MODELLING THE SYNOPTIC SCALE RELATIONSHIP BETWEEN EDDY HEAT FLUX AND THE MERIDIONAL TEMPERATURE GRADIENT

by

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ABSTRACT

A simple statistical-dynamical model relating synoptic scale changes in the meridional temperature gradient to changes in the meridional eddy sensible heat flux is developed. The model solution is compared with observations relating the flux to the stability parameter for the two layer model which, assuming the variance of the critical shear in the two layer model is negligible, is equivalent to the meridional temperature gradient. The comparison suggests that a diabatic time scale of about one day is appropriate for perturbations due to changes in the flux, and that roughly one half of the variance of the temperature gradient can be ascribed to processes with time scales much less than the synoptic time scale. The possibility that variations in the critical shear are important is discussed.

Feedback of the temperature gradient on the flux is added to the model. Three model parameters emerge which, if properly tuned, could yield significantly better results than the model without feedback. Although this provides supporting evidence for the presence of feedback, other more justifiable mechanisms yield similar results.

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TABLE OF CONTENTS

I INTRODUCTION

Modelling studies of the relationship between the meridional eddy sensible heat flux and the meridional temperature gradient have in the past concentrated on their time mean relationship. Ensemble averaging of at least as long as the time scale of the transporting eddies is implicit in mixing length parameterizations of the heat flux in terms of the temperature gradient (Green, 1970; Stone, 1972). An examination of the time dependent (synoptic scale) relationship between the heat flux and the temperature gradient is also warranted. Such a study should reveal something about the processes which maintain the time mean relationship. Recently Stone et al., (1981, hereafter denoted by S) have shown how the order of the equations governing the behavior of the heat flux and the temperature gradient can be estimated by examining their respective autocorrelation functions, i.e., their time dependent behavior. This information is then used to test the results of finite amplitude calculations of baroclinic instability (Pedlosky, 1979).

Modelling the observed correlation functions can also be fruitful. In particular, modelling the cross correlation function for the heat flux and the temperature gradient can reveal specific details about the processes relating the two variables. Model parameters can be tuned to match the observations, thus allowing one to measure both the relative importance of different processes and their respective time scales. Although the physics involved in such modelling is included in general circulation models, no attempt has previously been made to explicitly model the synoptic scale relationship between the meridional eddy sensible heat flux and the meridional temperature gradient as seen in their cross correlation function.

In this paper simple linearized models of the temporal relationship between the heat flux and the temperature gradient are developed. The emphasis is on reproducing the observed auto and cross correlation functions, although some attention is devoted to modelling variance. In section II the observed correlation functions are described. The model equations are developed in section III, while in section IV the basic model solution is presented. These solutions are discussed in section V. The possibility of feedback of the temperature gradient on the flux is considered in section VI. Conclusions are presented in section VII.

II OBSERVATIONS

The data used in this study is that used in S, consisting of twice daily observations for three consecutive Januaries (1973, 1974, 1975) of the total tropospheric mean eddy sensible heat flux across selected midlatitude circles

II.1 $$\mathcal{F}(\phi, t) = \frac{2\pi a c_p}{g} \cos\phi \int_{p_T}^{p_0} [v^* T^*] \, dp$$

and the stability parameter for the two layer model

II.2 $$\mathcal{S}(\phi, t) = [u_1 - u_2 - u_c]$$

where a is the earth's radius, c_p is the specific heat at constant pressure, g is the gravitational acceleration, ϕ is the latitude, p_0 and p_T are the surface and tropopause pressures, respectively, u and v are the zonal and meridional velocities, respectively, T is the temperature and

II.3 $$u_c = \frac{\beta R(\theta_1 - \theta_2)}{f^2}$$

is the critical shear for the two layer model, where f is the coriolis parameter, β is the meridional gradient of f, R is the ideal gas constant for dry air and θ is the potential temperature. Square brackets denote the zonal mean, asterisks deviations from the zonal mean. The subscripts one and two denote the mass weighted upper and lower tropospheric mean, respectively. The shear $u_1 - u_2$ is related to the meridional temperature gradient by the thermal wind relation for the two layer model

II.4 $$u_1 - u_2 = -\frac{g}{af}(z_1 - z_2)\left(\frac{1}{T}\frac{\partial T}{\partial \phi}\right)_{p=p_m}$$

where z is the geopotential height and p_m is the mid- tropospheric pressure. Because per cent variations of the thickness $z_1 - z_2$ are less than those of the temperature gradient we will often refer to the shear as the temperature gradient and the critical shear as the static stability, with the appropriate scaling factors implied.

S calculated the auto and cross correlation functions for the flux and the stability parameter, averaging the respective functions over the three Januaries and latitudes 46N, 50N and 54N. For the cross correlation function (figure 1) lag is defined such that the stability parameter lags the flux for positive lags. The most notable feature is the significant correlation for lags of zero, one half and one day. A cubic spline fit to the cross correlations indicates the strongest negative correlation occurs at a lag of about one half day. Note also that for no lag is the cross correlation significantly positive (the 95% confidence level correlation is 0.22). The auto correlation functions for the flux and the stability parameter are shown in figures 2 and 3, respectively. Also shown are auto correlation functions corresponding to first order Markov processes (red noise) with time scales of one and one and a half days, respectively. As discussed by S, although the auto correlation function for the flux is best fit by a second order Markov process, a first order process is still a good fit.

We may consider variations in the stability parameter

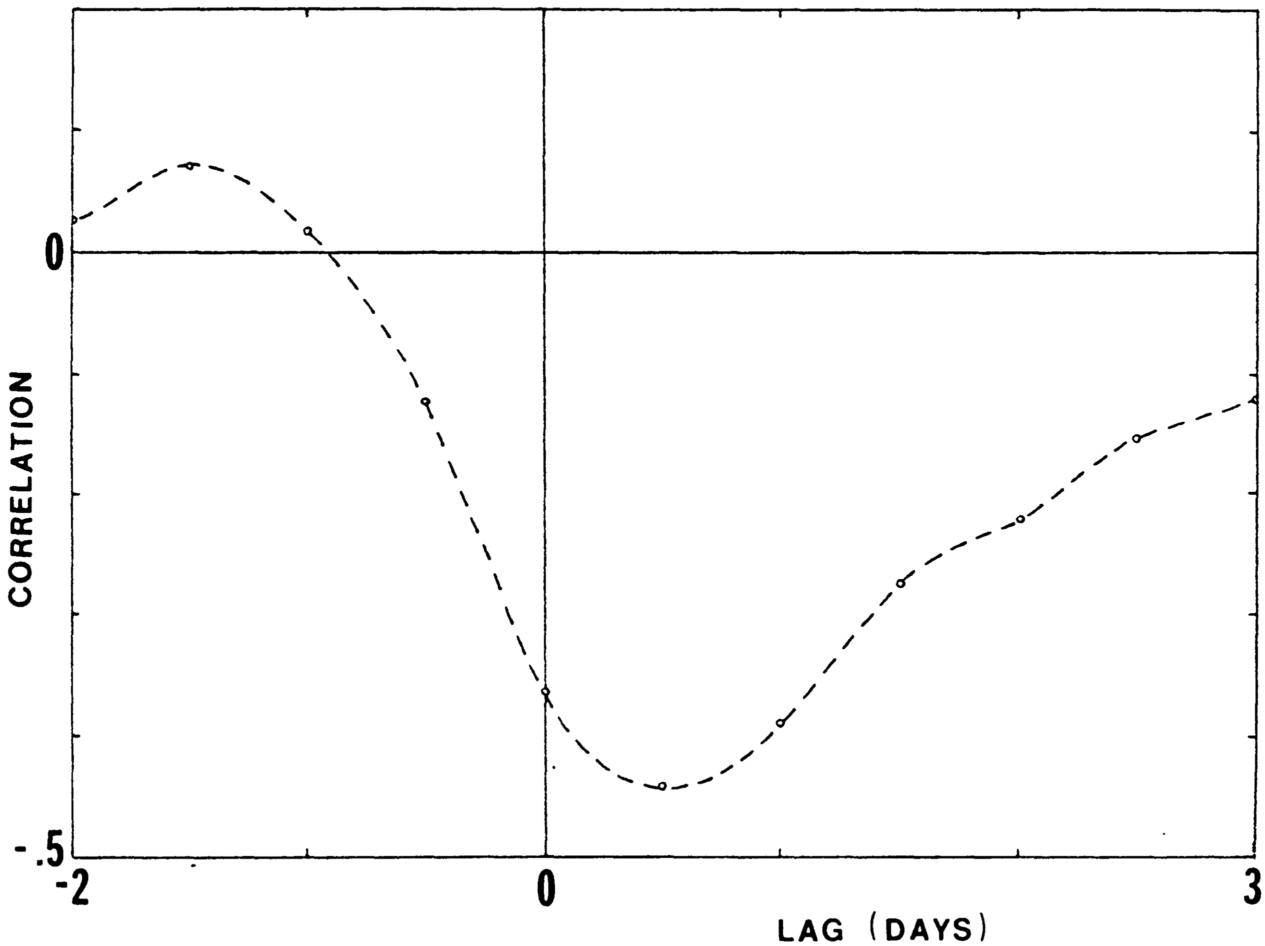

Figure 1. Cross correlation averaged over three Januaries and latitudes 46N, 50N and 54N between the flux $\mathcal{F}(t)$ and the stability parameter $\mathcal{S}(t + \tau)$ as a function of the lag τ (in days). Dashed line is a cubic spline fit to the data points. The 95% confidence level correlation (two-sided) is 0.22.

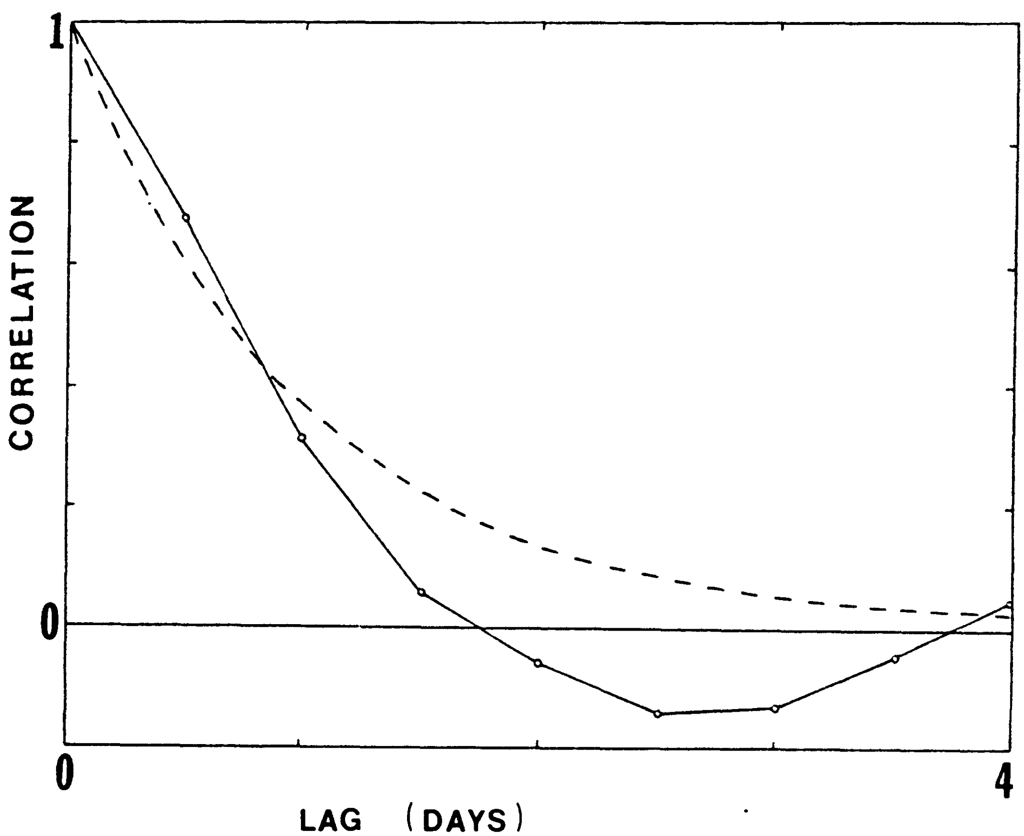

Figure 2. Auto correlation averaged over three Januaries and latitudes 46N, 50N and 54N of the flux as a function of the lag τ. Dashed line is the auto correlation for a red noise with a time scale of one day.

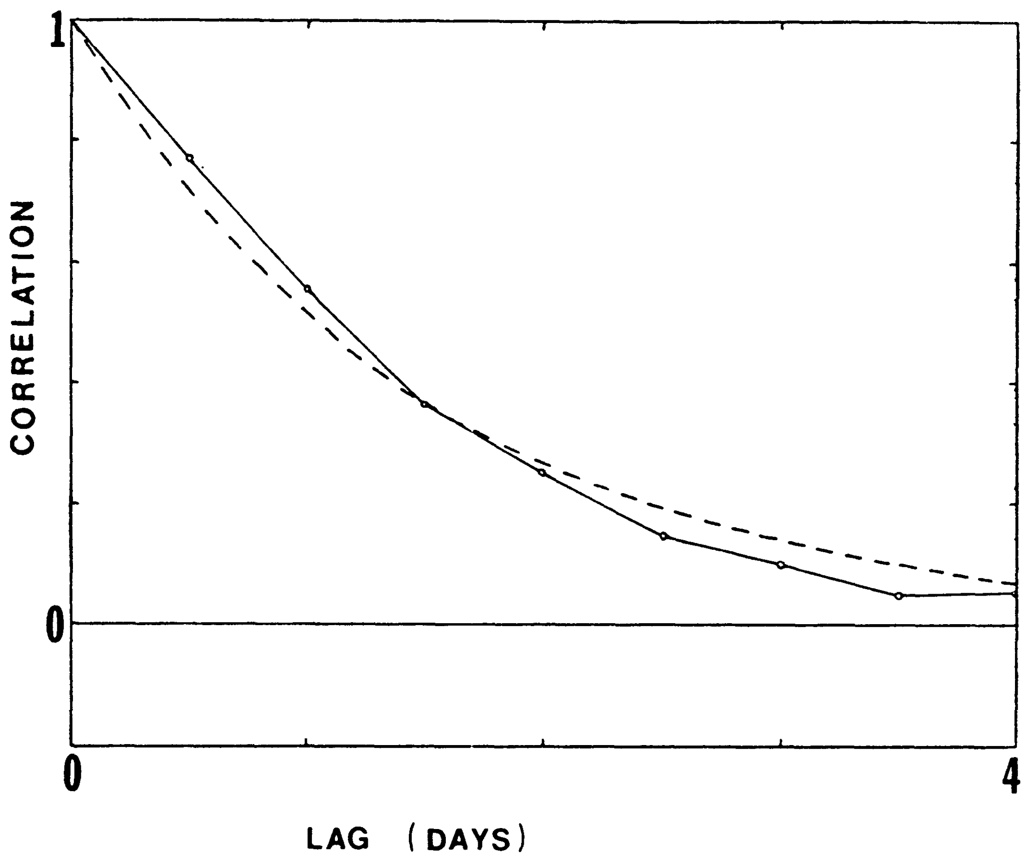

Figure 3. Auto correlation averaged over three Januaries and latitudes 46N, 50N and 54N of the stability parameter as a function of the lag τ. Dashed line is the auto correlation for a red noise with a time scale of one and a half days.

to be due to variations in the meridional temperature gradient provided the variance of the critical shear is negligible compared to the variance of the shear. According to S, the root variance of the stability parameter is about 1.7 m s^{-1}, comparable to the mean value. Since the January mean shear in midlatitudes (50N) is about 10 m s^{-1}, if all of the variance of the stability parameter was due to changes in the shear the rms deviation of the shear, or temperature gradient, would be no more than 20% of the January mean value. Since the January mean critical shear in midlatitudes is also about 10 m s^{-1} an rms deviation of the static stability $\theta_1 - \theta_2$ of only 20% of the January mean value is enough for variations in the critical shear to be important. Such synoptic variations are certainly conceivable, but have never been studied. Although a modelling study of the temporal relationship between the flux and the stability parameter would be illuminating in its own right and was the original motivation for this study, we shall tentatively assume that the variance of the critical shear is negligible, so that variations in the stability parameter are due to variations in the temperature gradient. The above observations are then relevant to the problem at hand, and we proceed with the modelling.

III MODEL EQUATIONS

This study examines the temporal relationship on the synoptic time scale of two variables: the zonal and vertical mean meridional eddy sensible heat flux and temperature gradient. Consequently, at least two time dependent equations are required to model such a relationship.

Although the form of the model equation (III.8) relating changes in the temperature gradient to changes in the flux is not new (Stone, 1972; Lorenz, 1979), a complete derivation is presented here to 1) point out some of the limitations of the model, 2) provide first estimates of model parameters and 3) suggest which approximations should be modified to yield more realistic results.

Assuming only that the atmosphere is a hydrostatic ideal gas in a thin shell, its kinetic energy negligible compared with its static energy, the zonal mean equation of energy conservation may be written in the form (Hantel, 1976)

III.1 $$\frac{\partial}{\partial t}[c_pT + Lq] + \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}[vh]\cos\phi + \frac{\partial}{\partial p}[\omega h + gF_r] = 0$$

where L is the latent heat of vaporization, q is the specific humidity, $h = c_pT + gz + Lq$ is the moist static energy and F_r is the net downward radiative flux. Decomposing the fields into their zonal mean and eddy components yields

III.2 $$\frac{\partial}{\partial t}[c_pT + Lq] + \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}[v^*h^*]\cos\phi + \frac{\partial}{\partial p}[\omega^*h^*] + \frac{1}{a}[v]\frac{\partial}{\partial\phi}[h] + [\omega]\frac{\partial}{\partial p}[h] + g\frac{\partial}{\partial p}[F_r] = 0.$$

To be useful a model equation derived from this energy equation must include both the meridional eddy sensible heat flux and the time change of the sensible heat. All terms larger than these must be retained, while all terms much smaller may be neglected. The results of Hantel (1976) indicate the dominant energy balance in the atmosphere to be between radiative flux divergence and vertical eddy moist static energy flux convergence. The remaining difference in midlatitudes is balanced largely by meridional eddy moist static energy flux convergence which, according to Oort (1971), is dominated in winter by meridional eddy sensible heat flux convergence. Radiative flux divergence and vertical eddy moist static energy flux convergence must the be retained, while meridional eddy latent heat (in winter) and geopotential (all seasons) flux convergence may be neglected. In summer meridional eddy latent flux convergence may not be neglected, but its effects can be incorporated by assuming the latent heat flux to be proportional to the meridional eddy sensible heat flux.

Although advection of moist static energy by the meridional circulation is generally smaller in midlatitudes than meridional eddy heat flux convergence, it may not be negligible (Newell et al., 1974). This is especially so when considering temporal variations because, as a consequence of thermal wind balance, the meridional circulation in midlatitudes is forced by friction, diabatics and eddy fluxes of heat and momentun, all of which have significant temporal variance. Although the effect of the meridional circulation forced by diabatics and eddy heat flux can be incorporated within the model, the effects of forcing by friction and eddy momentum flux can not. Therefore, the effects of the meridional circulation are tentatively neglected, but are discussed further in section V.

Neglecting spherical effects, the energy equation may now be written

III.3 $$\frac{\partial}{\partial t}\left[T + \frac{L}{c_p} q\right] + \frac{1}{a}\frac{\partial}{\partial \phi}\left[v^* T^*\right] + \frac{1}{c_p}\frac{\partial}{\partial p}\left[\omega^* h^*\right] + \frac{g}{c_p}\frac{\partial}{\partial p}\left[F_r\right] = 0.$$

Integrating from the top of the surface layer to the top of the atmosphere yields

III.4 $$\frac{\partial}{\partial t}\left\langle T + \frac{L}{c_p} q\right\rangle + \frac{1}{a}\frac{\partial}{\partial \phi}\left\langle v^* T^*\right\rangle + \frac{1}{c_p p_o}\left[\omega^* h^*\right]_{p_o} + \frac{g}{p_o c_p}\left[F_r(p_o) - F_r(0)\right] = 0$$

where $\langle (\ ) \rangle \equiv \frac{1}{p_o}\int_o^{p_o} [(\ )]\, dp.$

Temporal changes in the vertical mean latent heat may be neglected provided the water precipitates immediately upon convection from the surface. This is never strictly true of course; a certain time lag is involved. If this lag is much shorter than the synoptic time scale, the lag is negligible. Since convective clouds typically develop over time scales of a few hours, such an approximation is valid provided precipitation is convective in origin and moist convection within the troposphere begins immediately after surface convection. According to the GFDL climate model (Miyakoda et al., 1969), approximately half of the precipitation is subgrid scale. Unfortunately, since precipitation is such an important part of the moist static energy balance, the grid scale precipitation and its associated synoptic time scale can not be neglected. Whereas moist convection of an arbitrary lag can be treated as a white noise (and will be included later in the modelling), the synoptic scale precipitation can not. Although this suggests we consider the dry static energy budget to deal with latent heat release explicitly, we shall continue with the moist static energy budget because it affords us an a priori estimate of the diabatic time scale. Therefore, we assume all precipitation is convective, and that a significant amount of it follows immediately after convection from the surface (we can relax this constraint if we integrate only from the top of the mixed layer rather than from the surface layer).

The remaining assumptions are all directed at relating the diabatics, i.e., the surface convective fluxes and the radiative flux divergence, to the vertical mean temperature. Newtonian cooling after Spiegel (1957) models the radiative flux divergence:

III.5 $$\frac{g}{c_p p_0}\left[F_r(p_0) - F_r(0) \right] = -\frac{1}{\tau_N}\langle T_r - T \rangle$$

where τ_N is the radiative cooling time and T_r is the radiative equilibrium temperature consistent with the observed surface temperature. Appropriate values for τ_N will be discussed later in this section. Simple drag laws model the convective fluxes of sensible and latent heat at the top of the surface layer:

III.6
$$\frac{1}{p_o}\left[\omega^* T^*\right]_{p_o} = -\frac{1}{\tau_c}\left[T_g - T_o\right]$$
$$\frac{1}{p_o}\left[\omega^* q^*\right]_{p_o} = -\frac{1}{\tau_c}\left[q_g - q_o\right]$$

where $\tau_c = \frac{H}{c_D |u_o|}$ is the convective time scale, $H = \frac{RT}{g}$ is the scale height, c_D is the drag coefficient and $|u_o|$ is a typical wind speed at the top of the surface layer. The subscripts g and o denote values at the ground and at the top of the surface layer, respectively. Note that ensemble averaging longer than the time scale of the eddies within the surface layer but much shorter than the synoptic time scale is implicit in the mixing length expressions. The appropriate specific humidity at the ground is just the saturation specific humidity at the ground temperature $q_s(T_g)$. Assuming the relative humidity r at the top of the surface layer is constant and neglecting spatial variations in the radiative cooling and convective time scales, the energy equation differentiated with respect to $y = a\phi$ becomes

III.7
$$\frac{\partial}{\partial t}\frac{\partial}{\partial y}\langle T\rangle = -\frac{\partial^2}{\partial y^2}\langle v^* T^*\rangle + \frac{1}{\tau_c}\left[\frac{\partial}{\partial y}T_g - \frac{\partial}{\partial y}T_o\right] + \frac{1}{\tau_c}\frac{L}{c_p}\left[\frac{dq^s}{dT}(T_g)\frac{\partial}{\partial y}T_g - r\frac{dq^s}{dT}(T_o)\frac{\partial}{\partial y}T_o\right] + \frac{1}{\tau_N}\frac{\partial}{\partial y}\langle T_r - T\rangle.$$

The only remaining problem is to relate the temperature at the top of the surface layer to the vertical mean temperature. As a first approximation, we shall assume perturbations about the time mean temperature to be independent of height, so that the required relation is straightforward. In fact, temperature perturbations can be quite shallow. Since the model results are quite sensitive to this condition, the possibility of shallow perturbations is discussed in more detail in section V.

With these approximations the energy equation reduces to

III.8 $$\frac{\partial}{\partial t}\frac{\partial}{\partial y}\langle T\rangle = -\frac{\partial^2}{\partial y^2}\langle v^* T^*\rangle + \frac{1}{\tau_d}\frac{\partial}{\partial y}\langle T_e - T\rangle$$

where $\tau_d = \dfrac{\tau_c}{1 + r\frac{L}{c_p}\frac{dq_s}{dT}(T_0) + \tau_c/\tau_N}$ is the diabatic time scale. $\langle T_e\rangle$ is a sort of radiative-convective equilibrium temperature consistent with the observed ground temoperature and assumed constant (note here that we consider only unforced time scales, i.e., short enough so that the seasonal cycle in $\langle T_e\rangle$ is not resolved). Such an assumption can be justified for the synoptic time scale over the ocean as follows. The heat capacity of an infinite column of dry air is equivalent to that of a two meter column of water. T_e over the ocean can be considered constant if the temperature of this layer is constant for the time scale of interest, i.e., one day. This is true if the time scale in which this layer mixes with a much deeper layer is much less than one day. According to Ekman layer theory, the dynamical time scale of the entire mixed layer ($\sim$100 meters) in midlatitudes is about one day, so that the mixing time of a ten meter layer (much deeper than two meters) is one tenth of a day, clearly negligible. T_e may then be considered constant over the ocean for the synoptic time scale. Because land generally has a much smaller heat capacity than oceans for the synoptic time scale, T_e cannot be considered constant over land. However, the oceans comprise the greater part of the zonal mean surface, so that T_e in the zonal mean is approximately constant. The actual value of $\langle T_e \rangle$ is not important in the time dependent calculations; it need only be consistent with the time mean flux and temperature.

With only one additional assumption the observed negative correlation at zero lag between the eddy sensible heat flux and the temperature gradient may be derived from the energy equation, in the same manner as Lorenz (1979). This assumption is that the flux is well correlated in time with the laplacian of the flux. This is not altogether obvious because variations of the flux occur in general on a spectrum of spatial scales. If the temporal variance of the flux as a function of meridional scale decreased only slowly with decreasing meridional scale, the variance of the laplacian of the flux, which accentuates smaller scales, would peak at a scale well removed from that of the flux. One would then expect significant correlations between the flux and the flux laplacian only if fluxes of widely different scales were well correlated in time, an unlikely proposition. In fact, when we correlate the flux with the finite difference equivalent of the flux laplacian using the data of S, we find significant negative correlations at all latitudes (figure 4). One can argue that since the finite difference equivalent of the flux laplacian at a given latitude is heavily weighted by the flux at that latitude, if the data were in the noise level one should expect significant, non-physical correlations. However, calculation of the standard deviation of the finite difference equivalent of the flux laplacian from the same data (figure 5) indicates that the variance is sufficient to yield physically significant correlations. One must then come to the conclusion that, unless fluxes of widely different meridional scales are significantly correlated (a truly remarkable result), the variance of the flux is confined to a narrow band of meridional scales. We will henceforth assume that the variance of the flux is dominated by one meridional scale , so that the flux correlates perfectly with the laplacian of the flux.

The perturbation flux laplacian then equals a negative constant times the perturbation flux. The resulting perturbation energy equation becomes

III.9 $$\frac{\partial}{\partial t}\frac{\partial}{\partial y}\langle T\rangle' = D\langle v^*T^*\rangle' - \frac{1}{\tau_d}\frac{\partial}{\partial y}\langle T\rangle'$$

where primes denote deviations from the time mean. Note that the constant of proportionality between the perturbation flux and the perturbation flux laplacian is not necessarily the same as that corresponding to the time mean. Whereas the constant of proportionality for the time mean corresponds to the planetary scale (Stone, 1978), the proper

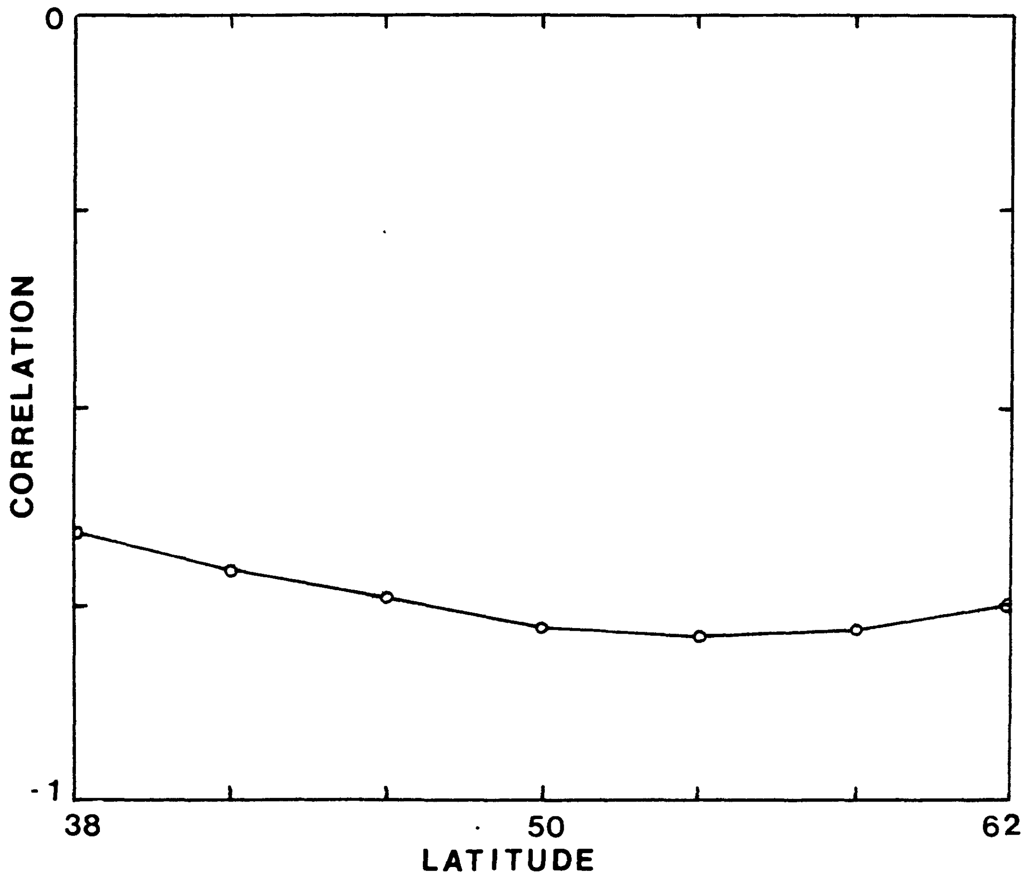

Figure 4. Correlation averaged over three Januaries between the flux $\mathcal{F}(\phi, t)$ and the finite difference equivalent of the flux laplacian $\mathcal{F}(\phi - \Delta\phi, t) - 2\mathcal{F}(\phi, t) + \mathcal{F}(\phi + \Delta\phi, t)$ as a function of the latitude ϕ, for $\Delta\phi = 8^{\circ}$ latitude. The 95% confidence level correlation (two-sided) is 0.22.

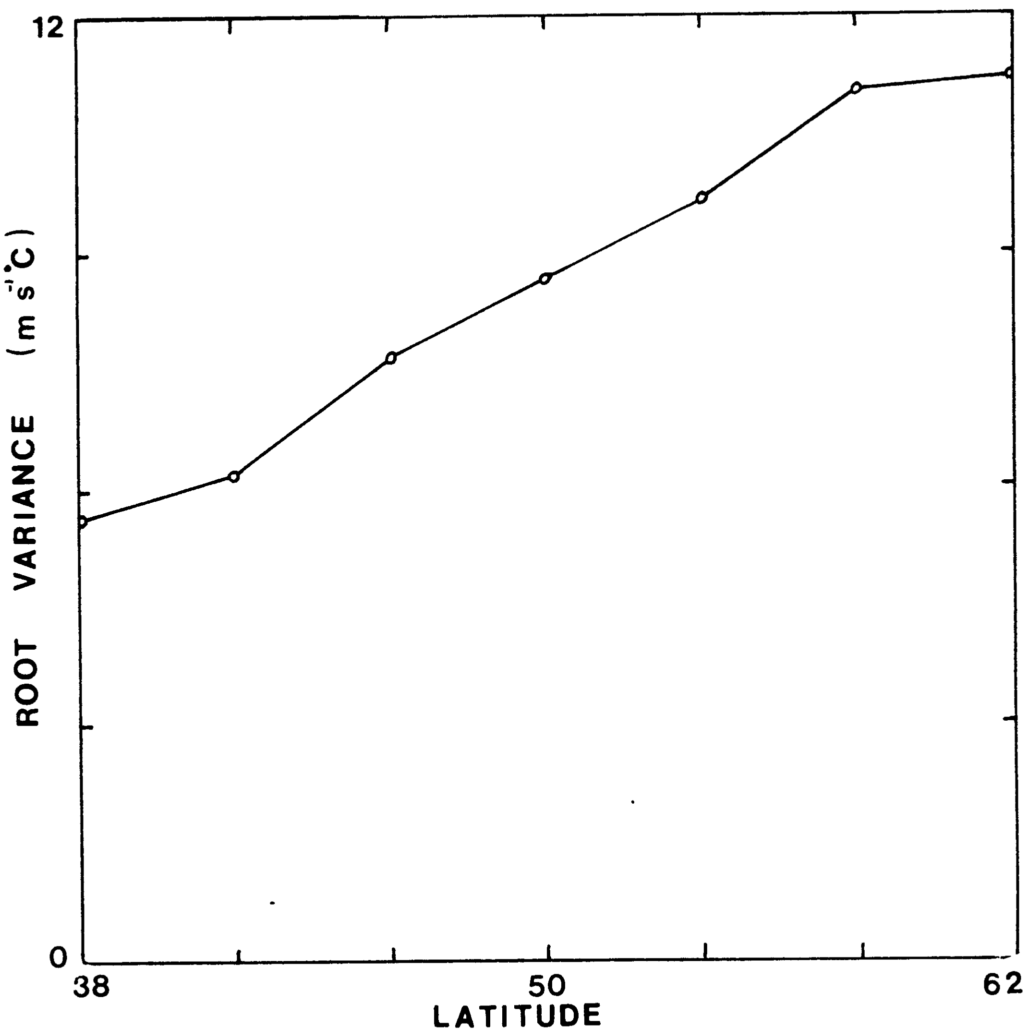

Figure 5. Standard deviation averaged over three Januaries of the finite difference equivalent of the flux laplacian as a function of latitude. Units are m s^{-1} °C, assuming a tropospheric depth of (generously) 1000 mb. The noise level is approximately four m s^{-1} °C.

value for the constant for perturbations is not well known. Although theoretical studies have considered the meridional scale of these perturbations (Stone, 1974; Simmons, 1974; Pedlosky, 1975a), the value for D used in our modelling will be empirically determined.

Multiplying equation III.9 by the perturbation temperature gradient and averaging in time yields

$$\text{III.10} \qquad D\,\overline{\langle v^{*}T^{*}\rangle'\,\frac{\partial}{\partial y}\langle T\rangle'} = \overline{\left(\frac{\partial}{\partial y}\langle T\rangle'\right)^2} > 0$$

where overbars denote the time mean. The heat flux and the magnitude of the temperature gradient must be negatively correlated, independent of whatever equation governs the behavior of the flux. Note that the negative correlation depends on the presence of diabatics.

According to S, the flux can be modelled as a first order Markov process. Although the flux is best fit in winter by a second order process, a first order process, or red noise, is also a good fit. Figure 2 shows that the auto correlation function for the flux in winter resembles that of a red noise with a time scale of about one day.

Theoretical justification for the red noise hypothesis derives from Pedlosky (1979), in which he considers finite amplitude dynamics of a weakly unstable baroclinic wave in a continuous atmosphere on a β-plane, with both Ekman friction at the surface and internal damping. Pedlosky derives an equation for the wave amplitude of the form

$$\text{III.11} \qquad \frac{dA}{dt} = \nu\left(A - A^3/A_e^2\right)$$

where ν is the growth rate from linear theory and A_e is the equilibrium amplitude. Since the flux is second order in the wave amplitude, the flux is governed by

III.12 $$\frac{dF}{dt} = 2\nu\,(F - F^2/F_e)$$

where $F \equiv \langle v^* T^* \rangle$. Linearization about the mean $F = F_e + F'$ yields

III.13 $$\frac{dF'}{dt} = -2\nu F'.$$

According to S, the rms deviation of the flux in winter is about 35% of the mean value. One expects this value to increase in the summer, when stationary eddy heat flux is relatively small. Linearization of the flux about the mean is then justified in winter but perhaps not in summer. The time scale of the flux can be identified with one half the inverse growth rate which, according to Eady's model, yields a value in midlatitudes in winter of about a day and a half. This is in approximate agreement with the observed time scale of about one day. One expects the time scale in summer to be larger because of the weaker temperature gradient.

Adding white noise forcing to both the equation for the flux and the equation for the temperature gradient, the model equations may be written

III.14 $$\frac{dF'}{dt} = -\frac{1}{\tau_b} F' + E_f$$
$$\frac{dG'}{dt} = -DF' - \frac{1}{\tau_d} G' + E_g$$

where $G \equiv -\frac{\partial}{\partial y}\langle T \rangle'$, $\tau_b = \frac{1}{2\nu}$ is the time scale of the flux, and E_f and E_g are white noises. The source of the white noise forcing of the temperature gradient was discussed earlier; the white noise forcing of the flux might be due to resonant triad interactions of baroclinic waves (Loesch, 1974).

Introducing the non-dimensional quantities

$$f' = F'/F_e$$

$$g' = G'/\bar{G}$$

III.15 $$t' = t/\tau_b$$

$$\epsilon_f = \tau_b E_f / F_e$$

$$\epsilon_g = \tau_b E_g / \bar{G}$$

yields the non-dimensional model equations

III.16
$$\frac{df'}{dt'} = -f' + \epsilon_f$$

$$\frac{dg'}{dt'} = -\delta f' - \gamma g' + \epsilon_g$$

where

III.17
$$\gamma \equiv \tau_b / \tau_d$$

$$\delta \equiv \tau_b F_e D / \bar{G}$$

Although the interpretation of γ is obvious, that of δ is not. In the case of the mixing length parameterization of the flux from Eady's model (Stone, 1972), δ is equivalent to an order one constant times the ratio squared of the deformation radius to the meridional scale of the flux. The scale of the flux may be found from

$$L_f = \pi D^{-1/2}$$

where L_f is the half-wavelength of the variance weighted scale of the flux. The value of D is found empirically by computing the ratio of the rms deviation of the flux laplacian to the rms deviation of the flux. For the data of S, D is found to be $1.8\times10^{-12}\ m^{-2}$. The corresponding half-wavelength is about 2300 km.

To calculate the diabatic time scale we must estimate the convective and radiative time scales. Hicks (1972) provides an average value for c_D of about 1.4×10^{-3} but finds values ranging from 4×10^{-4} to 4×10^{-3}, depending on the stability of the surface layer. Typical 10 meter wind speeds are 5 m s^{-1}, so that a first estimate of the convective time scale is about ten days. The radiative time scale varies widely, depending upon the height and vertical scale of the temperature perturbations, and the nature of the surface below. Prinn (1977) calculated values ranging from one half day for shallow perturbations immediately above a conducting surface to one month for deep perturbations well removed from the surface. Since we have assumed temperature perturbations to be independent of height we shall take the largest value, one month, as a first estimate of the radiative time scale. For a relative humidity of 90% the value of $r\frac{L}{c_p}\frac{dq_s}{dT}$ is 0.7 at 0°C, appropriate for the January mean. The corresponding diabatic time scale is then 5 days.

Although we could use the results from Eady's model to find δ, there really is no point in doing so since the value of the meridional scale is empirically determined. Therefore we use the observed values for January of

$$F = 20 \text{ m s}^{-1}\,{}^{\circ}\text{C}$$

$$G = 4\times10^{-6}\ {}^{\circ}\text{C m}^{-1}$$

$$\tau_b = 1 \text{ day}$$

to yield

$$\gamma = 0.2$$

$$\delta = 0.8.$$

With these first estimates of the model parameters we shall proceed to solve for the correlation functions.

IV BASIC MODEL SOLUTION

Although one could just as easily solve the finite difference equivalent of the model equations, we present the solution of the continuous system. This is reasonable provided the time scale of the white noise forcing is small but finite. The model equations are again (dropping primes)

IV.1 $$\frac{df}{dt} = -f + \epsilon_f$$

IV.2 $$\frac{dg}{dt} = -\delta f - \gamma g + \epsilon_g$$

The auto covariance function for the flux is found by multiplying equation IV.1 evaluated at time t + τ ($\tau > 0$) by the flux at time t and averaging in time, yielding

$$\frac{\partial}{\partial \tau} \Phi(f, f, \tau) = - \Phi(f, f, \tau)$$

or

IV.3 $$\Phi(f, f, \tau) = \Phi(f, f, 0)\, e^{-\tau}$$

where

$$\Phi(x, y, \tau) \equiv \int x(t)\, y(t+\tau)\, dt$$

To find the cross correlation, multiply equation IV.2 evaluated at time t + τ (any τ) by the flux at time t and average, yielding

$$\frac{\partial}{\partial \tau} \Phi(f, g, \tau) = -\delta \Phi(f, f, \tau) - \gamma \Phi(f, g, \tau)$$

Substituting $\Phi(f, f, \tau)$ from equation IV.3 into the general solution

$$\Phi(f, g, \tau) = e^{-\gamma\tau}\left\{ c - \delta \int^{\tau} e^{\gamma\tau'} \Phi(f, f, \tau')\, d\tau' \right\}$$

yields

$$\Phi(f,g,\tau)=\begin{cases} \left.\begin{array}{ll} c_1 e^{-\gamma\tau} - \frac{\delta}{\gamma-1}\Phi(f,f,0)\,e^{-\tau} & \gamma\neq 1 \\ c_2 e^{-\tau} - \delta\,\Phi(f,f,0)\,\tau e^{-\tau} & \gamma=1 \end{array}\right\} & \tau\geq 0 \\ -\frac{\delta}{\gamma+1}\Phi(f,f,0)\,e^{\tau} & \tau\leq 0 \end{cases}$$

Matching solutions at $\tau = 0$ yields

$$\Phi(f,g,\tau)=\begin{cases} \left.\begin{array}{ll} \frac{\delta}{\gamma-1}\Phi(f,f,0)\left(\frac{2}{1+\gamma}e^{-\gamma\tau} - e^{-\tau}\right) & \gamma\neq 1 \\ -\delta\,\Phi(f,f,0)\left(\tau + 1/2\right)e^{-\tau} & \gamma=1 \end{array}\right\} & \tau\geq 0 \\ -\frac{\delta}{\gamma+1}\Phi(f,f,0)\,e^{\tau} & \tau\leq 0 \end{cases} \qquad \text{IV.4}$$

The auto covariance function for the temperature gradient is found by multiplying equation IV.2 evaluated at time $t + \tau$ ($\tau > 0$) by the temperature gradient at time t and averaging, yielding

$$\frac{\partial}{\partial\tau}\Phi(g,g,\tau) = -\delta\,\Phi(g,f,\tau) - \gamma\,\Phi(g,g,\tau)$$

Substituting $\Phi(g,f,\tau)$ from equation IV.4 into the general solution

$$\Phi(g,g,\tau)= e^{-\gamma\tau}\left\{ c - \delta\int^{\tau} e^{\gamma\tau'}\Phi(g,f,\tau')\,d\tau'\right\}$$

yields

$$\Phi(g,g,\tau) = \begin{cases} \frac{\delta^2}{(\gamma+1)(\gamma-1)}\Phi(f,f,0)\left(e^{-\tau}-e^{-\gamma\tau}\right)+\Phi(g,g,0)\,e^{-\gamma\tau} & \gamma\neq 1 \\ \left\{\frac{\delta^2}{2}\Phi(f,f,0)\,\tau + \Phi(g,g,0)\right\}e^{-\tau} & \gamma=1 \end{cases} \qquad \text{IV.5}$$

To close the problem we need an expression relating the variance of the temperature gradient to that of the flux.

In the special case $\epsilon_g = 0$ the variance of the temperature gradient may be related to the variance of the flux by multiplying equation IV.2 by the temperature gradient and averaging, yielding

IV.6 $$\Phi(g,g,0) = -\frac{\delta}{\gamma}\Phi(f,g,0) = \frac{\delta^2}{\gamma(\gamma+1)}\Phi(f,f,0)$$

The auto and cross correlation functions then become

IV.7 $$\rho(f,f,\tau) = e^{-\tau} \qquad \tau \geq 0$$

IV.8 $$\rho(g,g,\tau) = \left\{ \begin{array}{ll} \dfrac{\gamma e^{-\tau} - e^{-\gamma\tau}}{\gamma - 1} & \gamma \neq 1 \\ (1+\tau)e^{-\tau} & \gamma = 1 \end{array} \right\} \quad \tau \geq 0$$

IV.9 $$\rho(f,g,\tau) = \begin{cases} \left. \begin{array}{ll} -\left(\dfrac{\gamma}{\gamma+1}\right)^{1/2} \dfrac{2e^{-\gamma\tau} - (\gamma+1)e^{-\tau}}{1-\gamma} & \gamma \neq 1 \\ -\dfrac{\sqrt{2}}{2}(1+2\tau)e^{-\tau} & \gamma = 1 \end{array} \right\} & \tau \geq 0 \\ -\left(\dfrac{\gamma}{\gamma+1}\right)^{1/2} e^{\tau} & \tau \leq 0 \end{cases}$$

where $\rho(x,y,\tau) \equiv \dfrac{\Phi(x,y,\tau)}{[\Phi(x,x,0)\Phi(y,y,0)]^{1/2}}$

Note that these solutions depend only on γ.

V BASIC MODEL RESULTS

Solutions of the auto correlation function for the temperature gradient and the cross correlation function for the flux and the temperature gradient are shown in figures 6 and 7, respectively, for different values of γ. Since the non-dimensional time has been scaled by the time scale of the flux, which is about one day, values of the non-dimensional lag may be thought of in terms of days. Comparison with the observed correlation functions reveals significant differences, given the first estimate of $\gamma \sim 0.2$. The modelled temperature gradient is much more persistent than is observed, while the modelled lag of maximum negative correlation between the flux and the temperature gradient is much later than observed. Either some other process must be included in the model or larger values of γ must be justified.

We therefore include the white noise forcing of the temperature gradient in the model. This adds additional variance $\Phi'(g,g,0)$ to the temperature gradient which depends on both the magnitude and time scale of the forcing. The total variance of the temperature gradient is then

V.1 $$\Phi(g,g,0) = \frac{\delta^2}{\gamma(\gamma+1)}\,\Phi(f,f,0)\,(1+\zeta)$$

where $\zeta \equiv \frac{\gamma(\gamma+1)}{\delta^2}\frac{\Phi'(g,g,0)}{\Phi(f,f,0)}$ is the ratio of the variance of the temperature gradient due to direct white noise forcing to the variance due to forcing by the flux. Although the flux auto covariance function and the cross covariance function

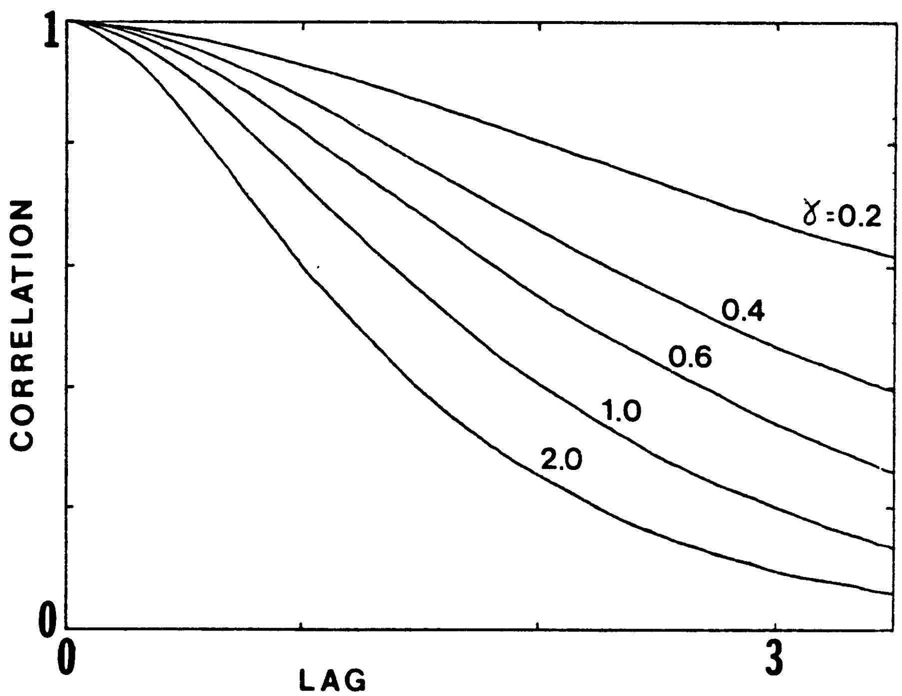

Figure 6. Model auto correlation for the temperature gradient as a function of the non-dimensional lag for $\zeta = 0$ and different values of γ .

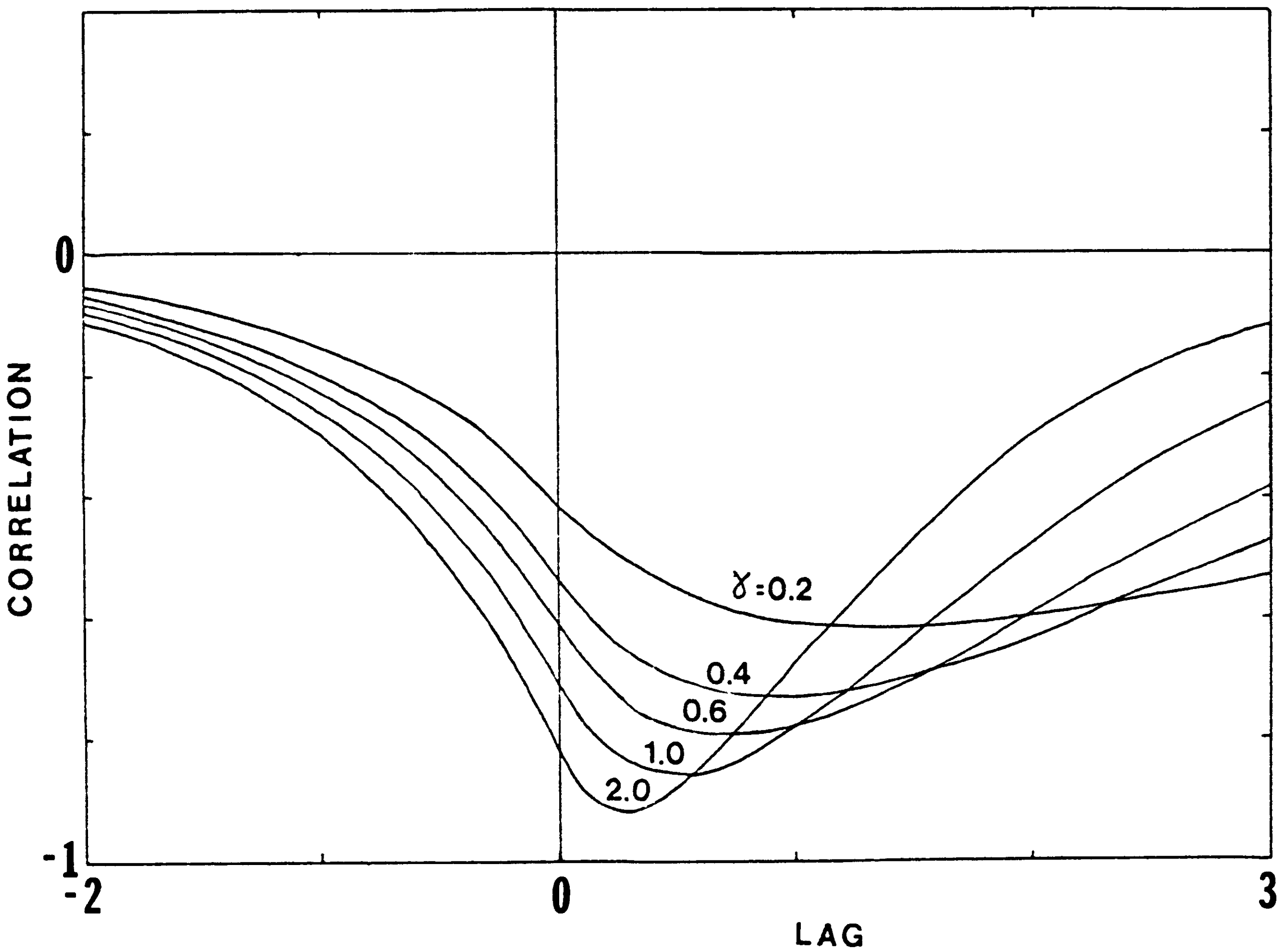

Figure 7. Model cross correlation for the flux and the temperature gradient as a function of the non-dimensional lag for $\zeta = 0$ and different values of γ .

are independent of $\Phi(g,g,0)$ and hence ζ, the structure of the auto covariance function and the magnitude of the cross covariance function will be altered by white noise forcing of the temperature gradient. As ζ increases, the magnitude of the cross correlation function decreases, and auto corrlation function for the temperature gradient becomes more like that of a red noise. In the limit $\zeta \to \infty$ the cross correlation function becomes zero while the auto correlation function for the temperature gradient is that of a red noise with time scale τ_d. Since the observed auto correlation function for the temperature gradient is similar to that of red noise with a time scale of one and a half days, our estimate of the diabatic time scale is clearly too large. The diabatic time scale must be at least as small as one and a half days and is probably smaller to allow for reasonable values of ζ.

Although other processes may still be important we shall for the moment assume they are not. Since the structure of the cross correlation function is fully determined by γ, the observed lag of the strongest negative correlation can be used to estimate γ. From equation IV.9 we find that this lag is given by

$$\text{V.2} \qquad \tau_0 = \begin{cases} \frac{1}{1-\gamma} \ln\left(\frac{1+\gamma}{2\gamma}\right) & \gamma \neq 1 \\ 1/2 & \gamma = 1 \end{cases}$$

This function is shown in figure 8. Since the observed lag is one half day, we estimate that γ is about one. Because

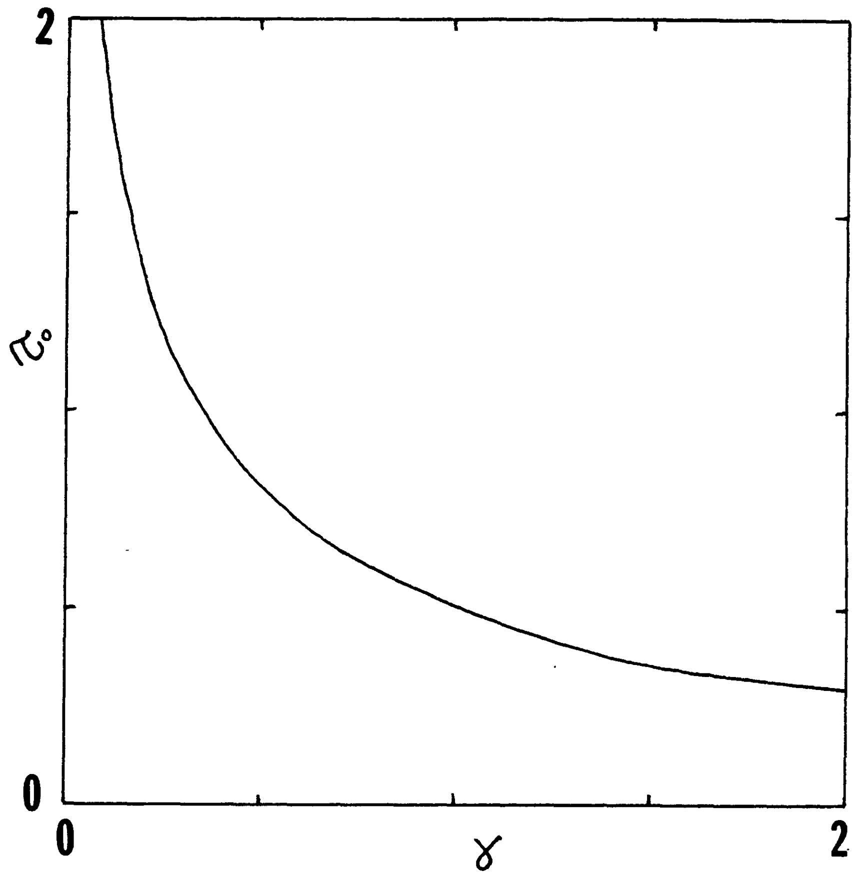

Figure 8. Non-dimensional lag of strongest correlation between the flux and the temperature gradient as a function of γ .

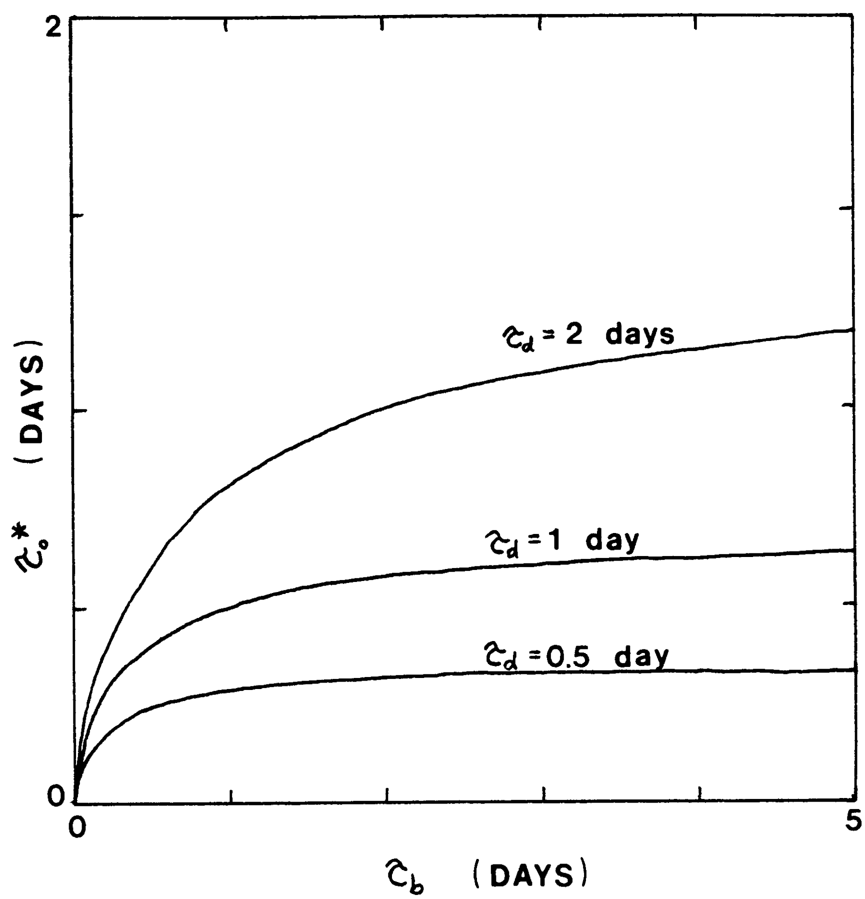

Figure 9. Dimensional lag of strongest correlation as a function of the time scale of the flux, for different values of the diabatic time scale. Units are in days.

there is some flexibility concerning the proper choice for the time scale of the flux we should check the effects of different τ_b on the dimensional lag of strongest correlation $\tau_o^* = \tau_b \tau_o$. Figure 9 shows τ_o^* as a function of τ_b for different values of the diabatic time scale. For flux time scales greater than or order the diabatic time scale ($\gamma \geq O(1)$) the lag of strongest correlation is nearly independent of the flux time scale and is given approximately by one half the diabatic time scale.

We can then be fairly confident that the diabatic time scale for perturbations in the atmosphere in midlatitudes in winter is about one day. This is considerably shorter than our first estimate of the diabatic time scale (five days). We consider two possible explanations. First, variations in the critical shear may not be negligible compared to variations in the shear. By definition we expect the diabatic time scale for the critical shear to be at least as small as one half that appropriate for the entire atmospheric depth. Furthermore, moist convection within the troposphere can be an extremely efficient source of diabatics. Although we have no way of a priori estimating the diabatic time scale associated with moist convection, a value of one day is certainly reasonable. The alternate explanation is that perturbations in the meridional temperature gradient associated with the flux are quite shallow. For perturbations which decay exponentially with a scale height h the convective time scale is reduced by the factor H/(h+H) from the time scale assuming vertical homogeneity. Observations of the zonal mean transient eddy heat flux from Oort & Rasmussen (1971) show that the heat flux in midlatitudes in winter decays exponentially from 850 mb with a scale height somewhat less than H. Therefore, a convective time scale of less than five days seems reasonable. In addition, the radiative time scale for shallow perturbations is also shorter. Prinn (1977) found a radiative time scale of about a day and a half for perturbations well removed from the surface with a vertical wavelength of 3 km (admittedly shallow). The corresponding diabatic time scale becomes about one day.

With the value of γ of about one well established and justified, we return to our discussion of ζ. Figure 10 shows the model auto correlation function of the temperature gradient for γ equal to one and different values of ζ. Comparison with the observed auto correlation function indicates that ζ of order unity is appropriate. We can independently estimate ζ by comparing the magnitudes of the observed vs. the modelled cross correlation functions. Since the modelled cross correlation function for $\gamma = 1.0$ is about twice the magnitude of the observed function, we expect ζ to be about three, in reasonable agreement with the previous estimate of ζ.

Such a value seems large for random forcing of the temperature gradient, suggesting that we should consider variations in the critical shear as well as the shear.

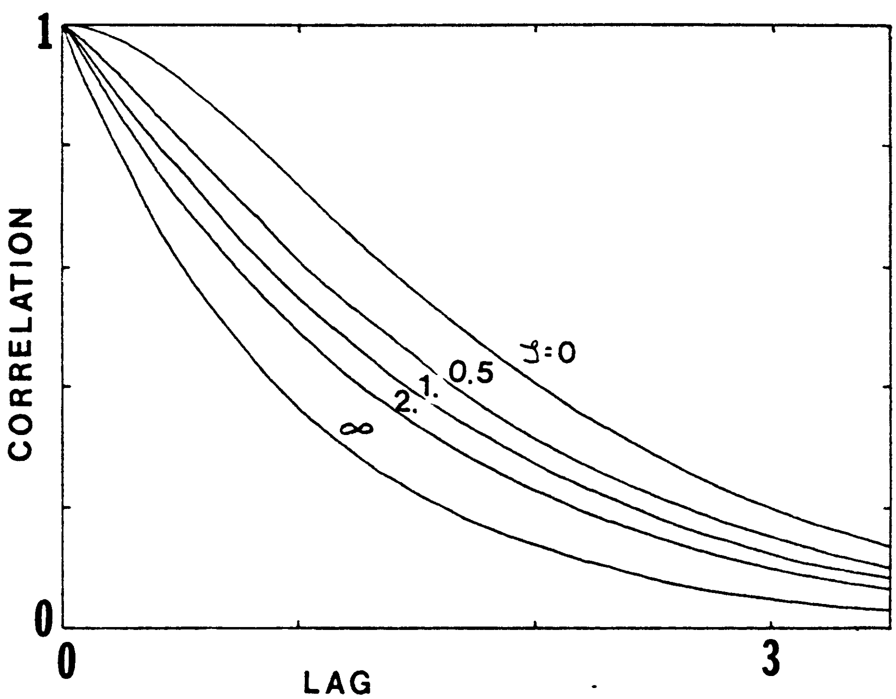

Figure 10. Model auto correlation for the temperature gradient as a function of the non-dimensional lag for $\gamma = 1$ and different values of ζ .

Random moist convection is often well organized in the vertical but poorly organized over the meridional scales of interest. However, it need not be well organized. Consider a random change $\Delta\theta_1$ in the temperature of the upper troposphere at a given latitude, corresponding to a change in the vertical mean temperature of $\Delta\theta_1/2$. The maximum scale L over which the change in the shear will be comparable to the change in the critical shear is given by

$$L_m = \frac{gf(z_1 - z_2)}{2\beta RT}$$

which in midlatitudes is about 2400 km. Since this is comparable to the observed scale of the flux we conclude that random moist convection affects the shear as well as the critical shear for the meridional scales of interest.

The model has reproduced the observed auto and cross correlation functions for winter fairly well with reasonable parameter values. We can with some confidence make some predictions for summer as well. We expect the diabatic time scale to be smaller in summer than in winter because latent heat convection is more efficient at the higher temperatures associated with summer. The lag of strongest correlation should then be smaller than it is in winter. Because of the weaker mean temperature gradient in summer we also expect the time scale of the flux to be larger than in winter, so that γ should be much larger than in winter (perhaps 2). This alone suggests the cross correlations should be stronger in summer. However, we also expect ζ to be larger, so that the correlations may in fact be weaker.

Another test of the model is how well the observed relative variances of the flux and temperature gradient agree with the relation expressed in equation V.1. If we assume all of the variance of the temperature gradient is due to forcing by the flux ($\zeta = 0$) then, using the empirically established values for δ and γ of 0.8 and 1.0, respectively, the root variance of the non-dimensional temperature gradient should be 0.6 that of the non-dimensional flux. The observed non-dimensional root variance of the flux was found to be 0.35 while, assuming the variance of the stability parameter is due only to variations in the temperature gradient, the observed non-dimensional root variance of the temperature gradient was found to be less than 0.2, or 0.6 that of the flux. The agreement is only valid for $\zeta = 0$; for $\zeta = 1$ the model predicts a non-dimensional root variance of the temperature gradient of 0.8 that of the flux. For $\zeta = 3$ the model predicts a value of 1.2. The model apparently overestimates the effectiveness of the flux in forcing the temperature gradient. The obvious solution to the problem is that our empirically determined value for D is too large. The noise level, although smaller than the variance of the flux laplacian, is not negligible. If the noise level is negligible compared with the variance of the flux the proper value for D could be as small as one half our empirically determined value.

Another possible explanation is that we must include the effects of the meridional circulation forced by the heat flux. This could significantly decrease the effectiveness of the flux in forcing the temperature gradient. We can roughly estimate this effect as follows. The zonally averaged zonal momentum and thermodynamic equations may be approximated by

V.3 $$\frac{\partial}{\partial t}[u] + \frac{\partial}{\partial y}[u^*v^*] - f_o[v] - [F_x] = 0$$

V.4 $$\frac{\partial}{\partial t}[T] + \frac{\partial}{\partial y}[v^*T^*] - \sigma[\omega] - [Q] = 0$$

where F_x is friction, Q is diabatics and σ(p) $= \left(\frac{R}{c_p p} - \frac{\partial}{\partial p}\right)[T]$ is a measure of the static stability. Substituting equations V.3 and V.4 into the thermal wind equation

V.5 $$\frac{\partial}{\partial p}[u] = \frac{R}{f_o p}\frac{\partial}{\partial y}[T]$$

yields

V.6 $$f_o\frac{\partial}{\partial p}[v] - \frac{R\sigma}{f_o p}\frac{\partial}{\partial y}[\omega] = \frac{\partial^2}{\partial p \partial y}[u^*v^*] - \frac{\partial}{\partial p}[F_x] - \frac{R}{f_o p}\frac{\partial^2}{\partial y^2}[v^*T^*] + \frac{R}{f_o p}\frac{\partial}{\partial y}[Q]$$

Continuity is identically satisfied if the meridional streamfunction ψ is defined by

$$[v] = \frac{\partial\psi}{\partial p} \quad ; \quad [\omega] = -\frac{\partial\psi}{\partial y}$$

Equation V.6 becomes

V.7 $$\frac{\partial^2}{\partial p^2}\psi + \frac{R\sigma}{f_o p}\frac{\partial^2}{\partial y^2}\psi = \frac{1}{f_o}\frac{\partial^2}{\partial p \partial y}[u^*v^*] - \frac{1}{f_o}\frac{\partial}{\partial p}[F_x] - \frac{R}{f_o^2 p}\frac{\partial^2}{\partial y^2}[v^*T^*] + \frac{R}{f_o^2 p}\frac{\partial}{\partial y}[Q].$$

The effects of forcing by individual terms may be considered separately. For an idealized heat flux

$$[v^* T^*] = F_o \sin\frac{m\pi p}{p_o} \sin 2\ell\phi$$

the equation for the meridional streamfunction is

$$\psi_{pp} + \frac{R\sigma}{a^2 f_o^2 p}\psi_{\phi\phi} = \frac{4\ell^2 R}{a^2 f_o^2 p} F_o \sin\frac{m\pi p}{p_o} \sin 2\ell\phi$$

Assuming σ is constant and $p = \frac{1}{2}p_o$ where it appears as a coefficient yields

$$\psi_{pp} + \frac{2R\sigma_o}{a^2 f_o^2 p_o}\psi_{\phi\phi} = \frac{8\ell^2 R}{a^2 f_o^2 p} F_o \sin\frac{m\pi p}{p_o} \sin 2\ell\phi$$

The particular solution

$$\psi = \psi_o \sin\frac{m\pi p}{p_o} \sin 2\ell\phi$$

satisfying the boundary conditions

$$\psi_p = 0 \quad \phi = 0, \pi/2 \; ; \quad \psi_\phi = 0 \quad p = 0, p_o$$

upon substitution yields

$$\psi_o = \frac{-\frac{C\ell^2}{\sigma_o} F_o}{m^2 + C\ell^2}$$

where

$$C = \frac{8R\sigma_o p_o}{a^2\pi^2 f_o^2} = \frac{8}{a^2\pi^2}\frac{N^2 p p_o}{\rho^2 g^2 f_o^2} \sim \frac{16 N^2 H^2}{a^2\pi^2 f_o^2} = 4\left(\frac{L_d}{L_p}\right)^2 \sim 0.04$$

$N = \left(\frac{g}{\theta}\frac{\partial\theta}{\partial z}\right)^{1/2}$ is the Brunt-Vaisala frequency, $L_d = \frac{NH}{f_o}$ is the deformation radius and $L_p = \frac{a\pi}{2}$ is the pole to equator distance. The ratio of the resulting adiabatic cooling $\sigma_o[\omega]$ to the eddy heat flux convergence is

$$\frac{\sigma_o\psi_o}{F_o} = \frac{-C\ell^2}{m^2 + C\ell^2}$$

The effect of the meridional circulation forced by the heat flux can be incorporated in the model by simply decreasing the value of δ:

$$\delta = \delta_0 \left(1 - \frac{Cl^2}{m^2 + Cl^2}\right) = \frac{m^2}{m^2 + Cl^2}\,\delta_0$$

The observed half-wavelength of the flux corresponds to a value of l between four and five. Since we have found the flux to be shallow, a value for m of two or three is probably more accurate than one. The effectiveness of the flux should then be reduced by no more than 25%. Although the effectiveness of the diabatics is also decreased by the meridional circulation, this effect has already been accounted for because the final value of γ was determined by the model results. With this adjustment the model predicts a non-dimensional root variance of the temperature gradient of 0.45 that of the flux for $\Im$ = 0, 0.6 for $\Im$ = 1, and 0.9 for $\Im$ = 3. Since the best estimate of $\Im$ is three, both explanations are required to resolve the difference. The proper value for δ should then be 0.4.

VI FEEDBACK MODEL

How might we further improve the model? The fact that, according to S, the flux is more accurately modelled as a second order Markov process suggests we should include feedback of the temperature gradient on the flux. In this section we include such feedback, so that the equation for the flux may be written

$$\frac{df'}{dt'} = -f' + \eta g' + \epsilon_f \tag{VI.1}$$

where η is a non-dimensional parameter presumed to be positive. One possible interpretation of η arises if we let the equilibrium flux F_e depend on the instantaneous rather than time mean value of the temperature gradient. In particular, if the equilibrium flux as parameterized by mixing length arguments is

$$F_e = a\, G^{\eta}$$

we have, upon linearization of equation III.12,

$$\frac{dF'}{dt} = -2\nu F' + \eta \bar{F}_e G' + E_f$$

which upon non-dimensionalization yields equation VI.1. Clearly such an instantaneous dependence is not valid for mixing length parameterizations which, as noted above, assume ensemble averaging over intervals comparable to the time scale of the flux. Although this leaves the interpretation of η as the power of dependence of the equilibrium flux on the temperature gradient in question, it does not prohibit the possibility of some form of dependence of the equilibrium flux on the instantaneous value of the temperature gradient. Therefore, although the appropriate value for η is not a priori known, we shall examine solutions when η is non-zero.

To simplify the problem we shall neglect white noise forcing of the temperature gradient. The model equations in non-dimensional form are then

$$\frac{df}{dt} = -f + \eta g + \epsilon_f \tag{VI.2}$$

$$\frac{dg}{dt} = -\delta f - \gamma g. \tag{VI.3}$$

Equations relating the covariance functions are

$$\left.\begin{aligned}
&\frac{\partial}{\partial\tau}\Phi(f,g,\tau) = -\delta\Phi(f,f,\tau) - \gamma\Phi(f,g,\tau) && \text{VI.4}\\
&\frac{\partial}{\partial\tau}\Phi(f,f,\tau) = -\Phi(f,f,\tau) + \eta\Phi(f,g,\tau) && \text{VI.5}\\
&\frac{\partial}{\partial\tau}\Phi(g,g,\tau) = -\delta\Phi(g,f,\tau) - \gamma\Phi(g,g,\tau) && \text{VI.6}\\
&\frac{\partial}{\partial\tau}\Phi(g,f,\tau) = -\Phi(g,f,\tau) + \eta\Phi(g,g,\tau) && \text{VI.7}
\end{aligned}\right\}\tau \geq 0$$

Equations VI.4 - VI.7 form two sets of coupled equations for the covariance functions. Solutions of the form $\Phi = e^{p\tau}$ yield characteristic roots $p = -\frac{1+\gamma}{2} \pm \sqrt{\left(\frac{1-\gamma}{2}\right)^2 - \delta\eta}$. These roots may be pure real and distinct, pure real and identical, or complex conjugates. Each case is considered separately in the appendix. The solutions are found to depend only on γ and the product $\kappa \equiv \delta\eta$, so that a parameter study is feasible.

We have already considered the case in which κ equals zero and γ is order unity, having established that γ equal to one is appropriate for the atmosphere. We now consider the case in which γ equals one and κ is order unity. Figure 11 shows solutions of the cross correlation function for γ = 1.0 and different values of κ . Feedback dramatically alters the form of the cross correlation function. This is not surprising since the characteristic roots for γ = 1.0 are double roots for κ = 0 and complex roots for positive κ . Feedback does not significantly alter the non-dimensional lag of strongest negative correlation, suggesting that our choice of γ is appropriate for whatever feedback might operate in the atmosphere. Comparison with the observed cross correlation function suggests a choice of κ near one would yield a more realistic modelled cross correlation function.

Figure 12 shows the modelled auto correlation function for the flux for γ = 1.0 and different values of κ . For a value of κ near unity the model solution is similar in form to the observed auto correlation function. However, the apparent time scale of the flux is much shorter when feedback is included (by apparent time scale we mean the lag at which the auto correlation reaches a value of 1/e). This suggests that our estimate of the time scale of the flux from the observed auto correlation function is too short.

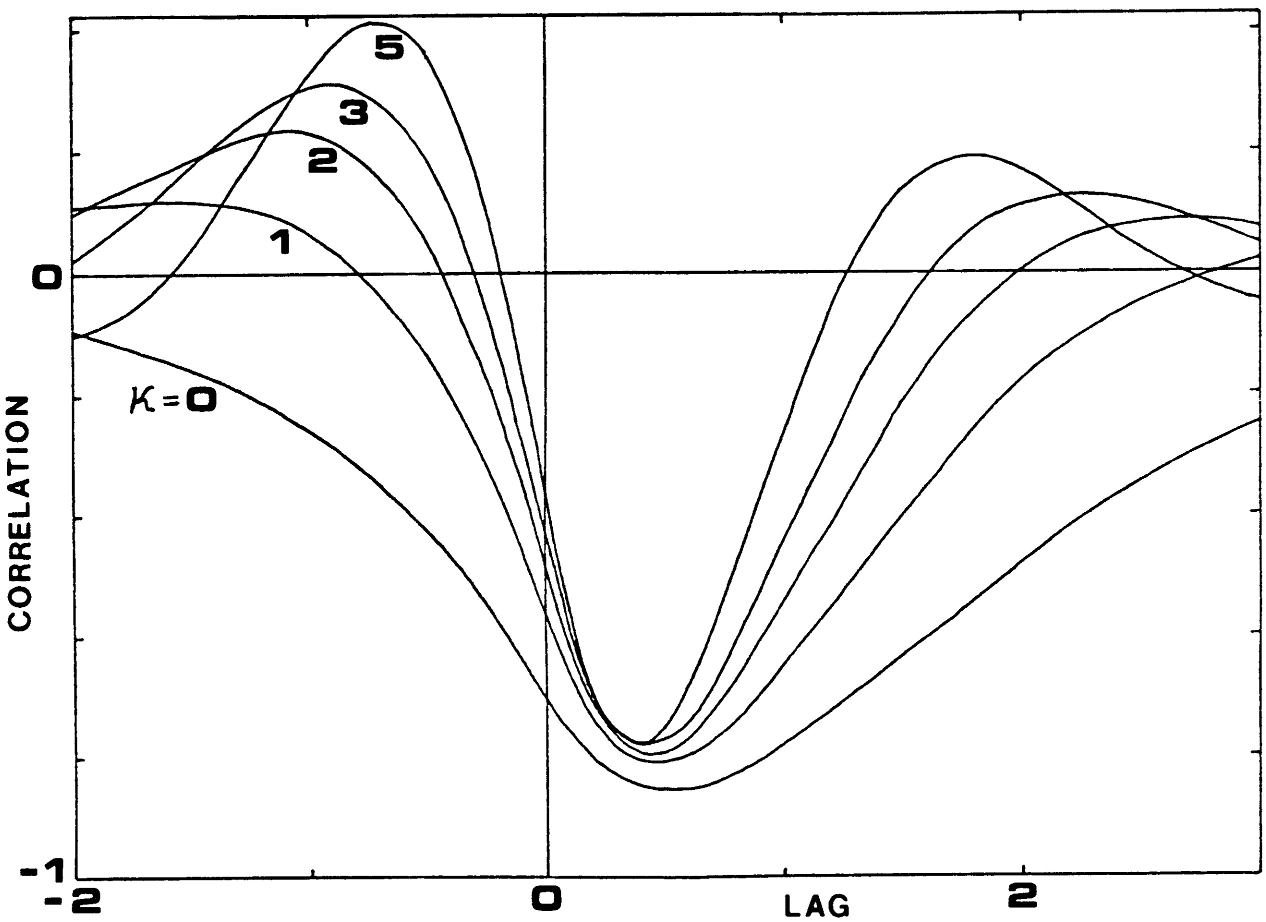

Figure 11. Model cross correlation for the flux and the temperature gradient as a function of the non-dimensional lag for ζ = 0, γ = 1 and different values of κ .

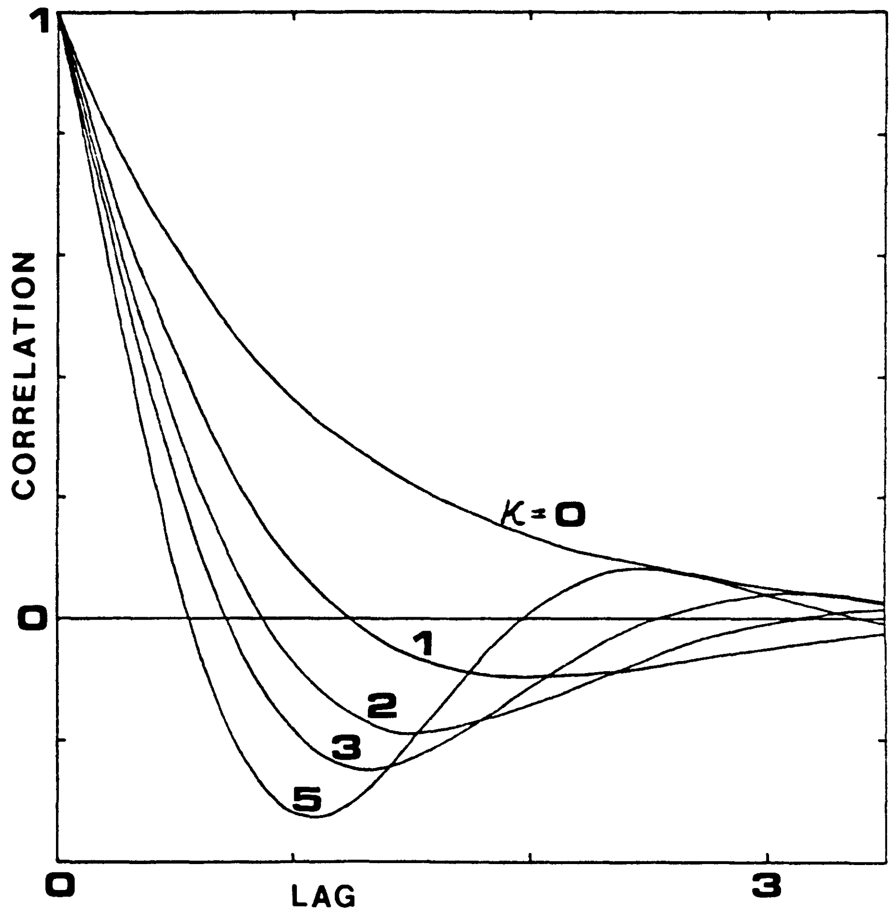

Figure 12. Model auto correlation for the flux as a function of the non-dimensional lag for ζ = 0, γ = 1 and different values of κ .

Since feedback does not significantly affect the non-dimensional lag of strongest negative correlation between the flux and the temperature gradient we expect our estimate of the diabatic time scale of one day to remain valid. The appropriate value for γ should then be larger than one. This then requires a larger value for κ to get complex characteristic roots. This suggests that a wide range of values for κ and the flux time scale should be considered for comparison with observations. Rather than carry out such a lengthy procedure we can go directly to the observations for the time scale for the flux. For the flux auto correlation function expressed in the form

$$\rho(f, f, \tau^*) = e^{-b\tau^*} \frac{\sin(\omega\tau^* + \psi)}{\sin\psi}$$

where τ^* is the dimensional lag (in days), S found values for

b = 1.316 day

ω = 1.189 day

ψ = 0.865 radians

by fitting to the observed flux auto correlation at lags of $1/2$ and 1 day. Matching the form of the model solution with the above form yields

$$\frac{1 + \tau_b/\tau_d}{2\tau_b} = b$$

or

$$\tau_b = \frac{1}{2b - 1/\tau_d}$$

For the above value for b and τ_d = 1 day we have τ_b = 0.61 days, actually less than our first estimate of one day. Thus, whereas the apparent time scale of the modelled flux underestimates the true time scale, the apparent time scale of the observed flux overestimates the true time scale. To resolve this problem we consider the phase ψ .

According to the model solution the phase may be expressed as

$$\psi = \tan^{-1}\left[- \frac{\gamma(1+\gamma)+\kappa}{\gamma(1-\gamma)+\kappa} \quad \frac{2p_i}{1+\gamma} \right]$$

where p $= \left[\kappa - \left(\frac{1-\gamma}{2}\right)^2\right]^{1/2}$. For γ = 0.61 and κ = 1.0 the model yields ψ = 2.045 radians. This is a significantly different phase from the observed phase of 0.865 radians, and explains why the modelled and observed apparent time scales of the flux relative to the true time scale are so different (note that, unless γ is much greater than one and κ is small, the modelled phase must be in the second or fourth quadrant, whereas the observed phase is in the middle of the first or third quadrant). We can not claim to have improved the model by adding feedback unless we can model the phase correctly. We can do this by including white noise forcing of the temperature gradient.

Consider the phase ϕ of the auto correlation function for the temperature gradient. Figure 13 shows this auto correlation function for γ = 1.0 and different values of κ . The phase ϕ is given by

$$\phi = \tan^{-1}\left[\frac{2p_i}{1+\gamma} \right]$$

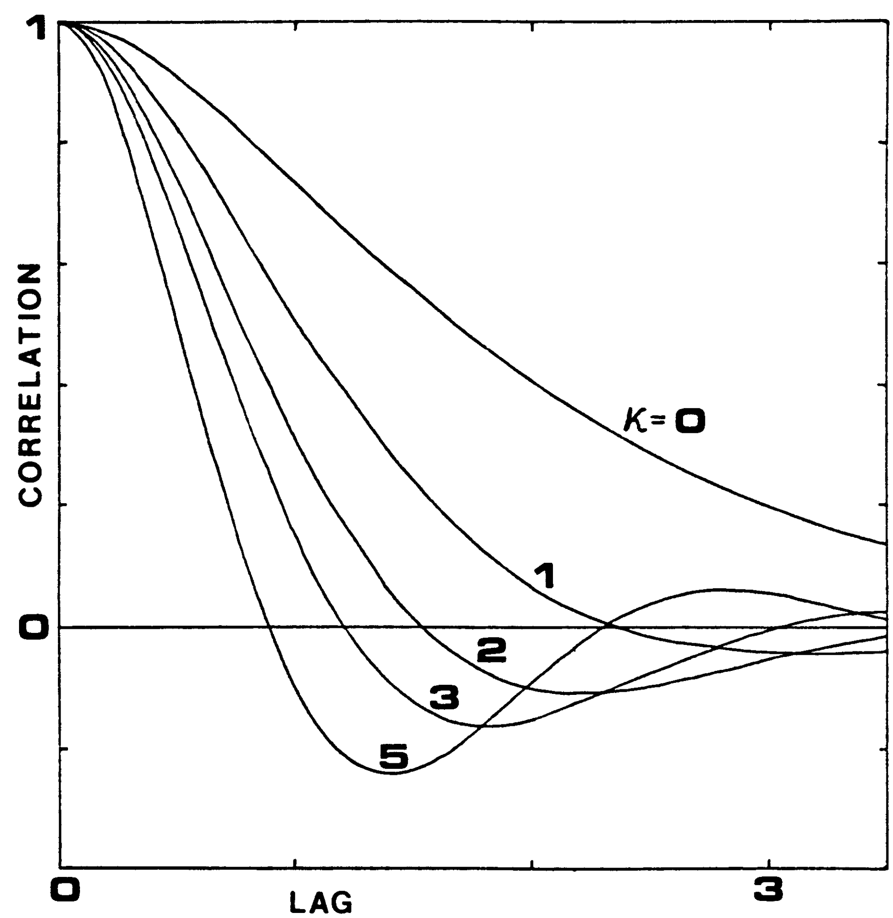

Figure 13. Model auto correlation for the temperature gradient as a function of the non-dimensional lag for $\zeta = 0$, $\gamma = 1$ and different values of κ .

which for $\kappa = 1.0$ and $\gamma = 0.61$ yields $\phi = 0.884$ radians. If the white noise forcing of the flux is very weak compared with the white noise forcing of the temperature gradient, we expect the auto correlation function for the flux to resemble that of the temperature gradient shown in figure 13. Thus, including white noise forcing of the temperature gradient will not only yield a more realistic auto correlation function for the temperature gradient, but a more realistic auto correlation function for the flux as well, complete with phase.

Although by tuning the three model parameters γ , κ and ζ one may find model solutions consistent with observations, we do not conclude that feedback of the temperature gradient on the flux operates in the atmosphere. The finite amplitude calculations of baroclinic stability in the absence of diabatics (Pedlosky, 1979) also yield second order equations for the behavior of the flux.

Pfeffer et al. (1980) calculated cross correlation functions for the flux and the temperature gradient in thermally driven rotating annulus experiments. The modelled cross correlation function (figure 11) for strong feedback ($\kappa \sim 5$) closely resembles the observed cross correlation function in the geostrophic turbulence regime. Although feedback may be stronger in the annulus experiments than in the atmosphere, it is more likely that the internal damping is too weak in the annulus experiments.

VII CONCLUSIONS

One can derive many of the model results from first principles. The fact that the flux behaves approximately as a red noise derives from Pedlosky (1979). The time scale for the flux can be identified with one half the inverse growth rate from baroclinic stability theory. Corrections to the first estimate for the diabatic time scale due to the vertical scale of perturbations follows from the vertical scale of the most unstable wave in baroclinic stability theory. One could roughly estimate the amount of white noise forcing of the temperature gradient from the typical scales of moist convection. The meridional scale of flux perturbations has been considered theoretically by Stone (1974), Simmons (1974) and Pedlosky(1975a). The only missing element is a model of the white noise forcing of the flux. Pedlosky (1975b) considered the amplitudes of interacting triads in a baroclinic current but found that the results depend on the initial conditions. Further work is clearly needed on this difficult problem.

The diabatic time scale which the modelling suggests is valid for perturbations due the eddy heat flux is surprisingly short. Since the diabatic time scale chosen in dynamical models is typically ten days or more, our results suggest that a much shorter diabatic time scale is appropriate. Since the diabatic time scale is comparable to the advective time scale, the importance of properly modelling diabatics in numerical models of atmospheric motions is readily apparent. We stress this because there is room for improvement in modelling diabatics at NMC.

Finally, although for the meridional scales of interest we have shown that random forcing of the meridional temperature gradient is as important as random forcing of the static stability, we cannot neglect the variance of the critical shear. One then has reason to question why the modelling has been so successful. We suspect that an equation very similar to that governing the behavior of the temperature gradient also governs the behavior of the static stability. According to baroclinic stability theory vertical eddy flux of sensible heat associated with synoptic disturbances must coincide with meridional eddy sensible heat flux. In addition, diabatics should act to restore the static stability, as well as the temperature gradient, to its respective equilibrium value. However, the diabatic time scale in the vertical may not correspond to the diabatic time scale in the horizontal. Therefore, as long as the variance of the critical shear cannot be neglected, our determination of the diabatic time scale may not strictly apply for meridional perturbations. Correlation functions involving the temperature gradient and the static stability individually should be examined to separate meridional from vertical perturbations.

APPENDIX

The eight unknowns associated with the four general solutions require eight constraints. Four constraints result from the requirement that equations VI.4 - VI.7 hold for all lags greater than or equal to zero, which is satisfied if the equations hold for zero lag. Another results from the requirement that the cross covariance functions $\Phi(f,g,\tau)$ and $\Phi(g,f,\tau)$ match at zero lag. Two more constraints follow from the requirement that

$$\frac{\partial}{\partial\tau}\Phi(g,g,\tau) = \delta\Phi(f,g,\tau) + \gamma\Phi(g,g,\tau)$$

(which is independent of equations VI.4 - VI.7 and is valid only when white noise forcing of the temperature gradient does not exist) hold for all positive lags. The final constraint is that the flux auto covariance function at zero lag match the flux variance. All constants are then expressed in terms of the flux variance.

A Real, Distinct Roots

For pure real and distinct characteristic roots $p = p^{\pm}$ the general solutions are

$$\Phi(f,g,\tau) = A_1 e^{p^+\tau} + B_1 e^{p^-\tau}$$

$$\Phi(g,f,\tau) = A_2 e^{p^+\tau} + B_2 e^{p^-\tau}$$

$$\Phi(f,f,\tau) = A_3 e^{p^+\tau} + B_3 e^{p^-\tau}$$

$$\Phi(g,g,\tau) = A_4 e^{p^+\tau} + B_4 e^{p^-\tau}.$$

The eight constraints are

$$(p^+ + \gamma)A_1 + (p^- + \gamma)B_1 + \delta(A_3 + B_3) = 0$$

$$(p^+ + 1)A_3 + (p^- + 1)B_3 - m(A_1 + B_1) = 0$$

$$(p^+ + \gamma)A_4 + (p^- + \gamma)B_4 + \delta(A_2 + B_2) = 0$$

$$(p^+ + 1)A_2 + (p^- + 1)B_2 - m(A_4 + B_4) = 0$$

$$A_1 + B_1 = A_2 + B_2$$

$$(p^+ - \gamma)A_4 = \delta A_1$$

$$(p^- - \gamma)B_4 = \delta A_4$$

$$A_3 + B_3 = \Phi(f,f,0).$$

The constants in terms of $\Phi(f,f,0)$ are

$$A_1 = -\frac{p^-(p^+ - \gamma)\delta}{(p^+ - p^-)(p^+p^- + \gamma^2)}\,\Phi(f,f,0)$$

$$B_1 = \frac{p^+(p^- - \gamma)\delta}{(p^+ - p^-)(p^+p^- + \gamma^2)}\,\Phi(f,f,0)$$

$$A_2 = \delta\left[\frac{\gamma(p^- + 1) + \delta m}{(p^+ - p^-)(p^+p^- + \gamma^2)}\right]\Phi(f,f,0)$$

$$B_2 = -\delta\left[\frac{\gamma(p^+ + 1) + \delta m}{(p^+ - p^-)(p^+p^- + \gamma^2)}\right]\Phi(f,f,0)$$

$$A_3 = -\left[\frac{(p^- + 1)(p^+p^- + \gamma^2) + \delta m\gamma}{(p^+ - p^-)(p^+p^- + \gamma^2)}\right]\Phi(f,f,0)$$

$$B_3 = \left[\frac{(p^+ + 1)(p^+p^- + \gamma^2) + \delta m\gamma}{(p^+ - p^-)(p^+p^- + \gamma^2)}\right]\Phi(f,f,0)$$

$$A_4 = -\frac{p^- \delta^2}{(p^+ - p^-)(p^+ p^- + \gamma^2)} \, \Phi(f,f,0)$$

$$B_4 = \frac{p^+ \delta^2}{(p^+ - p^-)(p^+ p^- + \gamma^2)} \, \Phi(f,f,0).$$

The correlation functions become

$$\rho(f,g,\tau) = \frac{1}{(p^+ - p^-)(p^+ p^- + \gamma^2)^{1/2}} \left\{ -p^-(p^+ - \gamma) e^{p^+ \tau} + p^+(p^- - \gamma) e^{p^- \tau} \right\}$$

$$\rho(g,f,\tau) = \frac{1}{(p^+ - p^-)(p^+ p^- + \gamma^2)^{1/2}} \Big\{ \left[\gamma (p^- + 1) + \delta m \right] e^{p^+ \tau}$$

$$- \left[\gamma (p^+ + 1) + \delta m \right] e^{p^- \tau} \Big\}$$

$$\rho(f,f,\tau) = -\left[\frac{p^- + 1}{p^+ - p^-} + \frac{\delta m \gamma}{(p^+ - p^-)(p^+ p^- + \gamma^2)} \right] e^{p^+ \tau}$$

$$+ \left[\frac{p^+ + 1}{p^+ - p^-} + \frac{\delta m \gamma}{(p^+ - p^-)(p^+ p^- + \gamma^2)} \right] e^{p^- \tau}.$$

$$\rho(g,g,\tau) = -\frac{p^-}{p^+ - p^-} e^{p^+ \tau} + \frac{p^+}{p^+ - p^-} e^{p^- \tau}.$$

B Double Roots

For real double characteristic roots the general solutions are

$$\Phi(f,g,\tau) = A_1 e^{p\tau} + B_1 \tau e^{p\tau}$$

$$\Phi(g,f,\tau) = A_2 e^{p\tau} + B_2 \tau e^{p\tau}$$

$$\Phi(f,f,\tau) = A_3 e^{p\tau} + B_3 \tau e^{p\tau}$$

$$\Phi(g,g,\tau) = A_4 e^{p\tau} + B_4 \tau e^{p\tau}.$$

The eight constraints are

$$(p+\gamma) A_1 + B_1 + \delta A_3 = 0$$

$$(p+1) A_3 + B_3 - m A_1 = 0$$

$$(p+\gamma) A_4 + B_4 + \delta A_2 = 0$$

$$(p+1) A_2 + B_2 - m A_4 = 0$$

$$A_1 = A_2$$

$$(p-\gamma) A_4 + B_4 = \delta A_1$$

$$(p-\gamma) B_4 = \delta B_1$$

$$A_3 = \Phi(f, f, 0).$$

The constants are

$$A_1 = -\frac{\gamma\delta}{p^2+\gamma^2} \Phi(f,f,0)$$

$$B_1 = -\frac{p(p-\gamma)\delta}{p^2+\gamma^2} \Phi(f,f,0)$$

$$A_2 = -\frac{\gamma\delta}{p^2+\gamma^2} \Phi(f,f,0)$$

$$B_2 = [(p+1)\gamma + \delta m] \frac{\delta}{p^2+\gamma^2} \Phi(f,f,0)$$

$$A_3 = \Phi(f,f,0)$$

$$B_3 = -\left(p+1 + \frac{\delta m \gamma}{p^2+\gamma^2}\right) \Phi(f,f,0)$$

$$A_4 = \frac{\delta^2}{p^2+\gamma^2} \Phi(f,f,0)$$

$$B_4 = -\frac{p\delta^2}{p^2+\gamma^2} \Phi(f,f,0).$$

The correlation functions are

$$\rho(f,g,\tau) = -\frac{\gamma}{(p^2+\gamma^2)^{1/2}} e^{p\tau} - \frac{p(p-\gamma)}{(p^2+\gamma^2)^{1/2}} \tau e^{p\tau}$$

$$\rho(g,f,\tau) = -\frac{\gamma}{(p^2+\gamma^2)^{1/2}} e^{p\tau} + \left[\frac{(p+1)\gamma+\delta m}{(p^2+\gamma^2)^{1/2}}\right] \tau e^{p\tau}$$

$$\rho(f,f,\tau) = e^{p\tau} - \left(p+1+\frac{\delta m\gamma}{p^2+\gamma^2}\right)\tau e^{p\tau}$$

$$\rho(g,g,\tau) = e^{p\tau} - p\tau e^{p\tau}.$$

C Complex Roots

For complex characteristic roots $p = p_r \pm i p_i$ the general solutions are

$$\Phi(f,g,\tau) = e^{p_r\tau}(A_1 \cos p_i\tau + B_1 \sin p_i\tau)$$

$$\Phi(g,f,\tau) = e^{p_r\tau}(A_2 \cos p_i\tau + B_2 \sin p_i\tau)$$

$$\Phi(f,f,\tau) = e^{p_r\tau}(A_3 \cos p_i\tau + B_3 \sin p_i\tau)$$

$$\Phi(g,g,\tau) = e^{p_r\tau}(A_4 \cos p_i\tau + B_4 \sin p_i\tau).$$

The eight constraints are

$$(p_r+\gamma)A_1 + p_i B_1 + \delta A_3 = 0$$

$$(p_r+1)A_3 + p_i B_3 - m A_1 = 0$$

$$(p_r+\gamma)A_4 + p_i B_4 + \delta A_2 = 0$$

$$(p_r+1)A_2 + p_i B_2 - \mu A_4 = 0$$

$$A_1 = A_2$$

$$(\gamma - p_r)A_4 - p_i B_4 + \delta A_1 = 0$$

$$(\gamma - p_r)B_4 + p_i A_4 + \delta B_1 = 0$$

$$A_3 = \Phi(f,f,0).$$

The constants become

$$A_1 = -\frac{\gamma\delta}{\gamma(1+\gamma)+\delta\mu}\,\Phi(f,f,0)$$

$$B_1 = -\left[\frac{\gamma^2+3\gamma+2\delta\mu}{\gamma(1+\gamma)+\delta\mu}\right]\frac{\delta}{2p_i}\,\Phi(f,f,0)$$

$$A_2 = -\frac{\gamma\delta}{\gamma(1+\gamma)+\delta\mu}\,\Phi(f,f,0)$$

$$B_2 = \left[\frac{\gamma(1-\gamma)+2\delta\mu}{\gamma(1+\gamma)+\delta\mu}\right]\frac{\delta}{2p_i}\,\Phi(f,f,0)$$

$$A_3 = \Phi(f,f,0)$$

$$B_3 = -\left[\frac{\gamma(1-\gamma)+\delta\mu}{\gamma(1+\gamma)+\delta\mu}\right]\frac{1+\gamma}{2p_i}\,\Phi(f,f,0)$$

$$A_4 = \frac{\delta^2}{\gamma(1+\gamma)+\delta\mu}\,\Phi(f,f,0)$$

$$B_4 = \left[\frac{\delta^2}{\gamma(1+\gamma)+\delta\mu}\right]\frac{1+\gamma}{2p_i}\,\Phi(f,f,0).$$

The correlation functions are then

$$\rho(f,g,\tau) = \frac{e^{p_r\tau}}{[\gamma(1+\gamma)+\delta\mu]^{1/2}}\left[-\gamma\cos p_i\tau - \frac{\gamma^2+3\gamma+2\delta\mu}{2p_i}\sin p_i\tau\right]$$

$$\rho(g,f,\tau) = \frac{e^{p_r\tau}}{[\gamma(1+\gamma)+\delta M]^{1/2}}\left[-\gamma\cos p_i\tau + \frac{\gamma(1-\gamma)+2\delta M}{2p_i}\sin p_i\tau\right]$$

$$\rho(f,f,\tau) = e^{p_r\tau}\left[\cos p_i\tau - \frac{\gamma(1-\gamma)+\delta M}{\gamma(1+\gamma)+\delta M}\,\frac{1+\gamma}{2p_i}\sin p_i\tau\right]$$

$$\rho(g,g,\tau) = e^{p_r\tau}\left[\cos p_i\tau + \frac{1+\gamma}{2p_i}\sin p_i\tau\right].$$

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