Particle Accelerators
1975, Vol. 6, pp. 193–200

AN ENVELOPE EQUATION IN THE DISPERSIVE PLANE OF A BENDING MAGNET†

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(*Received October 22, 1974*)

A second-order differential equation for the beam envelope, originally designed to calculate envelope changes inside non-dispersive magnets, has been adapted to permit calculation of the envelope of an off-central momentum beam in the dispersive plane inside any bending magnet.

1 INTRODUCTION

There is now available to the particle beam designer a variety of extremely powerful computational codes for following the motion of an individual particle or beam of particles through a magnetic channel. The most complete and undoubtedly the most generally employed code is the TRANSPORT code devised by K. L. Brown,[1] which contains dispersive terms, second-order coupling, and provision for other refinements in channel design, such as fringing field corrections, tilt and positioning errors, and finite momentum spread. The TRANSPORT code can be used only with a relatively large computer because of storage requirements. Another code which has (in its present form) only first-order terms is the TRACE code, the creation of K. R. Crandall,[2] which has fewer options than TRANSPORT but can be operated on a smaller computer; such a code can be extremely useful to the experimenter when adjusting the electrical parameters of a channel whose mechanical design is already fixed.

In its original form, TRACE could treat only the motions of particles of a central momentum, p_0, which precluded using it for a first-order determination of the effects of momentum spread, $p \neq p_0$, in an aperture of finite size. The interpretation given in Section 2, below, was developed to overcome this deficiency and arose rather incidentally from a previously published observation on the value of the use of a dispersive spatial offset, Δ, in a graphical solution for particle motions in a bending magnet.[3] The development to this point will treat only individual particle motion, although the particle can have a variable momentum. In Section 3, a few cases of particle motion in several types of dispersive magnets are discussed. But with this method, the concept of offset, Δ, was readily added to the TRACE code, which calculates, among other parameters, the *beam envelope* (the total lateral space occupied by a beam of finite size) a result which can be most helpful to the channel designer or experimenter. However, there is a method still simpler than the TRACE code for obtaining only the envelope of a beam of finite lateral size. This is a second-order differential equation which can be solved either analytically or by digital or analog computer. In its original form, the differential equation is unfortunately incapable of treating particles of momentum other than p_0. Its origin is described in Section 4, both because of its subsequent application and also because the development is not well known in spite of the value of the equation. Finally, in Section 5, the nondispersive envelope equation is modified to permit inclusion of off-momentum particles so that effects of momentum spread in a beam channel of finite aperture can be simply determined. (The author intends to devote a future paper to the details of momentum acceptance of a channel based upon solutions to the modified dispersion-envelope equation and truncation of the emittance ellipse by finite apertures.)

It is helpful to compare the three computational methods and the amount of computer memory

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† This work done under the auspices of the U.S. Atomic Energy Commission.

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needed in the following way: for calculations in the plane of dispersion, TRANSPORT requires at least a (3×3) matrix, TRACE a (2×2) matrix but the envelope differential equation can work with only one parameter. The envelope equation conveys of course much less information than the other two codes but is entirely adequate for a great variety of preliminary design purposes, and is moreover readily adaptable to small computers.

2 DYNAMICAL EQUATIONS FOR A PARTICLE

In nondispersive magnetic beam transport systems it is customary to employ equations of the form:

$$\xi'' \pm k^2 \xi = 0 \tag{1}$$

to describe the behavior of the particle in either direction, ξ, *perpendiculat to the direction of motion*, s. In Eq. (1), the prime indicates differentiation with respect to s. Clearly, if the sign is positive in Eq. (1), it represents focusing, while if it is negative, defocusing is implied. Using ξ_0, ξ'_0 for the initial displacement and slope, one obtains a (2×2) matrix solution of the type:

$$\begin{pmatrix} \xi \\ \xi' \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi'_0 \end{pmatrix} \tag{2}$$

for particle motion in the ξ direction, when $C(s)$ and $S(s)$ are respectively the "cosine-like" and "sine-like" solutions—trigonometric if focusing and hyperbolic if defocusing. The determinant of the transformation matrix of Eq. (2) is necessarily unity, in conformance with the Sturm-deLiouville equation. When a bending magnet of any type must be treated, a third row and column is usually added to the matrix, incorporating two dispersive terms usually called $D(s)$ and $D'(s)$.[1] By this method, it is possible to follow particles of a slightly different momentum, p, from the selected mean momentum, p_0.

To demonstrate that simplification can be achieved in a bending magnet, it is convenient to start with the equations of motion of a charged particle in a cylindrical coordinate system. Let the directions ρ and z be perpendicular to the direction of motion, ϕ, and let ρ, ϕ, z be right-handed coordinates:

$$\begin{aligned} m(\ddot{\rho} - \rho\dot{\phi}^2) &= \frac{q}{c}\rho\dot{\phi}B_z \\ m\ddot{z} &= \frac{q}{c}\rho\dot{\phi}B_\rho \end{aligned} \tag{3}$$

where $m = \gamma m_0$ is the relativistic mass and q the charge. Gaussian units are used. For a particle of positive charge, let the z-component of the magnetic field be directed in the negative z-direction, and also require that the ρ-component of B be *symmetrical* about the $z = 0$ plane. It is assumed that the z-component of B has a ρ-dependence given by the equation,

$$B_z = -B_0\left(\frac{\rho}{\rho_0}\right)^{-n} = -B(\rho) \tag{4}$$

in the near vicinity of a mean radius, ρ_0, to be defined subsequently. In general n may be either a positive or negative number.

The momentum is generally given by the relation:

$$p = m\rho\dot{\phi} = m\dot{s} \tag{5}$$

But it is conventional to eliminate time dependence, represented by the dot, in favor of dependence on position along the (mean) path of the particle, s, by the transformation $\dot{s}(\mathrm{d}/\mathrm{d}s) = \mathrm{d}/\mathrm{d}t$. The radial equation then takes the form:

$$\rho'' + \frac{1}{p\rho}\left[\frac{q}{c}\rho B(\rho) - p\right] = 0 \tag{6}$$

in which the prime represents differentiation with respect to s.

A bending magnet in a beam channel is always devised to bend a particle of "central" momentum p_0 at a radius ρ_0.† This condition is met when the second derivative, ρ'', is equal to zero and the expression in brackets in Eq. (6) goes to zero:

$$\frac{q}{c}\rho_0 B_0 = p_0 \tag{7}$$

But the absence of radial acceleration can also be achieved for a different momentum, $p = (1 + \varepsilon)p_0$, at a different radius, $\rho = \rho_0 + \Delta$, where the value of the offset, Δ, can be determined by Taylor's expansion of the bracketed term in Eq. (6):

$$\frac{q}{c}(\rho_0 + \Delta)\left(B_0 + \Delta\frac{\partial B(\rho)}{\partial \rho}\right) = (1 + \varepsilon)p_0 \tag{8}$$

in which the field derivative has the value, from Eq. (4), for ρ near ρ_0:

$$\frac{\partial B(\rho)}{\partial \rho} \cong -n\left(\frac{B_0}{\rho_0}\right) \tag{9}$$

† Strictly, momentum per unit charge, p_0/q.

with the result, from Eq. (6), that the radial offset, Δ, needed to achieve stability at the new momentum, p, is given by the relation

$$\Delta \cong \frac{\varepsilon \rho_0}{(1-n)} \tag{10}$$

The radial equation of a particle of arbitrary momentum p near p_0 around its stable position is most simply obtained by rewriting Eq. (5) in its more familiar form:

$$\rho'' + \frac{q}{c}\frac{B(\rho)}{p} - \frac{1}{\rho} = 0$$

Now, by expanding in a small displacement, x, about ρ_0, one obtains:

$$\frac{1}{\rho} = \frac{1}{\rho_0 + x} \cong \frac{1}{\rho_0}\left(1 - \frac{x}{\rho_0}\right) \qquad |x| \ll \rho_0$$

It is also noted that the momentum term can be expanded:

$$\frac{1}{\rho} = \frac{1}{p_0(1+\varepsilon)} \cong \frac{1}{p_0}(1-\varepsilon)$$

so that the equation of motion (now in x rather than ρ), may be written in the approximate form:

$$x'' + (1-n)\frac{x}{\rho_0^2} \cong \frac{\varepsilon}{\rho_0} \tag{11}$$

This can be readily interpreted as a simple-harmonic oscillator equation [or that of a "hyperbolic oscillator," if n exceeds unity, as suggested by Eq. (1)] but with an offset stable position. Thus, if the parameter, Δ, from Eq. (10) is introduced, it becomes, to first order:

$$x'' + \frac{(1-n)}{\rho_0^2}x = \frac{(1-n)}{\rho_0^2}\Delta \tag{11'}$$

The position x is the outward displacement from ρ_0, and for a particle of momentum p_0, oscillation must be about ρ_0. But if the following substitutions are made,

$$\begin{aligned} \eta(s) &= x(s) - \Delta \\ \eta'(s) &= x'(s) \end{aligned} \tag{12}$$

Eq. (11') takes the familiar form of Eq. (1),

$$\eta'' + \frac{(1-n)}{\rho_0^2}\eta = 0 \qquad \left[k^2 = \frac{(1-n)}{\rho_0^2}\right] \tag{13}$$

Oscillation must now be about the *displaced* position of radial stability, $(\rho_0 + \Delta)$. In this manner, it is possible to avoid the complexity of using a (3×3) matrix for determining the behavior of particles of variable momenta in bending magnets. As will be shown, certain conceptual matters are also simplified.

In the other direction perpendicular to s, the requirement of symmetry about the z-plane, together with the Maxwell equation applicable to a current-free region of space,

$$(\nabla \times \mathbf{H}) = (\nabla \times \mathbf{B}) = 0$$

can be shown to lead to the equation,

$$z'' + \frac{n}{\rho_0^2} z = 0 \tag{14}$$

which is correct (to first order) for all particles of momenta near p_0.

3 SOLUTIONS IN SPECIFIC TYPES OF BENDING MAGNETS

In focusing conditions, which require that n does not exceed unity when the radial motion is being considered, the elements of the transfer matrix of Eq. (2) are known to have the form:

$$\begin{aligned} C(s) &= \cos ks & S(s) &= k^{-1}\sin ks \\ C'(s) &= -k\sin ks & S'(s) &= \cos ks \end{aligned} \tag{15}$$

where the radial value of the constant k must necessarily be:

$$k = \frac{(1-n)^{1/2}}{\rho_0} \tag{16}$$

Defocusing in a magnet described by Eq. (4) results, in the radial direction, if n exceeds unity and thus leads to the matrix elements:

$$\begin{aligned} C(s) &= \cosh ks & S(s) &= k^{-1}\sinh ks \\ C'(s) &= k\sinh ks & S'(s) &= \cosh ks \end{aligned} \tag{17}$$

and here k is given by the relation of Eq. (16), because the lower sign applies in Eq. (1).

In both instances, $s/\rho_0 = \alpha$, the actual angle through which the particle of mean momentum p_0 is bent. Comparable formulas are valid in the z-direction, as is well-known, except that the condition $n > 0$ provides transverse focusing and $n < 0$ leads to defocusing, with respective application of Eqs. (15) and (17), and of the relations:

$$\begin{aligned} k &= \frac{(n)^{1/2}}{\rho_0}\ (n > 0,\ \text{focus}) \\ k &= \frac{(-n)^{1/2}}{\rho_0}\ (n < 0,\ \text{defocus}) \end{aligned} \tag{18}$$

For $n = 0$, a uniform-field magnet, the magnet simply acts as a drift space in the z-direction. Consider the solutions of Eq. (13) in a uniform-field magnet, where k is equal to ρ_0^{-1}:

$$\begin{aligned}\eta(s) &= \eta_0\left(\cos\frac{s}{\rho_0}\right) + \eta_0'\left(\rho_0 \sin\frac{s}{\rho_0}\right)\\ \eta'(s) &= -\eta_0\left(\frac{1}{\rho_0}\sin\frac{s}{\rho_0}\right) + \eta_0'\left(\cos\frac{s}{\rho_0}\right)\end{aligned} \tag{19}$$

which was obtained with the aid of Eqs. (15) and (16). Now if the relations of Eq. (12) are substituted in Eqs. (19) they may be rewritten:

$$\begin{aligned}x(s) &= x_0\left(\cos\frac{s}{\rho_0}\right) + x_0'\left(\rho_0 \sin\frac{s}{\rho_0}\right)\\ &\quad + \Delta\left(1 - \cos\frac{s}{\rho_0}\right)\\ x'(s) &= -x_0\left(\frac{1}{\rho_0}\sin\frac{s}{\rho_0}\right) + x_0'\left(\cos\frac{s}{\rho_0}\right)\\ &\quad + \Delta\left(\frac{1}{\rho_0}\sin\frac{s}{\rho_0}\right)\end{aligned} \tag{19$'$}$$

(with the relation $\eta_0 = x_0 - \Delta$ explicitly noted). Upon the use of the ε parameter instead of Δ, as given by Eq. (10), to represent a particle of momentum unequal to p_0, Eqs. (19′) now take the readily recognized form:

$$\begin{aligned}x(s) &= x_0\left(\cos\frac{s}{\rho_0}\right) + x_0'\left(\rho_0 \sin\frac{s}{\rho_0}\right)\\ &\quad + \varepsilon\rho_0\left(1 - \cos\frac{s}{\rho_0}\right)\\ x'(s) &= -x_0\left(\frac{1}{\rho_0}\sin\frac{s}{\rho_0}\right) + x_0'\left(\cos\frac{s}{\rho_0}\right) + \varepsilon\left(\sin\frac{s}{\rho_0}\right)\end{aligned} \tag{19$''$}$$

In the special case where $0 < n < 1$, the condition for which "betatron" oscillations occur, and especially for $n = 1/2$, where focusing of equal strength occurs in both directions, it can be shown that all familiar dispersive relations can be similarly retrieved. It is particularly noted that as n approaches unity from a lower value that the radial offset, Δ, becomes very large and such a magnet has an unusual amount of momentum dispersion. But when n becomes exactly equal to unity, the radial motion is simply that of a drift space with a bend through angle $\alpha = s/\rho_0$, because in this condition, all values of $B\rho$ are equal to $B_0\rho_0$.

An important case not commonly treated in the literature is that of dispersion in alternating-gradient (AG) magnets, in which n alternately assumes a value much less than and much greater than unity in successive magnets. Typical values of n might be -6 and $+6$. In the $n < 1$ magnet, there is obviously radial focusing and transverse defocusing. One can add the observation that the radial offset, Δ, for off-momentum particles, is relatively small, and from this develop a simple explanation of the high "momentum-compaction factor" that is characteristic of AG accelerators. In the $n > 1$ magnets, (which have comparably small values of the offset), the solution of Eq. (13) with the appropriate set of matrix elements of Eq. (17) and corresponding k-values of Eqs. (18), lead to the η,η' solutions:

$$\begin{aligned}\eta(s) &= \eta_0\left(\cosh\frac{(n-1)^{1/2}}{\rho_0}\frac{s}{\rho_0}\right)\\ &\quad + \eta_0'\left(\frac{\rho_0}{(n-1)^{1/2}}\sinh(n-1)^{1/2}\frac{s}{\rho_0}\right)\\ \eta'(s) &= \eta_0\left(\frac{(n-1)^{1/2}}{\rho_0}\sinh(n-1)^{1/2}\frac{s}{\rho_0}\right)\\ &\quad + \eta_0'\left(\cosh(n-1)^{1/2}\frac{s}{\rho_0}\right)\end{aligned} \tag{20}$$

The value of the offset of the stable orbit, Δ, is seen to be negative for positive ε in such magnets. That is, a particle of momentum higher than p_0 has its position of stability *at a radius inside* ρ_0. But since, from Eq. (4), $B\rho$ is seen to be greater than $B_0\rho_0$ at the smaller radii, this result is completely in agreement with the definition of Δ. When Eq. (2) is expanded in terms of x, x' and Δ, one obtains the following:

$$\begin{aligned}x(s) &= x_0\left(\cosh(n-1)^{1/2}\frac{s}{\rho_0}\right)\\ &\quad + x_0'\left(\frac{\rho_0}{(n-1)^{1/2}}\sinh(n-1)^{1/2}\frac{s}{\rho_0}\right)\\ &\quad + \Delta\left[1 - \cosh(n-1)^{1/2}\frac{s}{\rho_0}\right]\\ x'(s) &= x_0\left(\frac{(n-1)^{1/2}}{\rho_0}\sinh(n-1)^{1/2}\frac{s}{\rho_0}\right)\\ &\quad + x_0'\left(\cosh(n-1)^{1/2}\frac{s}{\rho_0}\right)\\ &\quad - \left(\Delta\frac{(n-1)^{1/2}}{\rho_0}\sinh(n-1)^{1/2}\frac{s}{\rho_0}\right)\end{aligned} \tag{20$'$}$$

In Eq. (20′) the coefficients of Δ are both intrinsically negative for a positive value of s. When they are further rewritten in the alternate form employing ε, the results are the following:

$$
\begin{aligned}
x(s) &= x_0\left(\cosh(n-1)^{1/2}\frac{s}{\rho_0}\right) \\
&\quad + x'_0\left(\frac{\rho_0}{(n-1)^{1/2}}\sinh(n-1)^{1/2}\frac{s}{\rho_0}\right) \\
&\quad + \varepsilon\left(\frac{\rho_0}{(n-1)}\left[\cosh(n-1)^{1/2}\frac{s}{\rho_0}-1\right]\right) \\
x'(s) &= x_0\left(\frac{(n-1)^{1/2}}{\rho_0}\sinh(n-1)^{1/2}\frac{s}{\rho_0}\right) \\
&\quad + x'_0\left(\cosh(n-1)^{1/2}\frac{s}{\rho_0}\right) \\
&\quad + \varepsilon\left(\frac{1}{(n-1)^{1/2}}\sinh\frac{s}{\rho_0}\right)
\end{aligned}
\qquad (20'')
$$

Equations (20″) imply a feature that at first appears to contradict the derived condition that sets stability at a lower radius: particles of momentum higher than p_0 still tend to move outward, as one would intuitively expect. But their position of stability, rather curiously, lies at a radius ρ smaller than ρ_0. However, it can be shown that if a particle of momentum $(1+\varepsilon)p_0$ is introduced under the conditions $x_0 = \Delta$, $x'_0 = 0$, it retains the position $x(s) = \Delta$ without deviation. The fundamental effect of the dispersion-related offset, Δ, in this particular case, is to induce rather large oscillations in the radial direction.

4 THE NONDISPERSIVE ENVELOPE EQUATION

To a beam of particles one can attribute an envelope, E, which is the maximum excursion of the outermost particle in the direction ξ at position s. The envelope is not allowed to exceed the aperture dimension at position s without the phase space suffering truncation. However, the equation of an envelope is not the same as that for an individual particle. In practice, an ensemble of particles is assumed to occupy a region of ξ, ξ' phase-space, bounded by a (centered) ellipse which has a constant area in ξ, ξ' phase-space equal to π times the emittance, e.† Emittance e is a constant throughout the system (unless truncation occurs). The ellipse equation is commonly given in the Courant–Snyder parameters[4] as:

$$\gamma\xi^2 + 2\alpha\xi\,\xi' + \beta\xi'^2 = e \qquad (21)$$

where α, β, and γ vary with s. In any channel element where ξ and ξ' motion can be described by Eq. (2), it can be shown that α, β, and γ obey the following matrix transformation equation [4]:

$$
\begin{bmatrix} \beta(s) \\ \alpha(s) \\ \gamma(s) \end{bmatrix}
= \begin{bmatrix} C^2(s) & -2C(s)S(s) & S^2(s) \\ -C(s)C'(s) & C(s)S'(s) + S(s)C'(s) & -S(s)S'(s) \\ C'^2(s) & -2C'(s)S'(s) & S'^2(s) \end{bmatrix}
\times \begin{bmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{bmatrix}. \qquad (22)
$$

with the concomitant result:

$$\beta'(s) = -2\alpha(s) \qquad (23)$$

Indeed, the parameters α, β, and γ are not independent but must satisfy the normalization relation:

$$\gamma\beta = 1 + \alpha^2 \qquad (24)$$

There are no dispersive terms in these equations (as they stand) and therefore they apply only to particles of momenta $p = p_0$. The ellipse shape at any s value may be computed by the above method. However, Steffen[5] presents a singularly elegant method (originated by Wünster[6]) for calculating the envelope value $E(s)$ by an ordinary second-order differential equation, and this is briefly recapitulated below because it must be extended to the dispersive case. In Figure 1 is shown the phase-space ellipse and its dimensions in terms of both the α, β, γ parameters and E, the envelope, A, the equivalent angular envelope, and E', the derivative of E with respect to s. The Courant–Snyder parameters are related to E, A, and E' in the following manner:

$$
\begin{aligned}
\gamma &= \frac{A^2}{e} \\
\beta &= \frac{E^2}{e} \\
\alpha &= -\frac{EE'}{e}
\end{aligned}
\qquad (25)
$$

† The notation "e" rather than the more common ε is used to indicate emittance to avoid confusion with $\Delta p/p$.

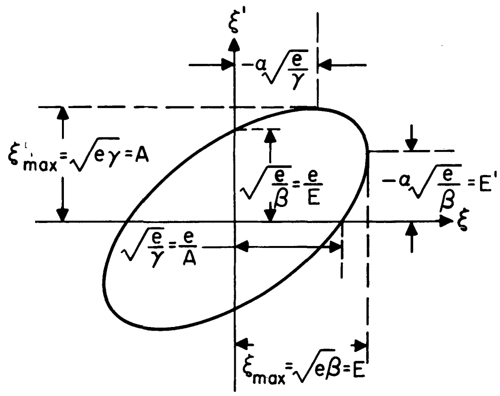

FIGURE 1 Emittance ellipse $\gamma\xi^2 + 2\alpha\,\xi\xi' + \beta\xi'^2 = e$ showing relation of parameters α, β, γ to E, E' A.

The relation of Eq. (23) between α and β' is seen to be consistent with that from Eqs. (25). The ellipse and the two different sets of parameters appear in Figure 1. When the values of Eq. (25) are inserted into the boundary ellipse equation, Eq. (21), one obtains

$$A^2\xi^2 - 2EE'\xi\xi' + E^2\xi'^2 = e^2 \tag{26}$$

and the normalization relation of Eq. (24) is similarly given by

$$A^2 = \frac{e^2}{E^2} + E'^2 \tag{27}$$

When this value for A^2 is substituted into Eq. (26), one obtains a mixed first order differential equation in ξ and E:

$$(E'\xi - E\xi')^2 = e^2\left(1 - \frac{\xi^2}{E^2}\right) \tag{28}$$

Now let Eq. (23) be differentiated by s. Because the Wronskian of E and ξ is not equal to zero (that is, because E and ξ must originate from distinct differential equations) the following second-order differential equation is developed:

$$(E''\xi - E\xi'') = \frac{e^2}{E^3}\,\xi \tag{29}$$

Upon substituting into Eq. (29) the value of ξ'' derived from Eq. (1), the desired differential equation for the envelope appears:

$$E'' \pm k^2E = \frac{e^2}{E^3}, \tag{30}$$

where again, the upper sign gives focusing and the lower, defocusing.

In regions of constant k, it is interesting to note that Eq. (30) can be completely integrated by analytic methods although the complete solutions are complicated and their final form is difficult to interpret. It is, however, helpful to note that the first integration is easily achieved by multiplication of Eq. (29) by an integrating factor $2E'$, which leads to this relation:

$$E'^2 \pm k^2E^2 = -\frac{e^2}{E^2} + \text{Constant}$$

and, by the use of Eq. (21), to obtain the following result:

$$A^2 \pm k^2E^2 = \text{Constant} \tag{31}$$

which provides a most useful relation between the envelope and the angular envelope.

5 THE DISPERSIVE ENVELOPE EQUATION

When a group of particles of mixed momentum enter a bending magnet, momentum dispersion will occur. Equations of motion for a particle of momentum $p \neq p_0$ have been presented [cf., Eqs. (19, 19′, 19″) for a uniform field magnet, Eqs. (20, 20′, 20″) for a magnet with $n > 1$] but the envelope equation for off-momentum particles is not yet known. To obtain this, first consider schematically what occurs when a particle beam of mixed momentum (but with all momenta having the same emittance ellipse) is injected into a bending magnet. For the purpose of illustration only, the x, x' plane for a focusing ($n < 1$) magnet is shown in Figure 2. Steffen has shown that if k is equal to unity in such a plot, the action of the magnet of a particle of $p = p_0$ can be represented as clockwise rotation of all points on the x, x' plane about the center. The angle, θ, through which the points are rotated on the plot is equal to ks and increases linearly as the particles are taken farther in s. The result is that the entire ellipse is rotated about its geometrical center, as shown in Figure 2.† Next, the behavior of a group of particles with momentum $p > p_0$ may be followed in the same bending magnet. In this example the offset, Δ, is positive and the emittance ellipse rotates through θ

† Had an arbitrary k-value been selected, both distortion and rotation would appear. The choice of $k = 1$ leads only to rotation.

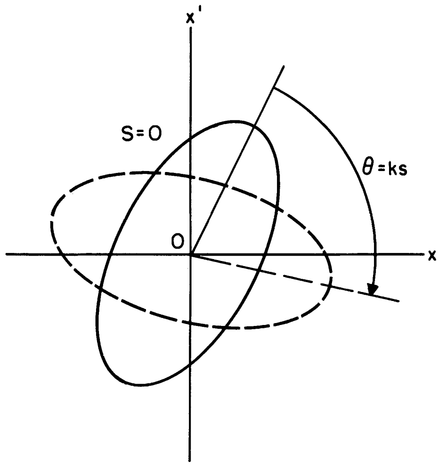

FIGURE 2 Rotation of emittance ellipse for particles of momentum $p = p_0$ about $x = x' = 0$. (Focusing magnet illustrated.)

about the point, P, ($x = \Delta$, $x' = 0$) as shown in Figure 3. This motion follows from Eqs. (12) and (13), the latter of which is equivalent to Eq. (1). Again for this particular example, the ellipse is on the average displaced toward a higher value of x' (as might be expected of a group of high-momentum particles in a focusing magnet) and the ellipse center has been displaced from O (0, 0) to a new position O′ (x_c, x_c'). It is convenient therefore to define new coordinates ζ and ζ', in the following way:

$$\begin{aligned} \zeta &= x - x_c \\ \zeta' &= x' - x_c' \end{aligned} \tag{32}$$

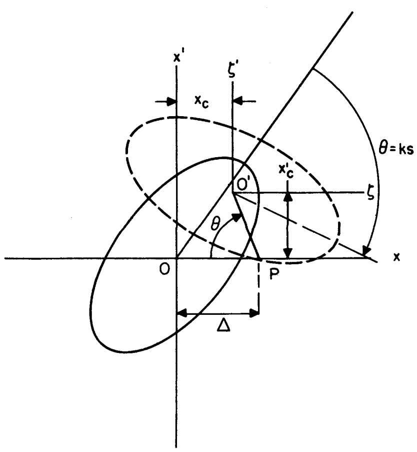

FIGURE 3 Rotation of emittance ellipse for particles of high momentum $p > p_0$ about $x = \Delta$, $x' = 0$. (Focusing magnet illustrated.)

in order to write the ellipse equation and also to obtain a reference point from which to calculate the envelope.

At this point, the focusing condition used for illustration can be abandoned and, in a very general way, the coordinates x_c and x_c' can be computed using Eqs. (2) (but with η in place of ξ) together with the definitions of Eq. (12):

$$\begin{aligned} \eta_c(s) &= C(s)\eta_{co} + S(s)\eta_{co}' = -\,C(s)\Delta \\ \eta_c'(s) &= C'(s)\eta_{co} + S'(s)\eta_{co}' = -\,C'(s)\Delta \end{aligned} \tag{33}$$

because the ellipse center O has the location $(-\Delta, 0)$ in η coordinates. Thus one obtains these equations for x_c and x_c':

$$\begin{aligned} x_c(s) &= \eta_c + \Delta = [1 - C(s)]\Delta \\ x_c'(s) &= \eta_c' = -\,C'(s)\Delta \end{aligned} \tag{34}$$

Therefore, the coordinates ζ and ζ' may be conveniently expressed in the following way:

$$\begin{aligned} \zeta &= x - x_c = x - [1 - C(s)]\Delta = \eta + C(s)\Delta \\ \zeta' &= x' - x_c' = x' + C'(s)\Delta = \eta' + C'(s)\Delta \end{aligned} \tag{35}$$

The equation for the emittance ellipse may consequently be written in coordinates ζ and ζ':

$$V^2\zeta^2 - 2U\,U'\zeta\zeta' + U^2\zeta^2 = e^2 \tag{36}$$

where U, U' and V replace E, E' and A, respectively, in the formalism of the previous section (and are, in fact, identical to them at $s = 0$). The size of the emittance ellipse remains constant (assuming that truncation does not occur, of course) and development of the U equation from Eq. (26) through Eq. (29) is exactly the same. One arrives at the relation:

$$(U''\zeta - U\zeta'') = \frac{e^2}{U^3}\zeta \tag{37}$$

by identical arguments to those used for obtaining Eq. (29). The distinction between U and V, E and A, is that the former are measured from x_c, x_c' positions rather than from 0, 0. It is now necessary to substitute for ζ'' in Eq. (37) and thus the second of Eqs. (35) is differentiated and related to the first.

$$\begin{aligned} \zeta'' &= \eta'' + C''(s)\Delta = \pm k^2\eta + C''(s)\Delta \\ &= \pm k^2\zeta + [C''(s) \pm k^2C(s)]\Delta \end{aligned}$$

But the coefficient of Δ—the term in square brackets—is identically equal to zero for either sign of k^2 or for k^2 equal to zero, because $C(s)$ is one solution of Eq. (1). Consequently, the differential equation for U is formally identical to that for E:

$$U'' \pm k^2 U = \frac{e^2}{U^3} \tag{38}$$

and may be solved by the same analytical method. But, as seen in Figure 4, once the solution for $U(s)$ has been found, the beam envelope E is no longer symmetrical about $x = x' = 0$ but must be obtained by adding and subtracting $U(s)$ from x_c:

$$\begin{aligned} E_1(s) &= [1 - C(s)]\Delta + U(s) \\ E_2(s) &= [1 - C(s)]\Delta - U(s) \end{aligned} \tag{39}$$

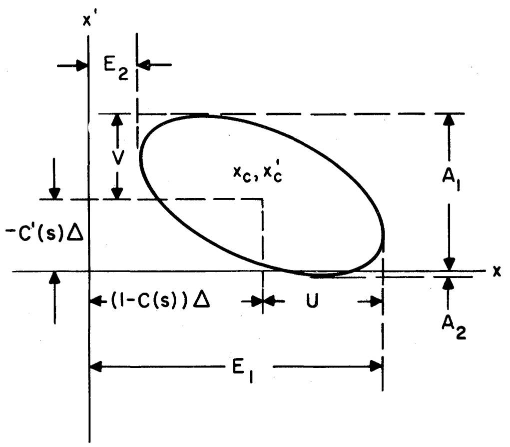

FIGURE 4 Relation of U, V to E, A.

The same treatment will yield $A_1(s)$ and $A_2(s)$, if desired, noting that $V(s)$ and $U(s)$ have the same interrelation as do $A(s)$ and $E(s)$ in Eq. (31),

$$V^2 \pm k^2 U^2 = \text{Constant} \tag{40}$$

In using any of these relations, especially in a defocusing ($n > 1$) magnet, care must be taken to ensure that the momentum offset, Δ, be given the correct sign.

6 CONCLUSION

The envelope equation can be solved for a particle beam of momentum unequal to p_0 as readily as can the equation for beam with p_0 momentum. The introduction of the offset distance, Δ, appears to be the key to achieving this simplicity. It is additionally useful in computational applications to note that (in first order) that Figure 3 shows not only the behavior of an x, x' emittance ellipse of momentum p greater than p_0 [such that the offset, Δ, is defined by Eq. (10)] but equally well the behavior of the emittance ellipse of momentum p_0 in the bending plane of a magnet which has been incorrectly positioned by a distance $(-\Delta)$ in the x direction or whose field value B has been set too low by an amount δB, given by:

$$\delta B = -B_0 \frac{\Delta}{\rho_0}(1 - n)$$

The positioning and field error equivalences had previously been incorporated into the TRACE code, which has not as yet been modified to compute dispersive envelopes for magnets of nonuniform field, but provides all other features needed for following beams in a magnetic beam channel.

Edge effects are of course not included intrinsically in this method but can be treated by the conventional thin-lens approximation.

ACKNOWLEDGMENTS

The author wishes to thank Professors R. M. Pearce and D. E. Lobb of the University of Victoria for their initial assistance in locating the source of the nondispersive beam envelope equation, and at Los Alamos Scientific Laboratory, Dr. K. R. Crandall for his reprogramming of the TRACE code and his helpful discussions, Dr. M. M. Minor for his adaptation of the modified TRACE code to the biomedical computer, and Drs. E. A. Knapp and L. Rosen for their support and assistance during 1973 and 1974.

REFERENCES

1. K. L. Brown, *A First- and Second-Order Matrix Theory for the Design of Beam Transport Systems and Charged Particle Spectrometers*, SLAC Report No. 75, Stanford Linear Accelerator Center, May 1969.
2. K. R. Crandall, TRACE: *An Interactive Beam-Transport Program*, LA-5332, UC-28, Los Alamos Scientific Laboratory, October 1973.
3. H. B. Knowles, *Nucl. Instrum. Methods*, **25**, 29 (1967).
4. E. D. Courant and H. S. Snyder, *Ann. Phys.*, **3**, 1 (1958).
5. K. G. Steffen, *High Energy Beam Optics*, (Interscience: John Wiley, New York, 1965), pp. 173–74.
6. *Ibid.*, p. 203, Ref. 11.
7. *Ibid.*, pp. 20–22.