

**Alternative large  $N_c$  schemes and chiral dynamics**

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We compare the dependences on the number of colors of the leading  $\pi\pi$  scattering amplitudes using the single index quark field and two index quark fields. These are seen to have different relationships to the scattering amplitudes suggested by chiral dynamics which can explain the long puzzling pion pion  $s$  wave scattering up to about 1 GeV. This may be interesting for getting a better understanding of the large  $N_c$  approach as well as for application to recently proposed technicolor models.

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**I. BACKGROUND**

Gaining control of QCD in its strongly interacting (low energy) regime constitutes a real challenge. One very attractive approach is based on studying the theory in the large number of colors ( $N_c$ ) limit [1,2]. At the same time one may obtain more information by requiring the theory to model the (almost) spontaneous breakdown of chiral symmetry [3,4]. A standard test case is pion pion scattering in the energy range up to about 1 GeV. Some time ago, an attempt was made [5,6] to implement this combined scenario. Since the leading large  $N_c$  amplitude contains only tree diagrams involving mesons of the standard quark-antiquark type, it is expected that the required amplitude should be gotten by calculating just the chiral tree diagrams for rho meson exchange together with the four point pion contact diagram. There are no unknown parameters in this calculation. The crucial question is whether the scattering amplitude calculated in this way will satisfy unitarity. When one compares the result with experimental data up to about 1 GeV on the real part of the (most sensitive to unitarity violation)  $J = I = 0$  partial wave, one finds (see Fig. 1 of [6]) that the result violates the partial wave unitarity bound by just a “little bit.” On the other hand, the pion contact term by itself violates unitarity much more drastically so one might argue that the large  $N_c$  approach, which suggests that the tree diagrams of all quark antiquark resonances in the relevant energy range be included, is helping a lot. To make matters more quantitative one might ask the question, By how much should  $N_c$  be increased in order for the amplitude in question to remain within the unitarity bounds for energies below 1 GeV?

This question was answered in a very simple way in [7], as we now briefly review. In terms of the conventional amplitude,  $A(s, t, u)$ , the  $I = 0$  amplitude is  $3A(s, t, u) + A(t, s, u) + A(u, t, s)$ . One gets the  $J = 0$  channel by pro-

jecting out the correct partial wave. The current algebra (pion contact diagram) contribution to the conventional amplitude is

$$A_{ca}(s, t, u) = 2 \frac{s - m_\pi^2}{F_\pi^2}, \quad (1)$$

where the pion decay constant,  $F_\pi$ , depends on  $N_c$  as  $F_\pi(N_c) = 131\sqrt{N_c}/\sqrt{3}$  so that  $F_\pi(3) = 131$  MeV. Furthermore,  $m_\pi = 137$  MeV is independent of  $N_c$ . The desired amplitude is obtained by adding to the current algebra term the following vector meson  $\rho(770)$  contribution:

$$A_\rho(s, t, u) = \frac{g_{\rho\pi\pi}^2}{2m_\rho^2} (4m_\pi^2 - 3s) - \frac{g_{\rho\pi\pi}^2}{2} \times \left[ \frac{u - s}{(m_\rho^2 - t) - im_\rho \Gamma_\rho \theta(t - 4m_\pi^2)} + \frac{t - s}{(m_\rho^2 - u) - im_\rho \Gamma_\rho \theta(u - 4m_\pi^2)} \right], \quad (2)$$

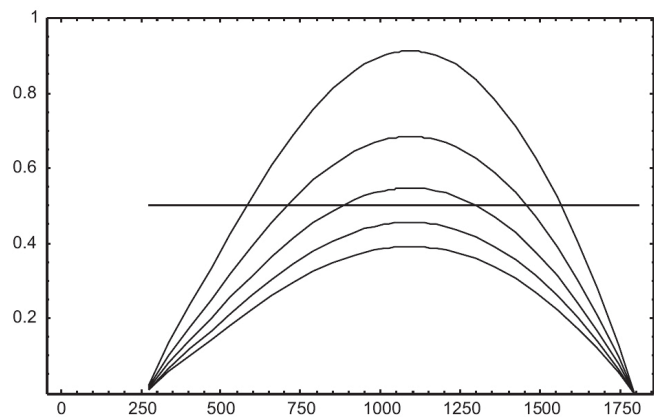


FIG. 1. Real part of the  $I = J = 0$  partial wave amplitude due to the current algebra +  $\rho$  terms, plotted for the following increasing values of  $N_c$  (from top to bottom), 3, 4, 5, 6, 7. The curve with largest violation of the unitarity bound corresponds to  $N_c = 3$  while the ones within the unitarity bound are for  $N_c = 6, 7$ . The horizontal axis shows the center of mass energy in MeV.

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where  $g_{\rho\pi\pi}(N_c) = 8.56\sqrt{3}/\sqrt{N_c}$  is the  $\rho\pi\pi$  coupling constant. Also,  $m_\rho = 771$  MeV is independent of  $N_c$  and

$$\Gamma_\rho(N_c) = \frac{g_{\rho\pi\pi}^2(N_c)}{12\pi m_\rho^2} \left( \frac{m_\rho^2}{4} - m_\pi^2 \right)^{3/2}. \quad (3)$$

It should be noted that the first term in Eq. (2), which is an additional nonresonant contact interaction other than the current algebra contribution, is required when we include the  $\rho$  vector meson contribution in a chiral invariant manner. In Fig. 1 we show the real part of the  $I = J = 0$  amplitude (denoted  $R_0^0$ ) due to current algebra plus the  $\rho$  contribution for increasing values of  $N_c$ . Since in this channel the vector meson is never on shell, we suppress the contribution of the width in the vector meson propagator in Eq. (2). One observes that the unitarity bound (i.e.,  $|R_0^0| \leq 1/2$ ) is satisfied for  $N_c \geq 6$  till well beyond the 1 GeV region. However, unitarity is still a problem for 3, 4, and 5 colors. At energy scales larger than the one associated with the vector meson, clearly other resonances are needed [5] but we shall not be concerned with that energy range here. It is also interesting to note that these considerations are essentially unchanged when the pion mass (i.e. explicit chiral symmetry breaking in the Lagrangian) is set to zero.

Note that essentially we are just using the scaling,

$$A(s, t, u) = \frac{3}{N_c} \tilde{A}(s, t, u), \quad (4)$$

where  $\tilde{A}(s, t, u)$  is defined replacing  $F_\pi$  and  $g_{\rho\pi\pi}$  with the  $N_c$  independent quantities  $\tilde{F}_\pi = F_\pi\sqrt{3}/\sqrt{N_c}$  and  $\tilde{g}_{\rho\pi\pi} = g_{\rho\pi\pi}\sqrt{N_c}/\sqrt{3}$ . Other authors [8] have found the same minimum value,  $N_c = 6$ , for the practical consistency of the large  $N_c$  approximation, by using different methods.

In order to explain low energy  $\pi\pi$  scattering for the physical value  $N_c = 3$  one must go beyond the large  $N_c$  approximation. It is attractive to keep the assumption of tree diagram dominance involving nearby resonances, however. One easily sees that adding a scalar singlet resonance (sigma) at the location where the unitarity bound on  $R_0^0(s)$  is first violated should restore unitarity. This is because the real part of a Breit Wigner resonance is zero at the pole location and negative just above it, so will bring  $R_0^0(s)$  below the bound, as required. In [7], the resonance mass was found to be around 550 MeV on this basis. Such a low value would make it different from a  $p$ -wave quark-antiquark state, which is expected to be in the 1000–1400 MeV range. We assume then that it is a four quark state (glueball states are expected to be in the 1.5 GeV range from lattice investigations). Four quark states of diquark-antidiquark type [9] and meson-meson type [10] have been discussed in the literature for many years. The arguments we are giving would seem to hold for either of these two types. Additional discussions using the 't Hooft extrapolation are given in [8]. Accepting this picture, how-

ever, poses a problem for the accuracy of the large  $N_c$  inspired description of the scattering since four quark states are predicted not to exist in the large  $N_c$  limit of QCD. We shall take the point of view that a four quark type state is present since it allows a natural fit to the low energy data. Of course, it is still necessary to fine-tune the parameters and shape of the sigma resonance to fit the experimental  $\pi\pi$  scattering data in detail. In practice, since the parameters of the pion contact and rho exchange contributions are fixed, the sigma is the most important one for fitting and fits may even be achieved [11] if the vector meson piece is neglected. However, the well-established, presumably four quark type,  $f_0(980)$  resonance must be included to achieve a fit in the region just around 1 GeV.

There is by now a fairly large recent literature [12–42] on the effect of light “exotic” scalars in low energy meson-meson scattering. There seems to be a consensus, arrived at using rather different approaches (keeping, however, unitarity), that the sigma exists.

## II. TWO INDEX QUARK FIELDS

Now, consider redefining the  $N_c = 3$  quark field with color index A (and flavor index not written) as

$$q_A = \frac{1}{2}\epsilon_{ABC}q^{BC}, \quad q^{BC} = -q^{CB}, \quad (5)$$

so that, for example,  $q_1 = q^{23}$  and similarly for the adjoint field,  $\bar{q}^1 = \bar{q}_{23}$  etc. This is just a trivial change of variables and will not change anything for QCD. However, if a continuation of the theory is made to  $N_c > 3$ , the resulting theory will be different since the two index antisymmetric quark representation has  $N_c(N_c - 1)/2$  rather than  $N_c$  color components. As was pointed out by Corrigan and Ramond [43], who were mainly interested in the problem of the baryons at large  $N_c$ , this shows that the extrapolation of QCD to higher  $N_c$  is not unique. Further investigation of the properties of the alternative extrapolation model introduced in [43] was carried out by Kiritsis and Papavassiliou [44]. Here, we shall discuss the consequences for the low energy  $\pi\pi$  scattering discussed above, of this alternative large  $N_c$  extrapolation, assuming for our purpose that all the quarks extrapolate as antisymmetric two index objects.

It may be worthwhile to remark that gauge theories with two index quarks have recently gotten a great deal of attention. Armoni, Shifman, and Veneziano [45–49] have proposed an interesting relation between certain sectors of the two index antisymmetric (and symmetric) theories at a large number of colors and sectors of super Yang-Mills. Using a supersymmetry inspired effective Lagrangian approach,  $1/N_c$  corrections were investigated in [50]. Information on the super Yang-Mills spectrum has been obtained in [51]. On the validity of the large  $N_c$  equivalence between different theories and interesting new possible phases, we refer the reader to [52–54]. The finite temperature phase transition and its relation with chiral

symmetry has been investigated in [55] while the effects of a nonzero baryon chemical potential were studied in [56].

When adding flavors, the phase diagram as a function of the number of flavors and colors has been provided in [57]. The complete phase diagram for fermions in arbitrary representations has been unveiled in [58]. The study of theories with fermions in a higher dimensional representation of the gauge group and the knowledge of the associated conformal window led to the construction of minimal models of technicolor [57,59,60] which are not ruled out by current precision measurements and lead to interesting dark matter candidates [61–63] as well as to a very high degree of unification of the standard model gauge couplings [64].

Besides these two limits, a third one for massless one-flavor QCD, which is in between the 't Hooft and Corrigan Ramond ones, has been recently proposed in [65]. Here one first splits the QCD Dirac fermion into the two elementary Weyl fermions and afterwards assigns one of them to transform according to a rank-two antisymmetric tensor while the other remains in the fundamental representation of the gauge group. For three colors one reproduces one-flavor QCD and for a generic number of colors the theory is chiral. The generic  $N_c$  is a particular case of the generalized Georgi-Glashow model [66].

To illustrate the large  $N_c$  counting for the  $\pi\pi$  scattering amplitude when quarks are designated to transform according to the two index antisymmetric representation of color SU(3) one may employ [1] the mnemonic where each tensor index of this group is represented by a directed line. Then the quark-quark gluon interaction is pictured as in Fig. 2. The two index quark is pictured as two lines with arrows pointing in the same direction, as opposed to the gluon which has two lines with arrows pointing in opposite directions. The coupling constant representing this vertex is taken to be  $g_t/\sqrt{N_c}$ , where  $g_t$  (the 't Hooft coupling constant) is to be held constant.

A “one point function,” like the pion decay constant,  $F_\pi$ , would have as its simplest diagram, Fig. 3.

The X represents a pion insertion and is associated with a normalization factor for the color part of the pion’s wave function,

$$\frac{\sqrt{2}}{\sqrt{N_c(N_c - 1)}}, \tag{6}$$

which scales for large  $N_c$  as  $1/N_c$ . The two circles each carry a quark index so their joint factor scales as  $N_c^2$  for large  $N_c$ ; more precisely, taking the antisymmetry into

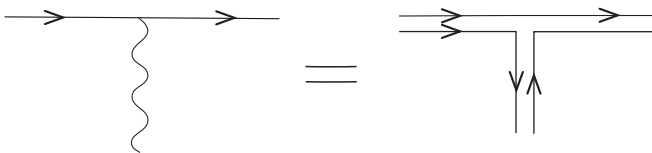


FIG. 2. Two index fermion-gluon vertex.

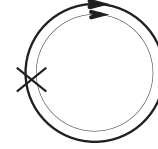


FIG. 3. Diagram for  $F_\pi$  for the two index quark.

account, the factor is

$$\frac{N_c(N_c - 1)}{2}. \tag{7}$$

The product of Eqs. (6) and (7) yields the  $N_c$  scaling for  $F_\pi$ :

$$F_\pi^2(N_c) = \frac{N_c(N_c - 1)}{6} F_\pi^2(3). \tag{8}$$

For large  $N_c$ ,  $F_\pi$  scales proportionately to  $N_c$  rather than to  $\sqrt{N_c}$  as in the case of the 't Hooft extrapolation.

Using this scaling together with Eq. (1) suggests that the  $\pi\pi$  scattering amplitude,  $A$  scales as,

$$A(N_c) = \frac{6}{N_c(N_c - 1)} A(3), \tag{9}$$

which, for large  $N_c$  scales as  $1/N_c^2$  rather than as  $1/N_c$  in the 't Hooft extrapolation. This scaling law for large  $N_c$  may be verified from the mnemonic in Fig. 4, where there is an  $N_c^2$  factor from the two loops multiplied by four factors of  $1/N_c$  from the X's.

It is interesting to find the minimum value of  $N_c$  for which the tree amplitude due to the pion and rho meson terms [given in Eqs. (1) and (2) above] is unitary in this two antisymmetric index quark extrapolation scheme. Figure 1 shows that the peak value of the partial wave amplitude,  $R_0^0$ , due to these two terms is numerically about 0.9. This is to be identified with  $A_{ca}(3) + A_\rho(3)$  in Eq. (9). Thus the condition that the extrapolated amplitude be unitary is

$$0.9 \frac{6}{N_c(N_c - 1)} < 1/2. \tag{10}$$

Clearly, the extrapolated amplitude is unitary already for  $N_c = 4$ , which indicates better convergence in  $N_c$  than for the 't Hooft case which became unitary at  $N_c = 6$ .

There is still another different feature; consider the typical  $\pi\pi$  scattering diagram with an extra internal (two index) quark loop, as shown in Fig. 5.

In this diagram there are four X's [factor from Eq. (6)], five index loops (each giving a factor  $N_c$ ) and six gauge

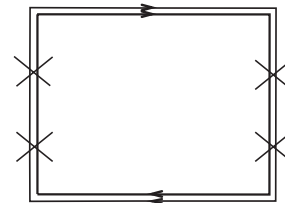


FIG. 4. Diagram for the scattering amplitude,  $A$ , with the two index quark.

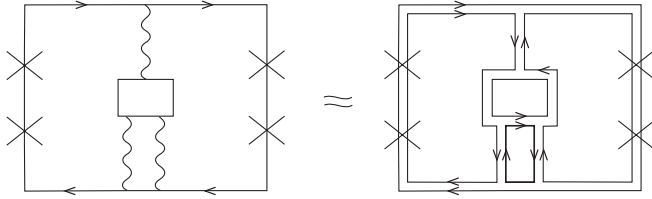


FIG. 5. Diagram for the scattering amplitude,  $A$ , including an internal two index quark loop.

coupling constants. These combine to give a large  $N_c$  scaling behavior proportional to  $1/N_c^2$  for the  $\pi\pi$  scattering amplitude. We see that diagrams with an extra internal 2 index quark loop are not suppressed compared to the leading diagrams. This is analogous, as pointed out in [44], to the behavior of diagrams with an extra gluon loop in the 't Hooft extrapolation scheme. Now, Fig. 5 is a diagram which can describe a sigma particle exchange. Thus in the two index quark scheme, exotic four quark resonances can appear at the leading order in addition to the usual two quark resonances. Given the discussion we reviewed above, the possibility of a sigma appearing at leading order means that one can construct a unitary  $\pi\pi$  amplitude already at  $N_c = 3$  in the two antisymmetric index scheme. From the point of view of low energy  $\pi\pi$  scattering, it seems to be unavoidable to say that the 2 index scheme is more realistic than the 't Hooft scheme given the existence of a four quark type sigma.

Of course, the usual 't Hooft extrapolation has a number of other things to recommend it. These include the fact that nearly all meson resonances seem to be of the quark-antiquark type, the Okubo-Zweig-Iizuka rule predicted holds to a good approximation and baryons emerge in an elegant way as solitons in the model.

A fair statement would seem to be that each extrapolation emphasizes different aspects of the true  $N_c = 3$  QCD. In particular, the usual scheme is not really a replacement for the true theory. That appears to be the meaning of the fact that the continuation to  $N_c > 3$  is not unique.

### III. QUARKS IN TWO INDEX SYMMETRIC COLOR REPRESENTATION

Clearly the assignment of quarks to the two index symmetric representation of color SU(3) looks very similar. We may denote the quark fields as

$$q_{AB}^{\text{sym}} = q_{BA}^{\text{sym}}, \quad (11)$$

in contrast to Eq. (5). There will be  $N_c(N_c + 1)/2$  different color states for the two index symmetric quarks. This means that there is no value of  $N_c$  for which the symmetric theory can be made to correspond to true QCD. For  $N_c = 3$  there are six color states of the quarks and eight color states of the gluon. If we choose  $N_c = 2$ , there are three color states of the quarks but unfortunately only three color

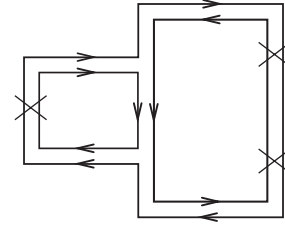


FIG. 6. Diagram for meson decay into two glueballs.

states of the gluon. On the other hand, for large  $N_c$  it would seem reasonable to make approximations like

$$A^{\text{sym}}(N_c) \approx A^{\text{asym}}(N_c), \quad (12)$$

for the  $\pi\pi$  scattering amplitude.

As far as the large  $N_c$  counting goes, the mnemonics in Figs. 2–5 are still applicable to the case of quarks in the two index symmetric color representation. For not so large  $N_c$ , the scaling factor for the pion insertion would be

$$\frac{\sqrt{2}}{\sqrt{N_c(N_c + 1)}}, \quad (13)$$

and the pion decay constant would scale as

$$F_{\pi}^{\text{sym}}(N_c) \propto \sqrt{\frac{N_c(N_c + 1)}{2}}. \quad (14)$$

With the identification  $A^{\text{QCD}} = A^{\text{asym}}(3)$ , the use of Eq. (12) enables us to estimate the large  $N_c$  scattering amplitude as

$$A^{\text{sym}}(N_c) \approx \frac{6}{N_c^2} A^{\text{QCD}}. \quad (15)$$

In applications to recently proposed minimal walking technicolor theories, this formula is useful for making estimates involving weak gauge bosons via the Goldstone boson equivalence theorem [67].

Finally we remark on the large  $N_c$  scaling rules for meson and glueball masses and decays in either the two index antisymmetric or two index symmetric schemes. Both meson and glueball masses scale as  $(N_c)^0$ . Furthermore, all six reactions of the type

$$a \rightarrow b + c, \quad (16)$$

where  $a$ ,  $b$ , and  $c$  can stand for either a meson or a glueball, scale as  $1/N_c$ . This is illustrated in Fig. 6 for the case of a meson decaying into two glueballs; note that the glueball insertion scales as  $1/N_c$  and that two interaction vertices are involved.

### IV. SUMMARY

We have investigated the dependences on the number of colors of the leading  $\pi\pi$  scattering amplitudes using the single and the two index quark fields.

We have seen that in the two index quark extension of QCD, *exotic* four quark resonances can appear at the lead-

ing order in addition to the usual two quark resonances. From the point of view of low energy  $\pi\pi$  scattering the two index scheme is more realistic than the 't Hooft one given the existence of a four quark type sigma. This allows one to explain the long puzzling pion pion  $s$  wave scattering up to about 1 GeV.

Of course, the usual 't Hooft extrapolation has a number of other important predictions to recommend it. A fair statement is that each large  $N_c$  extrapolation of QCD captures different aspects of the physical  $N_c = 3$  case.

We also have related the QCD scattering amplitude at large  $N_c$  with the one featuring two index symmetric quarks (similar connections exist for adjoint fermions). The results are interesting for getting a better understand-

ing of the large  $N_c$  approach as well as for application to recently proposed technicolor models.

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