the flows of the elementary output vector fields uniform can always be found. This transformation follows from the fact that the elementary output vector fields form a basis locally. To introduce the local transformation, we start with some notation.

Given a vector field f, denote by $\phi_t^f(x^0)$ the flow of f. This is the analytic function satisfying:

$$\frac{\partial}{\partial t}\phi_t^f(x) = f(\phi_t^f(x)) \quad \phi_0^f(x^0) = x^0.$$

Denote by $\phi_{-t}^f(x^0)$ the flow of $-f$.

For any x^0, $\exists T > 0$ and a neighborhood U of x^0 such that $\forall t \leq T$ $\phi_t^f(x)$ is defined for all $x \in U$ and is a local diffeomorphism. Additionally, $[\phi_t^f]^{-1} = \phi_{-t}^f$ and $\phi_{t+\tau}^f(x) = \phi_t^t\left(\phi_\tau^f(x)\right)$.

Given vector fields $f_1, ..., f_n$ such that

$$\text{span}\{f_1(x), ..., f_n(x)\} = \mathbb{R}^n, \quad \forall x \in U,$$

consider the mapping

$$F : U_\epsilon \rightarrow \mathbb{R}^n$$

$$(z_1, ..., z_n) \rightarrow \phi_{z_1}^{f_1} \circ \phi_{z_2}^{f_2} ... \circ \phi_{z_n}^{f_n}(x),$$

where $U_\epsilon = \{z \in \mathbb{R}^n : |z_i| < \epsilon\}$.

This mapping takes the "times" $z_1, ..., z_n$ to the point in $\mathbb{R}^n$ that is reached by following the flows of f_1 for time z_1, f_2 for time z_2 and so on.

Theorem 6.12. *$\exists\epsilon$ such that F is defined for all $z = (z_1, ..., z_n) \in U_\epsilon$ and is a diffeomorphism onto its image.*

If the inverse of the mapping is used as a change of coordinates transformation, the flows of the functions $f_1, ..., f_n$ are given by $\phi^{f_i}(z) = z_i$. Thus using this diffeomorphism, the directions of the flow of f_i can be assumed to be e_i.

Since the elementary output vector fields span the space locally, this transformation can be used locally to make the flows of the elementary output vector fields uniform. Note that since we are applying a transformation, the distribution of $Y_1, ..., Y_n$ will change. Thus if $Y_1, ..., Y_n$ were uniformly, independently and identically distributed, their distribution will not be uniform in general after the transformation. Thus all of the properties 1A-8A should be checked.

To generalize the results to general dynamic systems, we just have to prove that we can find a partition of the $[0,1]^d$ cube such that in every element of the partition, a local transformation can be applied to make the directions of the elementary output vector fields uniform. We can find a partition as follows:

There is a set of balls, $\{B(y) : y \in [0,1]^d\}$ such that the a transformation ϕ^{-1} can be found. This set is a cover for the $[0,1]^d$ cube since $\forall y \in [0,1]^d$, $y \in B(y)$. Since $[0,1]^d$ is compact, there is a finite sub-cover $B(y_1), ..., B(y_M)$ that covers $[0,1]^d$. Finally, the partition needed is the set of all of the non-empty intersections of $B(y_1), ..., B(y_M), B^C(y_1), ..., B^C(y_M)$, where B^C is the complement of $[0, 1^d$.

We now turn to the dynamic version of the TSP, the DTRP, and see how our results for the TSP can guarantee that the DTRP for a dynamic system is stabilizable. We deal with the DTRP under two scenarios, and study how to minimize the time a customer has to wait before being serviced.

Chapter 7

Dynamic Traveling Repairperson Problem for Dynamic Systems

In this chapter, we study the minimum customer waiting time for the DTRP, denoted T_{DTRP}. We repeat the problem formulation here for easy reference:

7.1 Problem Formulation

Given a dynamic system that is modeled as in (2.4), let R be compact region in the output space of the system (R is assumed to be a $d-$dimensional cuboid with dimensions $W_1, W_2, ..., W_q$). We study the DTRP, where "customer service requests" are arising according to a Poisson process with rate λ and, once a request arrives, it is randomly assigned a position in R uniformly and independently.

The repairperson is modeled as in (2.4) and is required to visit the customers and service their requests. At each customer's location, the repairperson spends a service time s which is a random variable with finite mean $\overline{s}$ and second moment $\overline{s^2}$. We study the expected waiting time a customer has to wait between the request arrival time and the time of service, and we mainly focus on how that quantity scales in terms of the traffic intensity $\lambda\overline{s}$ for low traffic ($\lambda\overline{s} \to 0$) and high traffic ($\lambda\overline{s} \to 1$). We also study the stability of the queuing system, namely whether the necessary condition for stability ($\lambda < \frac{1}{\overline{s}}$) is also sufficient.

It is known that the necessary condition for stability of the DTRP is that $\lambda\bar{s} < 1$ [18]. This condition simply means that the average time for a new customer request to arise should be less than the average time needed to service a customer. We study whether this necessary condition for stability is also sufficient, that is, whether there are schemes for every $\lambda < \frac{1}{\bar{s}}$ that will guarantee that the number of waiting customers is always bounded. Additionally, we study how T_{DTRP} scales in terms of the traffic intensity.

7.2 Low Traffic Intensity

We will start with results for low traffic intensity. This means that $\lambda\bar{s} \to 0$ and so almost all of the time can be used to move the system's output from one customer to another. Let y^* be a "time median" of R under the system's dynamical constraints (does not have to be unique). So y^* is the point in R that minimizes

$$E[T_v(y, y^*)],$$

where $T_v(y, y^*)$ is the time that the system needs to travel from y^* to y. Note that T_v doesn't have to be small, and therefore the steering and scaling results of the reachable sets we derived don't necessarily hold. Let

$$T_1 = E[T_v(y, y^*)],$$

and

$$T_2 = E[T_v^2(y, y^*)].$$

We have the following theorem:

Theorem 7.13. *The expected customer waiting time in the DTRP (T_{DTRP}) for a* *small time controllable dynamic system is equal to*

$$E[T_v(y, y^*)] + \overline{s}$$

as $\lambda \to 0$.

For dynamic systems that are not locally controllable, T_{DTRP} still scales as

$$E[T_v(y, y^*)] + \overline{s},$$

but there is a gap between the upper and lower bounds. This is because the system has to move around y^* while waiting for customers requests, and so when the customer requests arrive, the system will not necessarily be at y^* and therefore customers will have to wait longer.

Proving that $T_{DTRP} \geq E[T_v(y, y^*)] + \overline{s}$ is straightforward. When a customer service request arises, the system's output has to at least move from where it is already to the location of the new customer and service it. Thus the expected time a customer has to wait is at least $E[T_v(y, y^{sys})] + \overline{s}$, where y^{sys} is a random variable determining the location of the system's output when the customer service request arrived. From the definition of y^*, $E[T_v(y, y^{sys})] \geq E[T_v(y, y^*)]$, and thus

$$T_{DTRP} \geq E[T_v(y^*, y)] + \overline{s} = T_1 + \overline{s} = T^*.$$

Note that this lower bound holds for both small-time controllable systems and systems that are not small-time controllable.

To get the matching upper bound, the following policy can be followed: Service customers in a First Come First Serve fashion, and wait at y^* when there are no customers. For dynamic systems that can't stay at a certain point (because of drift for example), a looser upper bound can be achieved by moving around y^* ($T_1 + \overline{s} + T_3$, where T_3 is the expected time for the system to return to y^*). We will concentrate on small-time locally controllable (definition 2.2) systems here for simplicity.

Lemma 7.19. *The expected time of the previous policy T_{FCFS} satisfies the following relation:*

$$\frac{T_{FCFS}}{T^*} \rightarrow 1 \text{ as } \lambda \rightarrow 0.$$

Proof. The proof is similar to the one in [18].

By following the FCFS scheme, we have a single-server SQM system behaving like an $M/G/1$ queue with first moment $2T_1 + \overline{s}$ and second moment $4T_2 + 4\overline{s}T_1 + \overline{s^2}$. Thus the expected customer waiting time can be bounded by:

$$T_{FCFS} = \frac{\lambda(4T_2 + 4\overline{s}T_1 + \overline{s^2})}{2(1 - 2\lambda T_1 - \lambda\overline{s})} + T_1 + \overline{s}.$$

Therefore:

$$\begin{aligned} \lim_{\lambda\rightarrow 0} \frac{T_{FCFS}}{T^*} &= \lim_{\lambda\rightarrow 0} \frac{\frac{\lambda(4T_2+4\overline{s}T_1+\overline{s^2})}{2(1-2\lambda T_1-\lambda\overline{s})} + T_1 + \overline{s}}{T_1 + \overline{s}} \\ &= 1. \end{aligned} \tag{7.1}$$

□

Theorem (7.13) follows directly.

7.3 High Traffic Intensity

We now turn to the case where the traffic intensity is high. Heavy traffic intensity is when $\lambda\overline{s} \rightarrow 1$. This means that there is little time for travel ($1 - \lambda\overline{s}$ per customer on average.) Thus the system will need to follow a more complicated scheme to allow the number of waiting customers to be bounded.

We first produce a lower bound on the expected customer waiting time. This result will depend on the area of the small time reachable set discussed in Section 3.2, and is given by the following lemma:

Lemma 7.20. *T_{DTRP} is $\Omega\left((1 - \lambda\overline{s})^{-(\|r\|_1)}\right)$.*

Proof. The lower bound proof is in three steps:

1. *Bound the Expected time traveled per customer:*

   Let n be the average number of customers waiting in R to be serviced at a given time. Given that the output of system (2.4) is at any point in R, let the minimum time needed to travel to a customer be t^*. Then

$$\begin{aligned}\mathbf{E}[t^*] &\geq \int_0^\infty \mathbf{P}[t^* > \tau]d\tau \\ &\geq \int_0^\infty \max\{0, 1 - c\tau^{||r||_1}\}d\tau,\end{aligned}$$

   where $c = n\frac{C_U}{W_1 W_2}$, and C_U is from Theorem 3.5.

$$\begin{aligned}\mathbf{E}[t^*] &\geq \int_0^{c^{-\frac{1}{||r||_1}}} 1 - c\tau^{||r||_1}d\tau \\ &= c^{-\frac{1}{||r||_1}} - c\frac{1}{1+||r||_1}c^{-\frac{1+||r||_1}{||r||_1}} \\ &= \frac{||r||_1}{1+||r||_1}\left(\frac{C_U}{W_1W_2}\right)^{-\frac{1}{||r||_1}} n^{-\frac{1}{||r||_1}} \\ &= c_2 n^{-\frac{1}{||r||_1}}.\end{aligned}$$

2. *Upper bound the rate of arrival:*

   Recall that $\bar{s}$ is the average service time per customer needed, W is the average waiting time, and $T = W + \bar{s}$ is the system waiting time.

   The stability condition is that the average time spent traveling on the road plus the average service time is not greater than the average time for a customer to arrive:

$$\lambda(\bar{s} + \mathbf{E}[t^*]) \leq 1,$$

   where λ is the rate of arrival.

Therefore,

$$\bar{s} + c_2 n^{-\frac{1}{||r||_1}} \leq \frac{1}{\lambda}.$$

3. *Lower Bound the Customer Waiting Time:*

   From the previous bound, and using Little's formula to relate the average number of customers in a queue to the average waiting time: $n = \lambda W$ and $T^* = \bar{s} + W$ is the minimum system waiting time, we get:

$$n \geq \frac{(\lambda c_2)^{||r||_1}}{(1-\lambda\bar{s})^{||r||_1}},$$

   and

$$T_{DTRP} \geq \bar{s} + \frac{\lambda^{||r||_1 - 1} c_2^{||r||_1}}{(1-\lambda\bar{s})^{||r||_1}}.$$

   Therefore as $\lambda\bar{s} \to 1$, T_{DTRP} is $\Omega(1-\lambda\bar{s})^{-(||r||_1)}$.

□

To achieve the upper bound corresponding to the high traffic intensity lower bound, we use the TSP policy. Under this policy, the system waits until there are n customers, and then services them with a TSP tour. This means that it first waits for customers number 1 to n, service them using a TSP tour, then waits till the $2n^{th}$ customer arrives, and services customers $n+1$ to $2n$,... Denote the k^{th} set of n customers by $\mathbb{S}_k$ and the system waiting time under this policy by T_{TSP}, we have the following theorem:

Theorem 7.14. *As* $1 - \lambda\bar{s} \to 0$, $\frac{T_{TSP}}{T_{DTRP}} \leq c_3$.

Proof. We now consider $\mathbb{S}_k$ to be the k^{th} customer in a queue. Since the interarrival and service times are i.i.d, we have a $GI/G/1$ queue with an Erlang distribution of order n. The mean of the sets is $\frac{n}{\lambda}$ and the variance is $\frac{n}{\lambda^2}$.

Therefore, the expected value of the service time of a set is $E[T_{LA}(n)] + n\bar{s}$ and the variance is $var(T_{LA}(n)) + n\sigma^2$.

Therefore, we can bound the average waiting time of the sets by:

$$W_S \leq \frac{\frac{\lambda}{n}(\frac{n}{\lambda^2} + var(T_{LA}(n)) + n\sigma^2)}{2(1 - \frac{\lambda}{n}(E[T_{LA}(n)] + n\bar{s}))}$$
$$= \frac{\lambda(\frac{1}{\lambda^2} + \sigma^2)}{2(1 - \lambda(\bar{(s)} + C_{LA}n^{-\frac{1}{||r||_1}}))},$$

where C_{LA} is from lemma 6.17. For stability, we have:

$$1 - \lambda(\bar{s} + C_{LA}n^{-\frac{1}{||r||_1}}) > 0.$$

Therefore

$$\frac{(1-\lambda\bar{s})^{||r||_1}}{C_{LA}^{||r||_1}} > \frac{1}{n},$$

and

$$n > \frac{(\lambda C_{LA})^{||r||_1}}{(1-\lambda\bar{s})^{||r||_1}}.$$

This means that for high traffic $(1 - \lambda\bar{s} \to 0)$, n has to be large for the system to be stable. Our assumption for the Level Algorithm performance guarantee (that n is large) is thus satisfied.

The expected waiting time for a certain customer is the sum of the expected time it waits for its set to form, the waiting time for the set to get serviced, and the expected time it needs to wait to get serviced after the service of its set started.

Therefore,

$$T_{TSP} \leq \frac{\lambda(\frac{1}{\lambda^2} + \sigma^2)}{2(1 - \lambda(\bar{s} + C_{LA}n^{-\frac{1}{||r||_1}}))} + n\frac{1 + \lambda\bar{s}}{2\lambda} + C_{LA}n^{1-\frac{1}{||r||_1}}. \tag{7.2}$$

It can be shown that as $1 - \lambda\bar{s} \to 0$, the optimal n approaches $\frac{(\lambda C_{LA})^{||r||_1}}{(1-\lambda\bar{s})^{||r||_1}}$ (which is the stability bound).

Substituting the optimal value of n in 7.2 gives:

$$T_{TSP} \leq \frac{\lambda^{||r||_1 - 1} C_{LA}^{||r||_1}}{(1 - \lambda \bar{s})^{||r||_1}},$$

and using this with lemma (7.20) gives the result:

$$\frac{T_{TSP}}{T_{DTRP}} \leq c_3.$$

and thus proves that T_{DTRP} is $\Theta\left((1 - \lambda \bar{s})^{-||r||_1}\right)$. □

Thus the DTRP is stabilizable for any dynamic system that has basic reachability properties, that is, the expected waiting time for a customer can be guaranteed to be bounded as long as $\lambda \bar{s} < 1$. For the examples we are using, the average customer waiting time scales as $(1 - \lambda \bar{s})^{-4}$ and $(1 - \lambda \bar{s})^{-3}$ respectively, which is worse than the Euclidean case. This deterioration in behavior is due to the fact that there is a direction in which motions of the systems' output is slow (the direction with $r_i = 2$), and the increase in the dimension of the output space.

Chapter 8

Conclusion

8.1 Conclusions

This thesis has three goals: The first is to introduce a framework to study combinatorial problems under dynamic constraints. The second is to apply our framework to the study of the TSP and MBMP for dynamic systems and produce algorithms whose performance scales like the optimal in terms of the number of points. The final goal is a study of the DTRP, both in low traffic and high traffic regimes, produce lower bounds and algorithms whose performance in terms of the intensity scales as the optimal.

To study different combinatorial problems under dynamic constraints, we introduced a new class of functionals that we call Quasi-Euclidean functionals. These are a generalization of Euclidean functionals, which represent the cost of the combinatorial problems of interest when the dynamic constraints are ignored. We established the asymptotic properties of Quasi-Euclidean functionals, and produce results that parallel those available for Euclidean functional. Additionally, we provide some results simplifying the study of problems for dynamic systems, and some tests that show whether the cost function of a certain problem is a subadditive Quasi-Euclidean function or not. Therefore the work here offers tools for the study of problems for path planning for a dynamic system through a given number of points.

We established that the framework of Subadditive Quasi-Euclidean functionals can simplify the study a rich class of problems for dynamic systems. We did this by using the results for the Quasi-Euclidean functionals to study problems for dynamic systems and establish their asymptotic behavior. Namely, we studied the TSP and the MBMP for dynamic systems. We did this by studying the problems for systems that are locally controllable first, and then using those results to study the problems for systems that are not necessarily locally controllable but are locally reachable.

We then studied the TSP for dynamic systems in detail. We provided lower bounds on the time the system has to travel to visit all of the given points, and created an algorithm that allows the system to visit all of the points in a time that scales optimally. We used the results from the TSP to study the DTRP in the high customer arrival regime. We proved that the DTRP for dynamic systems is stabilizable and provided algorithm that perform order-optimally when the traffic intensity is high or low.

Appendix A

Appendix

A.1 Proofs for Dynamic Systems

A.1.1 Proof of theorem 2.2

We start by showing that the function given by

$$x_j(t) = x_j(0) + \sum_{k=0}^{\infty} \sum_{i_0=0}^{m} \dots \sum_{i_k=0}^{m} \mathcal{L}\left(g_{i_0}, \dots, \mathcal{L}\left(g_{i_k}, x_j(x^0)\right)\right) \int_0^t d\xi_{i_k} \dots d\xi_{i_0}, \qquad \text{(A.1)}$$

where, as before, $x_j : (x_1, \dots, x_n) \to x_j$ is the solution of the differential equation

$$\dot{x} = g_0(x) + \sum_{i=1}^{m} g_i(x) u_i, \quad x(0) = x^0. \qquad \text{(A.2)}$$

To show this, note that from the definition of $\int_0^t d\xi_{i_k} \dots d\xi_{i_0}$,

$$\frac{d}{dt} \int_0^t d\xi_0 d\xi_{i_{k-1}} \dots d\xi_{i_0} = \int_0^t d\xi_{i_{k-1}} \dots d\xi_{i_0},$$

and

$$\frac{d}{dt} \int_0^t d\xi_{i_k} d\xi_{i_{k-1}} \dots d\xi_{i_0} = u_{i_k} \int_0^t d\xi_{i_{k-1}} \dots d\xi_{i_0}, \quad 1 \leq i_k \leq m.$$

Taking the derivative of equation A.2 with respect to time, and rearranging the terms on the right side, we have

$$\begin{aligned}\dot{x}_j(t) =&\mathcal{L}(g_0, x_j(x^0)) + \sum_{k=0}^{\infty}\sum_{i_0=0}^{m}\cdots\sum_{i_k=0}^{m}\mathcal{L}\left(g_{i_0}, \dots, \mathcal{L}\left(g_{i_k}, \mathcal{L}\left(g_0, x_j(x^0)\right)\right)\right)\int_0^t d\xi_{i_{k-1}}\dots d\xi_{i_0}\\ &+\sum_{i=1}^{m}\left[\mathcal{L}(g_i, x_j(x^0)) + \sum_{k=0}^{\infty}\sum_{i_0=0}^{m}\cdots\sum_{i_k=0}^{m}\mathcal{L}\left(g_{i_0}, \dots, \mathcal{L}\left(g_{i_k}, \mathcal{L}\left(g_i, x_j(x^0)\right)\right)\right)\int_0^t d\xi_{i_k}\dots d\xi_{i_0}\right]u_i(t).\end{aligned} \tag{A.3}$$

Denote by $g_{i_k,j}$ the j^{th} component of g_{i_k}, and note that:

$$\mathcal{L}\left(g_{i_k}, x_j(x)\right) = g_{i_k,j}(x).$$

Therefore, for $0 \leq i \leq m$, we have:

$$\begin{aligned}&\mathcal{L}(g_i, x_j(x^0)) + \sum_{k=0}^{\infty}\sum_{i_0=0}^{m}\cdots\sum_{i_k=0}^{m}\mathcal{L}\left(g_{i_0}, \dots, \mathcal{L}\left(g_{i_k}, \mathcal{L}\left(g_i, x_j(x^0)\right)\right)\right)\int_0^t d\xi_{i_k}\dots d\xi_{i_0}\\ &= g_{i,j}(x^0) + \sum_{k=0}^{\infty}\sum_{i_0=0}^{m}\cdots\sum_{i_k=0}^{m}\mathcal{L}\left(g_{i_0}, \dots, \mathcal{L}\left(g_{i_k}, g_{i,j}(x^0)\right)\right)\int_0^t d\xi_{i_k}\dots d\xi_{i_0}\\ &= g_{i,j}(x(t)).\end{aligned} \tag{A.4}$$

Additionally, $x_j(t)$ in equation A.1 satisfies $x_j(0) = x_j^0$, and therefore they are the components of the solution of the differential equation A.2.

A similar calculation shows that the output given by $y = h(x)$ can be expressed as

$$y_j(t) = h_j(x^0) \tag{A.5}$$

$$+\sum_{k=0}^{\infty}\sum_{i_0=0}^{m}\cdots\sum_{i_k=0}^{m}\mathcal{L}\left(g_{i_0}, \dots, \mathcal{L}\left(g_{i_k}, h_j(x^0)\right)\right)\int_0^t d\xi_{i_k}\dots d\xi_{i_0}.$$

A.2 Proofs for the TSP for Dynamic Systems

A.2.1 Proof of lemma 6.16

This proof is similar to the one in [9]. We give every l-cuboid a unique identifier c, and let $c(y_i)$ be the l-cuboid y_i belongs to. We also denote $l(y_i)$ be the level at which y_i is visited, and assume that if $c(y_i) = c(y_j)$ and $i < j$, then $l(y_i) \leq l(y_j)$. Additionally, let $t_j(y_i)$ be the number of l-cuboids that have targets y_k such that $k \leq i$ at the beginning of the j^{th} level. Finally let the number of cuboids of the i^{th} level be m_i. The probability that y_i is not visited in the first level is:

$$\mathbf{P}[l(y_i) > 1 | t_1(y_i)] = \frac{t_1(y_i)}{m_1} \leq \frac{n}{2^{||r||_1} n} = \frac{1}{2^{||r||_1}}.$$

Similarly,

$$\begin{aligned}
\mathbf{P}[l(y_i) > j | t_j(y_{i-1}), ..., t_1(y_{i-1})] &= \mathbf{P}[l(y_i) > j | l(y_i) > j-1, t_j(y_{i-1})] \\
&\quad .\mathbf{P}[l(y_i) > j-1 | t_{j-1}(y_{i-1}), ..., t_1(y_{i-1})] \\
&\leq \prod_{k=1}^{j} \frac{t_k(y_{i-1})}{m_k} \\
&\leq \prod_{k=1}^{j} \frac{2^{k||r||_1 - 1} t_k(n)}{2n} \\
&= \left(\frac{2^{\frac{j||r||_1 - 3}{2}}}{n} \right)^j \prod_{k=1}^{j} t_k(n).
\end{aligned} \tag{A.6}$$

Now let $\beta_j = 2^{-j||r||_1 + \frac{||r||_1 + 1}{2}}, j \in \mathbb{N}$, and for a fixed $j > 0$ define the binary variables X_i such that $X_i = 1$ if $l(y_i) > j$ and $t_j(y_{i-1}) \leq \beta_i n$ and $X_i = 0$ otherwise. Now we have

$$\mathbf{P}[X_i = 1 | c(y_1), ..., c(y_{i-1})] \leq 2^{\frac{j^2||r||_1 - 3j}{2}} \prod_{k=1}^{j} \beta_k = 2^{-j}.$$

Thus the sum $\sum_{i=1}^{n} X_i$ is stochastically dominated by a binomial random variable with $p = 2^{-j}$, and therefore:

$$\mathbf{P}[\sum_{i=1}^{n} X_i > 2^{1-j}n] < 2^{-\frac{2^{-j}n}{3}}.$$

This is less than $\frac{1}{n^2}$ for $j < l^*$. Note that if $t_j(y_n) \leq \beta_j n$, then $t_{j+1}(n) \leq \sum_{i=1}^{n} X_i$ (because in this case the first term is the number of non-empty l-cuboids and the second one is the number of non-visited points.) Thus

$$\mathbf{P}[t_{j+1}(n) > \beta_{j+1}n | t_j(y_n) \leq \beta_j n]\mathbf{P}[t_j(y_n) \leq \beta_j n] \leq \mathbf{P}[\sum_{i=1}^{n} X_i > 2^{1-j}n] < \frac{1}{n^2},$$

and therefore

$$\begin{aligned} \mathbf{P}[t_{j+1}(n) > \beta_{j+1}n] &= \mathbf{P}[t_{j+1}(n) > \beta_{j+1}n | t_j(y_n) \leq \beta_j n]\mathbf{P}[t_j(y_n) \leq \beta_j n] + \mathbf{P}[t_{j+1}(n) > \beta_{j+1}n | t_j(y_n) > \\ &\leq \frac{1}{n^2} + \mathbf{P}[t_j(y_n) > \beta_j n] \\ &\leq \frac{j}{n^2}. \end{aligned} \tag{A.7}$$

Finally, at the l^* level, we have

$$\begin{aligned} \mathbf{P}[t_{l^*+1}(n) > n^{\frac{||r||_1-1}{2||r||_1}} | t_{l^*}(y_n) \leq \beta_j n]\mathbf{P}[t_{l^*}(y_n) \leq \beta_j n] &\leq \mathbf{P}[\sum_{i=1}^{n} X_i > n^{\frac{||r||_1-1}{2||r||_1}}] \\ &\leq 2^{-\frac{n^{\frac{||r||_1-1}{2||r||_1}}}{2}}, \end{aligned} \tag{A.8}$$

and therefore

$$\mathbf{P}[t_{l^*+1}(n) > n^{\frac{||r||_1-1}{2||r||_1}}] \leq 2^{-\frac{n^{\frac{||r||_1-1}{2||r||_1}}}{2}} + \frac{l^*}{n^2},$$

and the proof is complete.

A.3 Proofs for Quasi-Euclidean Functionals

A.3.1 Proof of theorem 4.7

Here we present the proofs for the theorems we produced. The first is of theorem 4.7, and it follows the proof of theorem 2.3 in [1].

Proof. We follow the notation in [1] and let Π denote a Poisson point process in $\mathbb{R}^d$ with a uniform intensity parameter equal to 1, $N(t)$ be a Poisson counting process on $[0, \infty)$, and $X_i, 1 \leq i < \infty$ be i.i.d. uniform random variables on $[0, 1]^d$. For every $A \subset \mathbb{R}^d$, we denote by $\Pi(A)$ the set of random points in A (A is a Borel set in $\mathbb{R}^d$), and introduce the random variable $\lambda_r(t) = L_r\left(\Pi\left(k^r(t) * [0, 1]^d\right)\right)$, which denotes the evaluation of L_r on random points in cuboids similar to the ones introduced in Section 2.4. Let $\phi_r(t) = \mathbf{E}\left[\lambda_r(t)\right]$ and $V(t) = \mathrm{VAR}\left(\lambda_r(t)\right)$.

Note that the distribution of the number of points in $\Pi\left(k^r(t) * [0, 1]^d\right)$ is Poisson with parameter $t^{||r||_1}$ (the volume of $k^r(t) * [0, 1]^d$), which is the same as the distribution of $N(t^{||r||_1})$. The conditional property of Poisson process states that the points of $\Pi\left(k^r(t) * [0, 1]^d\right)$ in $k^r(t) * [0, 1]^d$ are independently, identically and uniformly distributed given $|\Pi\left(k^r(t) * [0, 1]^d\right)| = n$, which again is true for $X_1, ..., X_{N(t^{||r||_1})}$ given $|N(t^{||r||_1})| = n$ (in $[0, 1]^d$).

Thus the points of $\Pi\left(k^r(t) * [0, 1]^d\right)$ can be generated as $k^r(t) * X_1, ..., X_{N(t^{||r||_1})}$, and from property 2A, we have that the distribution of $\lambda_r(t) = L_r\left(\Pi\left(k^r(t) * [0, 1]^d\right)\right)$ is the same as the distribution of $tL_r\left(X_1, ..., X_{N(t^{||r||_1})}\right)$.

So to study the distribution of $L_r\left(X_1, ..., X_{N(t^{||r||_1})}\right)$, we will study the distribution of $\lambda_r(t)$. The first step is to prove that the limit

$$\lim_{t \to \infty} \frac{\phi_r(t)}{t^{||r||_1}} \text{ exists.}$$

From the subadditivity property of L_r,

$$\lambda_r(t) \leq \sum_{i=1}^{m^{||r||_1}} L\left(\Pi\left(k^r(t) * Q_i(r, m)\right)\right) + Cm^{||r||_1 - 1}.$$

From the translation invariance property of L_r (property 1A) and the fact that Π has a uniform intensity, it follows that

$$\mathbf{E}\left[L_r\left(\Pi\left(k^r(t) * Q_i(r,m)\right)\right)\right] = \mathbf{E}\left[L_r\left(\Pi\left(k^r(t) * Q_1(r,m)\right)\right)\right] \forall 1 \leq i \leq m^{||r||_1}.$$

Now using the definitions of $k^r(.)$, $Q_1(r,m)$ and the $*$ operator, it is direct that

$$\mathbf{E}\left[L_r\left(\Pi\left(k^r(t) * Q_1(r,m)\right)\right)\right] = \mathbf{E}\left[L_r\left(\Pi\left(k^r\left(\frac{t}{m}\right) * [0,1]^d\right)\right)\right] = \phi_r\left(\frac{t}{m}\right)$$

Therefore,

$$\phi_r(t) \leq m^{||r||_1}\phi_r(\frac{t}{m}) + Cm^{||r||_1-1},$$

and thus,

$$\frac{\phi_r(t)}{t^{||r||_1}} \leq \frac{\phi_r\left(\frac{t}{m}\right)}{\left(\frac{t}{m}\right)^{||r||_1}} + C\left(\frac{t}{m}\right)^{1-||r||_1}. \tag{A.9}$$

We use equation (A.9) to prove that $\limsup_{t\to\infty}\frac{\phi(t)}{t^{||r||_1}} = \liminf_{t\to\infty}\frac{\phi(t)}{t^{||r||_1}}$.

Let $\liminf_{t\to\infty}\frac{\phi(t)}{t^{||r||_1}} = \beta$, we have

$$\begin{aligned}
\beta &\leq \limsup_{t\to\infty}\frac{\phi(t)}{t^{||r||_1}} = \limsup_{t\to\infty}\frac{\phi(u_0 t)}{u_0^{||r||_1}t^{||r||_1}} \ \forall\ u_0 \in \mathbb{R}^+ \\
&\leq \limsup_{t\to\infty}\frac{\phi(u_0\lceil t\rceil)}{u_0^{||r||_1}t^{||r||_1}} \text{ by the monotonicity of } \phi(.) \\
&= \limsup_{t\to\infty}\frac{\phi(u_0\lceil t\rceil)}{u_0^{||r||_1}t^{||r||_1}}\frac{\lceil t\rceil^{||r||_1}}{\lceil t\rceil^{||r||_1}} = \limsup_{t\to\infty}\frac{\phi(u_0\lceil t\rceil)}{u_0^{||r||_1}\lceil t\rceil^{||r||_1}}.
\end{aligned} \tag{A.10}$$

Note that equation (A.9) holds for any $m \in \mathbb{N}$, and thus applying it with $m = \lceil t\rceil$ gives that for all $t \in \mathbb{R}^+$:

$$\frac{\phi(u_0\lceil t\rceil)}{u_0^{||r||_1}\lceil t\rceil^{||r||_1}} \leq \frac{\phi(u_0)}{u_0^{||r||_1}} + Cu_0^{1-||r||_1}. \tag{A.11}$$

Using this is equation (A.10) yields:

$$\beta \leq \limsup_{t\to\infty}\frac{\phi(t)}{t^{||r||_1}} \leq \frac{\phi(u_0)}{u_0^{||r||_1}} + Cu_0^{1-||r||_1},\ \forall u_0 \in \mathbb{R}^+. \tag{A.12}$$

Since $\beta = \liminf_{t\to\infty} \frac{\phi(t)}{t^{||r||_1}}$, This means $\forall \epsilon > 0$ and any $\tau \in \mathbb{R}^+$, there is an infinite number of $u_0 \geq \tau$ that satisfy

$$\frac{\phi(u_0)}{u_0^{||r||_1}} \leq \beta + \epsilon. \tag{A.13}$$

Thus for any $\epsilon > 0$, $\exists\, u_0$ such that

$$\frac{\phi(u_0)}{u_0^{||r||_1}} + Cu_0^{1-||r||_1} \leq \beta + \epsilon,$$

and therefore

$$\liminf_{t\to\infty} \frac{\phi(t)}{t^{||r||_1}} = \limsup_{t\to\infty} \frac{\phi(t)}{t^{||r||_1}} = \lim_{t\to\infty} \frac{\phi(t)}{t^{||r||_1}} = \beta(L_r) < \infty.$$

Now to deal with proving that $\frac{V(t)}{t^{2||r||_1}}$ is bounded. Note that by using the subadditivity property (property 5A) of L_r with $m = \lceil t \rceil$ gives (with property 1A):

$$\lambda_r(t) \leq \lambda_r(\lceil t \rceil) \leq \sum_{i=1}^{\lceil t \rceil^{||r||_1}} L_r(\Pi([0,1]^d)) + C\lceil t \rceil^{||r||_1},$$

and therefore by squaring, taking expectations, subtracting $\phi^2(t)$ and dividing by $t^{2||r||_1}$

$$\frac{V(t)}{t^{2||r||_1}} \leq \frac{\mathbf{E}[\lambda_r^2(1)]}{t^{||r||_1}} + \phi^2(1) + 2C\phi_r(1) + C^2 - \frac{\phi^2(t)}{t^{2||r||_1}} < \infty\ \forall t \in \mathbb{R}^+.$$

We next aim to prove that

$$\sum_{k=1}^{\infty} \frac{V(2^k t)}{(2^k t)^{2||r||_1}} < \infty. \tag{A.14}$$

Before the proof, note that since $\lim_{t\to\infty} \frac{\phi(t)}{t^{||r||_1}} = \beta(L_r)$, applying Chebyshev's inequality to equation (A.14) gives:

$$\sum_{k=1}^{\infty} \mathbf{P}\left(\left|\frac{\lambda_r(2^k t)}{(2^k t)^{||r||_1}} - \beta(L_r)\right| > \epsilon\right) < \infty,$$

and thus since $\lambda_r(t)$ has the same distribution as $tL_r(X_1, ..., X_{N(t^{||r||_1})})$,

$$\sum_{k=1}^{\infty} \mathbf{P}\left(\left|\frac{2^k t L_r(X_1, ..., X_{N((2^k t)^{||r||_1})})}{(2^k t)^{||r||_1}} - \beta(L_r)\right| > \epsilon\right) < \infty,$$

and therefore for all $t \in \mathbb{R}^+$,

$$\lim_{k\to\infty} \frac{L_r(X_1, ..., X_{N((2^k t)^{||r||_1})})}{(2^k t)^{||r||_1 - 1}} = \beta(L_r) \text{ with probability 1.} \tag{A.15}$$

From property 5A, with $m = 2$, we have:

$$\lambda_r(2t) \leq \sum_{i=1}^{2^{||r||_1}} L_r\left(\Pi(k^r(2t)Q_i(r,2))\right) + C2^d t,$$

now let $\tilde{\lambda_r}(t) = \lambda_r(t) + 2Ct$ and $\tilde{\lambda_{ri}}(t) = L_r\left(\Pi(k^r(2t)Q_i(r,2))\right) + 2Ct, 1 \leq i \leq 2^{||r||_1}$, we have:

$$\tilde{\lambda_r}(2t) \leq \sum_{i=1}^{2^{||r||_1}} \tilde{\lambda_{ri}}(t) \tag{A.16}$$

and $\tilde{\lambda_{ri}}(t)$ have the same distribution as $\tilde{\lambda_r}(t)$. Now let $\tilde{\phi}(t) = \mathbf{E}[\tilde{\lambda_r}(t)]$ and $\psi(t) = \sqrt{\mathbf{E}[\tilde{\lambda_r}^2(t)]}$, now

$$V(t) = \psi^2(t) - \tilde{\phi}^2(t).$$

Therefore by squaring equation (A.16), taking expectations, subtracting $\phi^2(t)$ and dividing by $(2t)^{2||r||_1}$, we get:

$$\frac{V(2t)}{(2t)^{2||r||_1}} - \frac{2^{||r||_1} V(t)}{(2t)^{2||r||_1}} \leq \frac{\tilde{\phi}^2(t)}{t^{2||r||_1}} - \frac{\tilde{\phi}^2(2t)}{2t^{2||r||_1}}$$

By iterating this for $2t, 4t, ..., 2^M t$ and summing, we get

$$(1 - 2^{-||r||_1}) \sum_{i=1}^{M} \frac{V(2^i t)}{(2^i t)^{2||r||_1}} \leq \frac{V(t)}{t^{2||r||_1}} + \frac{\tilde{\phi}^2(t)}{t^{2||r||_1}} \; \forall M \in \mathbb{N}, \; t \in \mathbb{R}^+$$

and thus for every $t \in \mathbb{R}^+$

$$\sum_{i=1}^{\infty} \frac{V(2^i t)}{(2^i t)^{2\|r\|_1}} \leq \frac{1}{1-2^{-\|r\|_1}} \left(\frac{V(t)}{t^{2\|r\|_1}} + \frac{\tilde{\phi}^2(t)}{t^{2\|r\|_1}} \right) < \infty.$$

Let $p \in \mathbb{N}$ be fixed, and note that for all $s \in \mathbb{R}, s \geq 2^p$, there is an integer t $2^p \leq t < 2^p + 1$, and an integer k such that $2^k t \leq s \leq 2^k(t+1)$. By the monotonicity of L_r,

$$L_r(X_1, ..., X_{N((2^k t)^{\|r\|_1})}) \leq L_r(X_1, ..., X_{N(s^{\|r\|_1})}) \leq L_r(X_1, ..., X_{N((2^k(t+1))^{\|r\|_1})}),$$

and therefore,

$$\begin{aligned}
\limsup_{s\to\infty} \frac{L_r(X_1, ..., X_{N(s^{\|r\|_1})})}{s^{\|r\|_1 - 1}} &\leq \limsup_{s\to\infty} \frac{L_r(X_1, ..., X_{N((2^k(t+1))^{\|r\|_1})})}{s^{\|r\|_1 - 1}} \\
&\leq \limsup_{s\to\infty} \frac{L_r(X_1, ..., X_{N((2^k(t+1))^{\|r\|_1})})}{(2^k t)^{\|r\|_1 - 1}} \frac{(t+1)^{\|r\|_1 - 1}}{(t+1)^{\|r\|_1 - 1}} \\
&= \limsup_{s\to\infty} \frac{L_r(X_1, ..., X_{N((2^k(t+1))^{\|r\|_1})})}{(2^k(t+1))^{\|r\|_1 - 1}} \frac{(t+1)^{\|r\|_1 - 1}}{(t)^{\|r\|_1 - 1}} \\
&\leq \limsup_{s\to\infty} \frac{L_r(X_1, ..., X_{N((2^k(t+1))^{\|r\|_1})})}{(2^k(t+1))^{\|r\|_1 - 1}} \frac{(2^p+2)^{\|r\|_1 - 1}}{(2^p)^{\|r\|_1 - 1}} \\
&\leq \beta(L_r)(1+2^{1-p})^{\|r\|_1 - 1}.
\end{aligned} \tag{A.17}$$

Similarly,

$$\liminf_{s\to\infty} \frac{L_r(X_1, ..., X_{N(s^{\|r\|_1})})}{s^{\|r\|_1 - 1}} \geq \beta(L_r)(1+2^{1-p})^{1-\|r\|_1},$$

and therefore, since p was arbitrary, we have

$$\lim_{s\to\infty} \frac{L_r(X_1, ..., X_{N(s^{\|r\|_1})})}{s^{\|r\|_1 - 1}} = \beta(L_r).$$

Now let $\tau(n)$ be the random variable defined so that $N(\tau(n)^{\|r\|_1}) = n$ and note that

$$\lim_{n\to\infty} \frac{\tau(n)}{n^{1/\|r\|_1}} = 1 \text{ almost surely,}$$

and therefore

$$\lim_{n\to\infty}\frac{L_r(X_1,...,X_n)}{n^{1-\frac{1}{||r||_1}}}=\lim_{n\to\infty}\frac{L_r(X_1,...,X_{N(\tau^{||r||_1}(n))})}{\tau^{||r||_1-1}(n)}\frac{\tau^{||r||_1-1}(n)}{n^{1-\frac{1}{||r||_1}}}=\beta(L_r).$$

□

A.3.2 Proof of lemma 4.7

This proof is similar to the one in [1]. Assume without loss of generality that E is a subset of $[0,1]^d$. $\exists$ disjoint cubes $Q_i, i \in \{1,...,M\}$ such that the Lebesgue measure of Q_i, $V(Q_i)$ satisfies $\sum_{i=1}^M V(Q_i) < \epsilon$ and

$$\mathbf{P}[Y_l \in E, Y_l \notin \cup_{i=1}^M Q_i] < \epsilon.$$

Denote by N_j the random variable that is equal to the number of points Y_i that are in Q_j, and by N_0 the number of points $Y_l \in E, Y_l \notin \cup_{i=1}^M Q_i$. By simple subadditivity,

$$L_r(\{Y_1,...,Y_n\}\cap E) \le MB + \sum_j L_r(\{Y_i : Y_i \in Q_j\}) + L_r(\{Y_i : Y_i \in E, Y_i \notin U_{j=1}^M Q_j\}).$$

By scale boundedness,

$$L_r(\{Y_i : Y_i \in Q_j\}) \le BN_j^{1-\frac{1}{||r||_1}}(V(Q_j))^{\frac{1}{||r||_1}}.$$

Thus

$$\begin{aligned}\sum_j L_r(\{Y_i : Y_i \in Q_j\}) &\le B\sum_j N_j^{1-\frac{1}{||r||_1}}(V(Q_j))^{\frac{1}{||r||_1}} \\ &\le B(\sum_j N_j)^{1-\frac{1}{||r||_1}}(\sum_j V(q_j))^{\frac{1}{||r||_1}} \qquad \text{(A.18)} \\ &\le Bn^{1-\frac{1}{||r||_1}}\epsilon^{\frac{1}{||r||_1}}\end{aligned}$$

Similarly,

$$\begin{aligned}L_r(\{Y_i : Y_i \in E, Y_i \notin U_{j=1}^M Q_j\}) &\le BN_0^{1-\frac{1}{||r||_1}} \\ &\le B(n\epsilon)^{1-\frac{1}{||r||_1}} \quad \text{a.s. as } n\to\infty. \qquad \text{(A.19)}\end{aligned}$$

Thus

$$L_r(\{Y_1, ..., Y_n\} \cap E) \leq MB + Bn^{1-\frac{1}{\|r\|_1}}(\epsilon^{\frac{1}{\|r\|_1}} + \epsilon^{1-\frac{1}{\|r\|_1}}).$$

Since $\epsilon > 0$ was arbitrary, the proof is complete.

A.3.3 Proof of lemma 4.8

This proof is similar to the one in [1]. Assume without loss of generality that the support of $Y_i \subset [0,1]^d$ and denote the singular support of Y_i by E. As above, denote by N_j the random variable that is equal to the number of Y_i in $Q_j(r,m)$. By simple subadditivity:

$$L_r(\{Y_1, ..., Y_n\}) \leq L_r(\{Y_1, ..., Y_n\} \cap E) + \sum_{i=1}^{s} L_r(\{Y_1, ..., Y_n\} \cap Q_i(r,m)) + sB. \quad \text{(A.20)}$$

By theorem 4.7 and property 1A, we have:

$$\lim_{n\to\infty} \frac{L_r(\{Y_1, ..., Y_n\} \cap Q_i(r,m))}{N_i^{1-\frac{1}{\|r\|_1}}} = \beta(L_r)(V(Q_i(r,m)))^{\frac{1}{\|r\|_1}}, \quad \text{a.s.}$$

since $\{Y_1, ..., Y_n\} \cap Q_i(r,m)$ has a uniform distribution. Now since as

$$n \to \infty, N_i = a_i n V(Q_i(r,m)) \quad \text{a.s.},$$

we have

$$\lim_{n\to\infty} \frac{L_r(\{Y_1, ..., Y_n\} \cap Q_i(r,m))}{n^{1-\frac{1}{\|r\|_1}}} = \beta(L_r)V(Q_i(r,m))a_i^{1-\frac{1}{\|r\|_1}}, \quad \text{a.s.},$$

and therefore

$$\limsup_{n\to\infty} \frac{L_r(\{Y_1, ..., Y_n\})}{n^{1-\frac{1}{\|r\|_1}}} \leq \beta(L_r) \int_{\mathbb{R}^d} a_i^{1-\frac{1}{\|r\|_1}}, \quad \text{a.s}$$

Using the same proof with the fact that by monotonicity and upper-linearity,

$$L_r(\{Y_1, ..., Y_n\}) \geq + \sum_{i=1}^{s} L_r(\{Y_1, ..., Y_n\} + o(n^{1-\frac{1}{||r||_1}})$$

gives a similar lower bound for the lim inf and completes the proof.

A.3.4 Proof of theorem 4.8

This proof is similar to the one in [1]. We first assume that the support of Y_i is a subset of $[0,1]^d$ and denote the singular part of the support by E. Now choose

$$\phi(y) = \sum_{i=1}^{s} a_i 1_{Q_i}(y),$$

where Q_i are cuboids as described in Section 4.18. The "thinning domain" A is defined as:

$$y : f(y) \leq \phi(y).$$

A sequence Y_i^1 can be generated from Y_i, such that the probability distribution of Y_i^1 on $(A \cup E)^C$ is $\phi(y)$ (here B^C is the complement of the set B.) This is done as follows: If $Y_i \in (A \cup E)^C$, then Y_i^1 is taken to be Y_i with probability $p = \frac{\phi(y)}{f(y)}$ or a fixed point $y_0 \in A$ with probability $1 - p$. This is done independently for every i. Note that this means that the probability distribution of Y_i^1 on $(A \cup E)^C$ is $\phi(y)$. If $Y_i \in A \cup E$, then Y_i^1 is chosen to be y_0.

Another sequence of i.i.d. random variables Y_i^2 can be generated from a distribution with bounded support and absolutely continuous part $\phi(y)$. Now $\{Y_i^1\} \cap (A \cup E)^C$ and $\{Y_i^2\} \cap (A \cup E)^C$ have the same distribution, and thus the two processes

$$L_r(\{Y_i^1\} \cap (A \cup E)^C) = L_n^1,$$

and

$$L_r(\{Y_i^2\} \cap (A \cup E)^C) = L_n^2$$

also have the same distribution. Since $\phi(y)$ satisfies the conditions for lemma 4.8, we have

$$\lim_{n\to\infty}\frac{L_n^1}{n^{1-\frac{1}{||r||_1}}}=\lim_{n\to\infty}\frac{L_n^2}{n^{1-\frac{1}{||r||_1}}}.$$

By simple subadditivity,

$$L_n^2\geq L_r(Y_1^2,...,Y_n^2)-L_r(\{Y_1^2,...,Y_n^2\}\cap(A\cup E))-B,$$

and thus

$$\lim_{n\to\infty}\frac{L_n^2}{n^{1-\frac{1}{||r||_1}}}\geq\beta(L_r)\int_{\mathbb{R}^d}\phi(y)^{1-\frac{1}{||r||_1}}dy-B(\int_{A\cup E}\phi(y)dy)^{1-\frac{1}{||r||_1}}\quad\text{a.s.}$$

Now by monotonicity,

$$L_r(Y_1,...,Y_n)\geq L_r(\{Y_1,...,Y_n\}\cap(A\cup E)^C)\geq L_n^1,$$

and thus

$$\liminf_{n\to\infty}\frac{L_r(Y_1,...,Y_n)}{n^{1-\frac{1}{||r||_1}}}\geq\beta(L_r)\int_{\mathbb{R}^d}f(y)^{1-\frac{1}{||r||_1}}\quad\text{a.s.}$$

Now, to get the upper bound, consider $\phi(y)$ as before, but let $A=y:f(y)\geq\phi(y)$. Let Y_i^1 be an i.i.d. sequence of random variables whose distribution has an absolutely continuous part $\phi(y)$, and a singular support at $y_0\in A$ with probability $1-\int_{\mathbb{R}^d}\phi(y)dy$ when $\int_{\mathbb{R}^d}\phi(y)dy<1$.

We now aim to produce a subsequence of Y_i^1 that has the same distribution as $Y_i\cap(A\cup E)^C$. This can be done as follows: If $Y_i^1\in(A\cup E)^C$, let $Y_i^2=Y_i^1$ with probability $p=\frac{f(y)}{\phi(y)}$, and $Y_i^2=y_0$ with probability $1-p$ (this is done independently for each i.) Note that the distribution of $Y_i^2\cap(A\cup E)^C$ is $f(y)$. Now since E is singular, we have

$$\limsup_{n\to\infty} \frac{L_r(Y_1,...,Y_n)}{n^{1-\frac{1}{||r||_1}}} \leq \limsup_{n\to\infty} \frac{L_r(\{Y_1,...,Y_n\}\cap E^C)}{n^{1-\frac{1}{||r||_1}}}$$
$$\leq \limsup_{n\to\infty} \frac{L_r(\{Y_1,...,Y_n\}\cap (A\cup E)^C)}{n^{1-\frac{1}{||r||_1}}} + B\mathbf{P}[Y_i \in E^C \cap A]^{1-\frac{1}{||r||_1}}$$
$$= \limsup_{n\to\infty} \frac{L_r(\{Y_1^2,...,Y_n^2\}\cap (A\cup E)^C)}{n^{1-\frac{1}{||r||_1}}} + B\mathbf{P}[Y_i \in E^C \cap A]^{1-\frac{1}{||r||_1}} \quad \text{a.s.}$$
$$= \limsup_{n\to\infty} \frac{L_r(\{Y_1^1,...,Y_n^1\}\cap (A\cup E)^C)}{n^{1-\frac{1}{||r||_1}}} + B\mathbf{P}[Y_i \in E^C \cap A]^{1-\frac{1}{||r||_1}} \quad \text{a.s.}$$
$$= \beta(L)\int_{\mathbb{R}^d} \phi(y)^{1-\frac{1}{||r||_1}} dy + B\mathbf{P}[Y_i \in E^C \cap A]^{1-\frac{1}{||r||_1}} \quad \text{a.s.} \tag{A.21}$$

Since $\phi(y)$ can be chosen arbitrarily close to $f(y)$, and A can be chosen to be arbitrarily small, then $\forall\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{L_r(Y_1,...,Y_n)}{n^{1-\frac{1}{||r||_1}}} \leq \beta(L_r)\int_{\mathbb{R}^d} f(y)^{1-\frac{1}{||r||_1}} + \epsilon \quad \text{a.s.},$$

and therefore

$$\limsup_{n\to\infty} \frac{L_r(Y_1,...,Y_n)}{n^{1-\frac{1}{||r||_1}}} \leq \beta(L_r)\int_{\mathbb{R}^d} f(y)^{1-\frac{1}{||r||_1}} \quad \text{a.s.}$$

A.3.5 Proof of lemma 4.10

We will first prove that the MBMP is subadditive almost surely for the case where $m = 2^k, k \in \mathbb{N}$. Thus the $[0,1]^d$ cube is divided into cuboids of sides $\frac{1}{2^{kr_i}}$. Let $n^x(i,k) = \{y_1,...,y_n\} \cap Q_i(r,2^k)$ and $n^y(i,k) = \{y_{n+1},...,y_{2n}\} \cap Q_i(r,2^k)$. Consider the following matching algorithm:

For $k = k_0,...,1$, let $n_l^x(i,k)$ be the number of points of $y_1,...,y_n$ that weren't matched yet, and $n_l^y(i,k)$ be the number of points of $y_{n+1},...,y_{2n}$ that weren't matched yet.

1. For every $Q_i(r,2^k)$ such that $n_l^x(i,k) > 0$ and $n_l^y(i,k) > 0$, match as many pairs as possible ($\min(n_i^x, n_i^y)$ pairs.

First note that given a cuboid with sides $a^{r_1}, ..., a^{r_d}$, if we denote the cost of the optimal matching by L, and we divide the cuboid into $2^{||r||_1}$ cuboids having sides $(\frac{a}{2})^{r_1}, ..., (\frac{a}{2})^{r_d}$, then

$$L \leq \sum_{i=1}^{2^{||r||_1}} L_i + \frac{a\sqrt{d}}{2} \sum_{i=1}^{||r||_1}.$$

Applying this recursively to the algorithm above gives us:

$$L_r(y_1, ..., y_{2n}) \leq \sum_{i=1}^{2^{k||r||_1}} L_i + \sum_{j=1}^{k} \frac{\sqrt{d}}{2^j} \sum_{i=1}^{2^{j||r||_1}} |n_l^x(i,j) - n_l^y(i,j)|.$$

The next step is to prove the subadditivity for the mean of $L_r(y_1, ..., y_{2n})$ (we call it $M(n)$), when $Y_1, ..., Y_{N_1}$ and $Y_{N_1+1}, ..., Y_{N_1+N_2}$ don't have a fixed number, but N_1 and N_2 are independent Poisson processes with mean n.

It is immediate that $\mathbf{E}[L_i] = 2^{-k} M(\frac{n}{2^{k||r||_1}})$ and that

$$|n_l^x(i,j) - n_l^y(i,j)| \leq \sqrt{2}(\frac{n}{2^{k||r||_1}})^{1/2}.$$

Thus

$$M(n) \leq 2^{k(||r||_1 - 1)} M(\frac{n}{2^{k||r||_1}}) + \sqrt{2dn} \sum_{j=1}^{k} 2^{j(\frac{||r||_1}{2} - 1)}.$$

A similar inequality can be derived for the case where the $[0,1]^d$ is divided into $m^{||r||_1}$ cuboids. This is done by considering the cuboid whose sides are $[0, 2^{k+1}]$, where k is such that $2^k < m < 2^{k+1}$ and applying the previous algorithm. The resulting inequality is

$$M(n) \leq m^{||r||_1 - 1} M(\frac{n}{m^{||r||_1}}) + 2^{||r||_1} \sqrt{2dn} \sum_{j=0}^{k} 2^{1 - \frac{k||r||_1}{2}}.$$

By comparing it to the process above, it follows that for a given n, the mean of the MBMP is bounded by

$$M(n) \leq m^{||r||_1 - 1} M(\frac{n}{m^{||r||_1}}) + 2^{||r||_1} \sqrt{2dn} \sum_{j=0}^{k} 2^{1 - \frac{k||r||_1}{2}} + 2\sqrt{2dn}.$$

Finally, since for all $t > 0$,

$$\mathbf{P}\left[\left|\frac{L_r(Y_1,...,Y_n)}{n^{1-\frac{1}{||r||_1}}} - \frac{M(n)}{n^{1-\frac{1}{||r||_1}}}\right| > t\right] \leq 2e^{\frac{n^{1-\frac{2}{||r||_1}}t^2}{8||r||_1}},$$

it follows that L_r is subadditive almost surely.

A.3.6 Proof of lemma 5.11 and 6.18

We will prove a more general statement that lemmas 5.11 and 6.18 are special cases of:

Theorem A.15. *Let L be a monotone functional (as described before), and $Q_1, ..., Q_M$ be a partition of the $[0,1]^d$ cube. Let $L^i(\{y_1, ..., y_n\}) = L(\{y_1, ..., y_n\} \cap Q_i)$. Let $B(M)$ be any function of M that is independent of n. Assume*

$$L(Y_1, ..., Y_n) \leq \sum_{i=1}^{M} L^i(Y_1, ..., Y_n) + B(M) \quad a.s.,$$

when $Y_1, ..., Y_n$ are i.i.d. with distribution $f(y)$. If $\forall 1 \leq i \leq M$, $L^i(Y_1, ..., Y_n)$ is

$$\Theta(n^{1-\frac{1}{s^i}}) \quad a.s.,$$

then L is

$$\Theta(n^{1-\frac{1}{s^{i^*}}}) \quad a.s.$$

when $Y_1, ..., Y_n$ are i.i.d. with distribution $f(y)$. Here $s^{i^*} = \max_i s^i$.

Note that this theorem relaxes the subadditivity assumption in that it only requires that L is subadditive for one partition of the $[0,1]^d$ cube and that the term $B(M)$ is arbitrary. By monotonicity, we have

$$L(Y_1, ..., Y_n) \geq L^{i^*}(Y_1, ..., Y_n),$$

and therefore L is $\Omega(n^{1-\frac{1}{s^{i*}}})$. By subadditivity, we have:

$$\begin{aligned} L(Y_1, ..., Y_n) &\leq \sum_{i=1}^{M} L^i(Y_1, ..., Y_n) + B(M) \quad \text{a.s.} \\ &leqM \max(L^i(Y_1, ..., Y_n)) + B(M) \quad \text{a.s.} \\ &\leq ML^{i^*} + B(M), \end{aligned} \tag{A.22}$$

and therefore L is $O(n^{1-\frac{1}{s^{i*}}})$.

A.3.7 Proof of lemma 5.13

This is similar to the proof of lemma 4.10.

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